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On the Compactness and Countable Compactness of $2^{\mathbb{R}}$ in ZF

by

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Summary. In the framework of ZF (Zermelo–Fraenkel set theory without the Axiom of Choice) we provide topological and Boolean-algebraic characterizations of the statements "$2^{\mathbb{R}}$ is countably compact" and "$2^{\mathbb{R}}$ is compact".

1. Notation and terminology

1. Let X be a non-empty set. 2^X will denote the Tychonoff product of the discrete space $2 = \{0, 1\}$. Likewise, $\mathcal{B}^X = \{[p] : p \in \mathrm{Fn}(X, 2)\}$, where $\mathrm{Fn}(X, 2)$ is the set of all finite partial functions from X into 2 and

$$[p] = \{f \in 2^X : p \subset f\},$$

will denote the standard clopen (= simultaneously closed and open) base for the topology on 2^X. If $A \subset X$, and $p \in \mathrm{Fn}(A, 2)$, then $[p]_A = \{f \in 2^A : p \subset f\}$. If $Y \subset 2^X$, then $\mathcal{B}^X|_Y = \{O \cap Y : O \in \mathcal{B}^X\}$. A non-empty collection $\mathcal{H} \subset \mathcal{P}(X)\backslash\{\emptyset\}$ has the *finite intersection property*, FIP for abbreviation, if $\forall \mathcal{Q} \in [\mathcal{H}]^{<\omega}, \bigcap \mathcal{Q} \neq \emptyset$.
2. Let (X, T) be a topological space.
   (a) X is said to be *compact* if every open cover of X has a finite subcover. Equivalently, X is compact iff for every family $\mathcal{G}$ of closed subsets of X having the FIP, $\bigcap \mathcal{G} \neq \emptyset$.

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(b) X is said to be *countably compact* if every countable open cover of X has a finite subcover. Equivalently, X is countably compact iff for every countable family $\mathcal{G}$ of closed subsets of X having the FIP, $\bigcap\mathcal{G} \neq \emptyset$.

(c) X is said to be *Lindelöf* if every open cover of X has a countable subcover.

(d) Let $\mathcal{E} \subset \mathcal{P}(X)\backslash\{\emptyset\}$. A non-empty collection $\mathcal{F} \subset \mathcal{E}$ is an *$\mathcal{E}$-filter* iff

  (i) if $F_1, F_2 \in \mathcal{F}$ then $F_1 \cap F_2 \in \mathcal{F}$,

  (ii) if $F \in \mathcal{F}$, $F' \in \mathcal{E}$ and $F \subset F'$, then $F' \in \mathcal{F}$.

  In particular, if $\mathcal{E}$ is the collection of all non-empty closed (respectively, clopen, open) sets of X, then we say that $\mathcal{F}$ is a *closed* (respectively, *clopen*, *open*) *filter*. If $\mathcal{E} = \mathcal{P}(X) \setminus \{\emptyset\}$ then an $\mathcal{E}$-filter is called simply a *filter* on X. An $\mathcal{E}$-filter $\mathcal{F}$ is *free* if $\bigcap\mathcal{F} = \emptyset$. An $\mathcal{E}$-filter $\mathcal{F}$ is an *$\mathcal{E}$-ultrafilter* if for every $\mathcal{E}$-filter $\mathcal{G}$ with $\mathcal{F} \subset \mathcal{G}$ we have $\mathcal{F} = \mathcal{G}$.

3. If A and B are any sets, then $|A| \leq |B|$ means that there exists a one-to-one function $f : A \to B$, and $|A| = |B|$ means that there exists a bijection $f : A \to B$.
4. AC($\mathbb{R}$) (Form 79 in [1]): Every family of non-empty subsets of $\mathbb{R}$ has a choice function.
5. CAC($\mathbb{R}$) (Form 94 in [1]): AC($\mathbb{R}$) restricted to countable families.
6. TP($2^{\mathbb{R}}$): The Tychonoff product $2^{\mathbb{R}}$ is compact.
7. TPC($2^{\mathbb{R}}$): The Tychonoff product $2^{\mathbb{R}}$ is countably compact.
8. $\mathrm{Pr}_X(2^{\mathbb{R}})$: If G is a closed subset of $2^{\mathbb{R}}$ then $\{g|_X : g \in G\}$ is a closed subspace of 2^X.
9. $\mathrm{Pr}(2^{\mathbb{R}})$: For all $X \subset \mathbb{R}$, $\mathrm{Pr}_X(2^{\mathbb{R}})$.
10. BPI(ω) (Form 225 in [1]): Every proper filter of $\mathcal{P}(\omega)$ can be extended to an ultrafilter.

2. Introduction and some preliminary results. The Tychonoff product of $\aleph_0$ copies of the two-element set $2 = \{0, 1\}$ with the discrete topology, i.e., the Cantor cube $\mathcal{K}_\omega = 2^{\aleph_0}$, is a universal space in the sense that every zero-dimensional, separable, metrizable space embed into it. Likewise, the Cantor cube $\mathcal{K}_{\mathbb{R}} = 2^{\mathbb{R}}$ of weight $\mathbb{R}$ is a universal space for the class $\mathcal{S}$ of all zero-dimensional spaces of weight $\leq |\mathbb{R}|$. It is well-known that $\mathcal{K}_\omega$ in ZF, and $\mathcal{K}_{\mathbb{R}}$ in ZFC, are compact spaces. In ZF, however, $\mathcal{K}_{\mathbb{R}}$ need not be compact. Indeed, in [4] we have shown:

THEOREM 1 ([4]). *The following statements are equivalent in* ZF:

(i) *In a Boolean algebra $\mathcal{B}$ of size $\leq |\mathbb{R}|$ every filter can be extended to an ultrafilter.*

(ii) BPI(ω).
(iii) *For every separable compact* T_2 *space* (X, T) *the product* $X^{\mathbb{R}}$ *is compact.*
(iv) *The product* $[0,1]^{\mathbb{R}}$ *is compact.*
(v) *Tychonoff products of finite subspaces of* $\mathbb{R}$ *are compact.*
(vi) $\mathcal{K}_{\mathbb{R}}$ *is compact.*

Since in Feferman's model (Model $\mathcal{M}2$ in [1]), ω has no free ultrafilters, it follows that "$\mathcal{K}_{\mathbb{R}}$ is compact" fails to hold in $\mathcal{M}2$, and consequently it is not a theorem of ZF. At this point one may ask whether the weaker statement "$\mathcal{K}_{\mathbb{R}}$ is countably compact" is a theorem of ZF. The answer is again no (Theorem 7).

In this paper we shall continue the research of [4] and find equivalent conditions under which "$\mathcal{K}_{\mathbb{R}}$ is compact" or "$\mathcal{K}_{\mathbb{R}}$ is countably compact".

THEOREM 2 ([5, Theorem 16.4(c)]). *A product of Hausdorff spaces with at least two points each is separable iff each factor is separable and there are* $\leq |\mathbb{R}|$ *factors. In particular, if* (X, T) *is a separable Hausdorff space then, in* ZF, *the product* $X^{\mathbb{R}}$ *is separable.*

PROPOSITION 3. (ZF) *If* $|X| = |Y|$, *then the Tychonoff products* 2^X *and* 2^Y *are homeomorphic.*

Proof. Let $h : X \to Y$ be a bijection. Then $H : 2^X \to 2^Y$ given by $H(f)(x) = f(h^{-1}(x))$ is easily seen to be a bijection. Since $H([(x,i)]) = [(h(x), i)]$ for all $i \in 2$ and $x \in X$, it follows that H maps a basic open set of 2^X to a basic of set of 2^Y. Thus, 2^X and 2^Y are homeomorphic as required. ∎

THEOREM 4 ([3]). (ZF) *For any well-ordered cardinal* $\aleph$, *the Tychonoff product* $2^{\aleph}$ *is compact.*

THEOREM 5 ([2], Cantor–Bernstein theorem). (ZF) *If* $|A| \leq |B|$ *and* $|B| \leq |A|$, *then* $|A| = |B|$.

3. Main results

THEOREM 6. *In* ZF, TPC($2^{\mathbb{R}}$) *implies that every family* $\mathcal{A} = \{A_n : n \in \omega\}$ *of non-empty finite subsets of* $\mathcal{P}(\mathbb{R})$ *such that* $\bigcup \mathcal{A}$ *is disjoint, has a choice function.*

Proof. This can be proved just as Theorem 2 in [3]. ∎

THEOREM 7. TPC($2^{\mathbb{R}}$) *is not provable in* ZF.

Proof. In the second Cohen model (model $\mathcal{M}7$ in [1]) there exists a countable family $\mathcal{A} = \{A_n : n \in \omega\}$ of two-element subsets of $\mathcal{P}(\mathbb{R})$ which admits no choice function in the model. Therefore, we may assume without loss of generality that for all $n \in \omega$, if $A_n = \{X, Y\}$, then $X \setminus Y \neq \emptyset$ and

$Y \setminus X \neq \emptyset$. Since $|\mathbb{R} \times \omega| = |\mathbb{R}|$, we may view the family $\mathcal{B} = \{B_n : n \in \omega\}$, where

$$B_n = \Big\{\Big(\Big(\bigcup A_n\Big) \setminus X\Big) \times \{n\} : X \in A_n\Big\},$$

as a family of two-element subsets of $\mathcal{P}(\mathbb{R})$ such that $\bigcup \mathcal{B}$ is disjoint. It follows that $\mathcal{B}$ has no choice function in $\mathcal{M}7$, hence by Theorem 6, $\mathrm{TPC}(2^{\mathbb{R}})$ fails to hold in $\mathcal{M}7$. ■

Clearly, for every $t \in \mathbb{R}$, the canonical projection of $2^{\mathbb{R}}$ on the t-th coordinate is a closed map. However, the statement $\mathrm{Pr}_X(2^{\mathbb{R}})$ need not be provable in ZF. In particular, we show in Theorem 8 below that $\mathrm{TPC}(2^{\mathbb{R}})$ and $\mathrm{Pr}_\omega(2^{\mathbb{R}})$ are equivalent.

THEOREM 8. *The following statements are equivalent in* ZF:

(i) $\mathrm{TPC}(2^{\mathbb{R}})$.
(ii) *Let G be a closed subset of $2^{\mathbb{R}}$. Then every countable family $\mathcal{F} \subset \mathcal{B}^{\mathbb{R}}|_G$ with the* FIP *has a non-empty intersection.*
(iii) *For every countably infinite subset X of* $\mathbb{R}$, $\mathrm{Pr}_X(2^{\mathbb{R}}.)$

Proof. (i)→(ii). This is straightforward.

(ii)→(iii). Let X be a countable subset of $\mathbb{R}$ and let G be a closed subset of $2^{\mathbb{R}}$. Without loss of generality assume that $X = \omega$. Suppose that $G_\omega = \{g|_\omega : g \in G\}$ is not closed in 2^ω, so let $f \in \overline{G}_\omega \setminus G_\omega$. For every $n \in \omega \setminus \{0\}$, let $f_n = f|_{\{0,1,\ldots,n-1\}}$. We set $V_n = [f_n] \cap G$ and $\mathcal{F} = \{V_n : n \in \omega \setminus \{0\}\}$. Clearly $V_n \neq \emptyset$ for all $n \in \omega \setminus \{0\}$ and $\mathcal{F}$ has the FIP. By hypothesis there exists $g \in \bigcap \mathcal{F}$. But then $f = g|_\omega \in G_\omega$, which is a contradiction.

(iii)→(i). Suppose that $2^{\mathbb{R}}$ is not countably compact and let $\mathcal{G} = \{G_n : n \in \omega \setminus \{0\}\}$ be a descending family of closed subsets of $2^{\mathbb{R}}$ having empty intersection. It is straightforward to verify that $G = \{g_n \in 2^\omega : n \in \omega\}$ is a closed subset of 2^ω, where for every $n \in \omega \setminus \{0\}$, g_n is the characteristic function of $A_n = \{n\}$, and g_0 is the characteristic function of the empty set. Since $|\mathbb{R}| = |\mathbb{R} \setminus \omega|$ we may assume without loss of generality that $\mathcal{G}$ is a family of closed subsets of $2^{\mathbb{R}\setminus\omega}$. For every $n \in \omega \setminus \{0\}$ put

$$B_n = \{f \in 2^{\mathbb{R}} : (f|_\omega = g_n) \wedge (f|_{\mathbb{R}\setminus\omega} \in G_n)\}.$$

CLAIM. *The set $H = \bigcup\{B_n : n \in \omega \setminus \{0\}\}$ is a closed subspace of* $2^{\mathbb{R}}$.

Proof of claim. Fix $f \in H^c$. We consider the following cases:

(a) $f|_\omega = g_n$ for some $n \in \omega \setminus \{0\}$. Then $f|_{\mathbb{R}\setminus\omega} \in G_n^c$. Let $p \in \mathrm{Fn}(\mathbb{R} \setminus \omega, 2)$ be such that $f|_{\mathbb{R}\setminus\omega} \in [p]_{\mathbb{R}\setminus\omega} \subset G_n^c$. Clearly, $[q]$, where $q = p \cup \{(n, 1)\}$, is a neighborhood of f avoiding H.

(b) $f|_\omega \neq g_n$ for all $n \in \omega$. Since G is a closed subset of 2^ω it follows that there exists a $p \in \mathrm{Fn}(\omega, 2)$ such that $f|_\omega \in [p]_\omega \subset G^c$. Then $f \in [p] \subset H^c$.

(c) $f|_\omega = g_0$. Since $\bigcap \mathcal{G} = \emptyset$ it follows that $f|_{\mathbb{R}\setminus\omega} \in G_n^c$ for some $n \in \omega$. Then $[\{(i,0) : i \le n\} \cup p]$, where $p \in \mathrm{Fn}(\mathbb{R}\setminus\omega, 2)$ and $f|_{\mathbb{R}\setminus\omega} \in [p]_{\mathbb{R}\setminus\omega} \subset G_n^c$, is clearly a neighborhood of f avoiding H.

Thus, H is closed in $2^{\mathbb{R}}$ as required.

Clearly, the projection of H to 2^ω is $G \setminus \{g_0\}$, which is not closed in 2^ω. This contradiction finishes the proof. ∎

Theorem 9. (ZF) *Each of the following statements implies the one beneath it*:

- (i) $\mathrm{TPC}(2^{\mathbb{R}})$.
- (ii) $2^{\mathbb{R}}$ *has no countably infinite closed relatively discrete subsets.*
- (iii) *Every countable clopen cover of* $2^{\mathbb{R}}$ *has a finite subcover.* (*Equivalently, every countable descending family of clopen subsets of* $2^{\mathbb{R}}$ *has a non-empty intersection*).
- (iv) *Every countable family of clopen subsets of* $2^{\mathbb{R}}$ *having the* FIP *can be extended to a clopen ultrafilter.*

Proof. (i)→(ii). This is straightforward.

(ii)→(iii). Let $\mathcal{G} = \{G_n : n \in \omega\}$ be a descending family of clopen subsets of $2^{\mathbb{R}}$. Assume that $\bigcap \mathcal{G} = \emptyset$. Let $D = \{d_m : m \in \omega\}$ be a countable dense subset of $2^{\mathbb{R}}$. Using the fact that D is countable we fix a $d_{m_n} \in (G_n \setminus G_{n+1}) \cap D$ for each $n \in \omega$. Since $\bigcap \mathcal{G} = \emptyset$, the set $K = \{d_{m_n} : n \in \omega\}$ is a closed relatively discrete subset of $2^{\mathbb{R}}$, contradicting (ii). Thus, $\bigcap \mathcal{G} \neq \emptyset$.

(iii)→(iv). Let $\mathcal{F} = \{F_n : n \in \omega\}$ be a descending family of clopen subsets of $2^{\mathbb{R}}$. By our hypothesis there exists an $f \in \bigcap \mathcal{F}$. Clearly, $\mathcal{G} = \{G \subset 2^{\mathbb{R}} : G \text{ is clopen and } f \in G\}$ is a clopen ultrafilter of $2^{\mathbb{R}}$ which includes $\mathcal{F}$. ∎

We next show that statements (iii) and (iv) of Theorem 9 are equivalent in ZF. First we need the following lemma. We denote by $\mathcal{S}$ the standard subbase of $2^{\mathbb{R}}$, that is, $\mathcal{S} = \{[(t,i)] : (t,i) \in \mathbb{R} \times 2\} \cup \{\emptyset, 2^{\mathbb{R}}\}$.

Lemma 10. (ZF)

- (i) *Every cover* $\mathcal{U}$ *of* $2^{\mathbb{R}}$ *consisting of standard subbasic open sets has a finite subcover* $\mathcal{V}$ *with* $|\mathcal{V}| \le 2$.
- (ii) $2^{\mathbb{R}}$ *has no free clopen ultrafilters.*

Proof. (i) Let $\mathcal{U} \subset \mathcal{S}$ be a cover of $2^{\mathbb{R}}$. Suppose that $2^{\mathbb{R}} \notin \mathcal{U}$. For each $t \in \mathbb{R}$, put $\mathcal{U}_t = \{i \in \{0,1\} : [(t,i)] \in \mathcal{U}\}$. We assert that there exists $t_0 \in \mathbb{R}$ such that $\mathcal{U}_{t_0} = \{0,1\}$. Assume not; then for each $t \in \mathbb{R}$ consider the least element $f(t) \in \{0,1\} \setminus \mathcal{U}_t$ and let $f = (f(t))_{t \in \mathbb{R}}$. It is evident that $f \notin \bigcup \mathcal{U}$, a contradiction. Clearly $\{[(t_0,0)], [(t_0,1)]\}$ is a finite subcover of $\mathcal{U}$.

(ii) Assume that $2^{\mathbb{R}}$ has a free clopen ultrafilter $\mathcal{F}$. Let $\mathcal{U} = \{C \in \mathcal{S} : C \notin \mathcal{F}\}$. Since $\bigcap \mathcal{F} = \emptyset$, it follows that $\mathcal{U}$ is a cover of $2^{\mathbb{R}}$. Indeed, let $f \in 2^{\mathbb{R}}$. Then there exists an $F \in \mathcal{F}$ such that $f \notin F$. Let $O = [\{(t_1, i_1), \ldots, (t_n, i_n)\}]$ be a basic neighborhood of f such that $O \cap F = \emptyset$. Since $O \notin \mathcal{F}$ and $\mathcal{F}$ is a filter, it follows that $[(t_j, i_j)] \notin \mathcal{F}$ for some $j \leq n$. Since $f \in [(t_j, i_j)]$ we infer that $\mathcal{U}$ is a cover of $2^{\mathbb{R}}$ as asserted. By (i), let $\mathcal{V}$ be a finite subcover of $\mathcal{U}$. Since $\mathcal{F}$ is a clopen ultrafilter, $V^c \in \mathcal{F}$ for all $V \in \mathcal{V}$, and as $\mathcal{F}$ is a filter it follows that $2^{\mathbb{R}} \neq \bigcup \mathcal{V}$, a contradiction. ∎

Theorem 11. *The following statements are equivalent in* ZF:

(i) *Every countable clopen cover of* $2^{\mathbb{R}}$ *has a finite subcover.*
(ii) *Every countable family of clopen subsets of* $2^{\mathbb{R}}$ *having the* FIP *can be extended to a clopen ultrafilter.*

Proof. (i)→(ii). This is shown in Theorem 9.

(ii)→(i). Let $\mathcal{U} = \{U_n : n \in \omega\}$ be a clopen cover of $2^{\mathbb{R}}$. Assume that $\mathcal{U}$ has no finite subcover. Then $\mathcal{V} = \{U_n^c : n \in \omega\}$ is a family of clopen subsets of $2^{\mathbb{R}}$ having the FIP. By our hypothesis, $\mathcal{V}$ can be extended to a clopen ultrafilter $\mathcal{W}$. By Lemma 10(ii) we have $\bigcap \mathcal{W} \neq \emptyset$. Hence $\mathcal{U}$ is not a cover of $2^{\mathbb{R}}$, a contradiction. ∎

Theorem 12. *The following statements are equivalent in* ZF:

(i) $\mathrm{TP}(2^{\mathbb{R}})$.
(ii) *Every closed filter of* $2^{\mathbb{R}}$ *can be extended to a closed ultrafilter.*
(iii) *Every clopen filter of* $2^{\mathbb{R}}$ *can be extended to a clopen ultrafilter.*
(iv) *Every open filter of* $2^{\mathbb{R}}$ *can be extended to an open ultrafilter.*
(v) *Every regular-open filter of* $2^{\mathbb{R}}$ *can be extended to a regular-open ultrafilter.*
(vi) $2^{\mathbb{R}}$ *is Lindelöf.*
(vii) *Every open cover of* $2^{\mathbb{R}}$ *has a well-ordered subcover.*

Proof. (i)→(ii). Fix a filter $\mathcal{C}$ of closed subsets of $2^{\mathbb{R}}$. Since $2^{\mathbb{R}}$ is compact it follows that $\bigcap \mathcal{C} \neq \emptyset$. Fix $g \in \bigcap \mathcal{C}$. Clearly, $\mathcal{F} = \{F \subset 2^{\mathbb{R}} : F$ is closed and $g \in F\}$ is a closed ultrafilter of $2^{\mathbb{R}}$ including $\mathcal{C}$.

(ii)→(iii). Let $\mathcal{G}$ be a family of clopen subsets of $2^{\mathbb{R}}$ having the FIP. By hypothesis, let $\mathcal{F}$ be a closed ultrafilter of $2^{\mathbb{R}}$ which includes $\mathcal{G}$. Clearly, $\mathcal{H} = \{F \in \mathcal{F} : F$ is clopen$\}$ is a clopen ultrafilter extending $\mathcal{G}$.

(iii)→(i). This can be proved as in (ii)→(i) of Theorem 11 using arbitrary basic open covers.

(i)→(v). Let $\mathcal{R}$ be the complete Boolean algebra of all regular-open subsets of $2^{\mathbb{R}}$. Let $\mathcal{D} = \{d_n : n \in \omega\}$ be a countable dense subset of $2^{\mathbb{R}}$ (see Theorem 2). It is not hard to verify that the function $f : \mathcal{R} \to \mathcal{P}(\mathcal{D})$, $f(O) = O \cap \mathcal{D}$ for all $O \in \mathcal{R}$, is one-to-one. Therefore, $|\mathcal{R}| \leq |\mathcal{P}(\mathcal{D})| = |\mathbb{R}|$.

Since clopen sets are regular-open and the standard clopen base $\mathcal{B}^{\mathbb{R}}$ of $2^{\mathbb{R}}$ has size $|\mathbb{R}|$, it follows from Theorem 5 that $|\mathcal{R}| = |\mathbb{R}|$. The conclusion now follows from Theorem 1.

(v)$\to$(iv). Fix a filter $\mathcal{G}$ of open subsets of $2^{\mathbb{R}}$ and let $\mathcal{H} \subset \mathcal{R}$ be the filter generated by the family $\{\mathrm{int}(\overline{G}) : G \in \mathcal{G}\}$, where $\mathrm{int}(A)$ denotes the interior of the set A. By hypothesis there exists an ultrafilter $\mathcal{W}$ of $\mathcal{R}$ which includes $\mathcal{H}$. Put

$$\mathcal{F} = \{O \subset 2^{\mathbb{R}} : O \text{ is open and } \mathrm{int}(\overline{O}) \in \mathcal{W}\}.$$

In order to complete the proof of the implication, it suffices to show:

Claim. *$\mathcal{G} \subset \mathcal{F}$ and $\mathcal{F}$ is an open ultrafilter of* $2^{\mathbb{R}}$.

Proof of Claim. The first assertion is straightforward. We next show that $\mathcal{F}$ is filter. Fix $O, Q \in \mathcal{F}$. Then $\mathrm{int}(\overline{O}) \cap \mathrm{int}(\overline{Q}) \in \mathcal{W}$. We show that

$$(*) \qquad \overline{\mathrm{int}(\overline{O}) \cap \mathrm{int}(\overline{Q})} = \overline{O \cap Q}.$$

The $\supseteq$ inclusion is clear. Conversely, fix $x \in \overline{\mathrm{int}(\overline{O}) \cap \mathrm{int}(\overline{Q})}$. Assume that $x \notin \overline{O \cap Q}$. Fix a neighborhood V_x of x such that $V_x \cap (O \cap Q) = \emptyset$ and let $W = V_x \cap (\mathrm{int}(\overline{O}) \cap \mathrm{int}(\overline{Q}))$. Clearly, $\emptyset \neq W \subset \overline{O} \cap \overline{Q}$. Fix $y \in W$ and let V_y be a basic neighborhood of y such that $V_y \subset W$. Then $P = V_y \cap O \neq \emptyset$. Fix now $z \in P$ and let V_z be a basic neighborhood of z such that $V_z \subset P$. Then $V_z \subset W$ (since $P \subset W$) and $S = V_z \cap Q \neq \emptyset$. It follows that $S \subset V_x$ satisfies $S \cap (O \cap Q) \neq \emptyset$, contradicting $V_x \cap (O \cap Q) = \emptyset$; this proves $(*)$. Consequently, $\mathrm{int}(\overline{O}) \cap \mathrm{int}(\overline{Q}) = \mathrm{int}(\overline{\mathrm{int}(\overline{O}) \cap \mathrm{int}(\overline{Q})}) = \mathrm{int}(\overline{O \cap Q})$, meaning that $O \cap Q \in \mathcal{F}$.

Now fix $O \in \mathcal{F}$ and let Q be an open set such that $O \subset Q$. Since $\mathrm{int}(\overline{O}) \subset \mathrm{int}(\overline{Q})$ and $\mathrm{int}(\overline{O}) \in \mathcal{W}$, it follows that $\mathrm{int}(\overline{Q}) \in \mathcal{W}$ and consequently $Q \in \mathcal{F}$.

We next show that $\mathcal{F}$ is not contained properly in any other open filter. Let H be an open set of $2^{\mathbb{R}}$ such that $H \cap O \neq \emptyset$ for all $O \in \mathcal{F}$. It can be readily verified that $\mathrm{int}(\overline{H}) \cap Q \neq \emptyset$ for all $Q \in \mathcal{W}$, and since $\mathcal{W}$ is not contained properly in any other regular-open ultrafilter, it follows that $\mathrm{int}(\overline{H}) \in \mathcal{W}$. Hence, $H \in \mathcal{F}$ and $\mathcal{F}$ is an open ultrafilter, finishing the proof of the claim and of the implication.

(iv)$\to$(iii). This can be proved similarly to (ii)$\to$(iii).

(i)$\to$(vi)$\to$(vii). These are straightforward.

(vii)$\to$(i). Fix a cover $\mathcal{U} \subset \mathcal{B}^{\mathbb{R}}$ of $2^{\mathbb{R}}$. By (vii) we may assume that $\mathcal{U}$ is well-ordered. Clearly, $A = \bigcup\{\mathrm{Dom}(p) : [p] \in \mathcal{U}\}$ is well-ordered (being a well-ordered union of finite subsets of $\mathbb{R}$). By Theorem 4, 2^A is compact. It is easy to see that $\mathcal{U}_A = \{[p]_A : [p] \in \mathcal{U}\}$ is an open cover of 2^A. Let $\mathcal{W}_A = \{[p_i]_A : i \leq n\}$ be a finite subcover. Clearly $\mathcal{W} = \{[p_i] : i \leq n\}$ is a finite subcover of $\mathcal{U}$, and so $2^{\mathbb{R}}$ is compact as required. ∎

Remark 13. 1. Note that the complete Boolean algebra $\mathrm{RO}(2^{\mathbb{R}})$ of all regular-open subsets of $2^{\mathbb{R}}$ does not coincide with the Boolean algebra $\mathrm{Clop}(2^{\mathbb{R}})$ of all clopen subsets of $2^{\mathbb{R}}$. Otherwise, $2^{\mathbb{R}}$ would be extremally disconnected (i.e., the closure of every open set would be open), which is not true. Indeed, let

$$V=\{f\in 2^{\mathbb{R}}: f^{-1}(0)\cap\omega\neq\emptyset \text{ and if } n_f \text{ is the least natural number such that } f(n_f)=0, \text{ then } f(n_f+1)=1\}.$$

It is straightforward to verify that V is an open set of $2^{\mathbb{R}}$. However, $\overline{V}=V\cup\{f\in 2^{\mathbb{R}}:(\forall n\in\omega)(f(n)=1)\}$ and the latter set is not open in $2^{\mathbb{R}}$.

2. Clearly, one can prove in ZF that $2^{\mathbb{R}}$ has closed as well as clopen ultrafilters. Indeed, for any $f\in 2^{\mathbb{R}}$, $\mathcal{F}=\{G\subset 2^{\mathbb{R}}: G$ closed (resp., clopen) and $f\in G\}$ is a closed (resp., clopen) ultrafilter of $2^{\mathbb{R}}$. If Q is a closed (resp., clopen) subset of $2^{\mathbb{R}}$ meeting non-trivially each member of $\mathcal{F}$ then $f\in Q$. If not then for some clopen neighborhood V_f of f we have $V_f\cap Q=\emptyset$, which is a contradiction. Thus, $Q\in\mathcal{F}$ and $\mathcal{F}$ is a closed (resp., clopen) ultrafilter of $2^{\mathbb{R}}$. However, we cannot prove in ZF that $2^{\mathbb{R}}$ has an open ultrafilter. The reason, as one can easily verify, is that in ZF an open ultrafilter $\mathcal{F}$ of $2^{\mathbb{R}}$ is always free. (If $\bigcap\{\overline{F}:F\in\mathcal{F}\}\neq\emptyset$, then $\bigcap\{\overline{F}:F\in\mathcal{F}\}=\{x\}$. Since $\{x\}^c\in\mathcal{F}$, we see that $\bigcap\mathcal{F}=\emptyset$.)

We leave the proofs of the next two theorems as an easy exercise for the reader.

Theorem 14. *The following statements are equivalent in* ZF:

- (i) $\mathrm{TPC}(2^{\mathbb{R}})$.
- (ii) *Every countable family of closed subsets of* $2^{\mathbb{R}}$ *having the* FIP *can be extended to a closed ultrafilter.*

Theorem 15. *The following statements are equivalent in* ZF:

- (i) *For every set X, the Tychonoff product 2^X is compact.*
- (ii) *For every set X, every closed filter of 2^X can be extended to a closed ultrafilter.*
- (iii) *For every set X, every clopen filter of 2^X can be extended to a clopen ultrafilter.*
- (iv) *For every set X, every open filter of 2^X can be extended to an open ultrafilter.*
- (v) *For every set X, every regular-open filter of 2^X can be extended to a regular-open ultrafilter.*
- (vi) BPI.
- (vii) *For every set X, 2^X is Lindelöf* $+\,\mathrm{CAC_{fin}}$ ($=$ AC *for countable families of non-empty finite sets*).

(viii) *For every set X, every open cover of 2^X has a well-ordered subcover* $+ \mathrm{AC}(\mathrm{WO}, < \aleph_0)$ (= AC *for well-ordered families of non-empty finite sets*).

In view of Theorem 1 we find that $\mathrm{TP}(2^{\mathbb{R}})$ implies that every Boolean algebra of size $\leq |\mathbb{R}|$ has an ultrafilter. In the following theorem we give a characterization of the latter statement.

THEOREM 16. *The following statements are equivalent in* ZF:

(i) *Every Boolean algebra of size $\leq |\mathbb{R}|$ has an ultrafilter.*
(ii) *Every family $\mathcal{B}$ of regular-open* (*respectively*, *clopen*, *closed*, *open*) *subsets of $2^{\mathbb{R}}$ which is closed under finite intersections contains a $\mathcal{B}$-ultrafilter.*

Proof. (i)→(ii). Fix a non-empty family $\mathcal{B}$ of regular-open subsets of $2^{\mathbb{R}}$ closed under finite intersections. Let B be the subalgebra of the Boolean algebra of all regular-open subsets of $2^{\mathbb{R}}$ which is generated by $\mathcal{B}$. By hypothesis B has an ultrafilter $\mathcal{F}$. Clearly, $\mathcal{F} \cap \mathcal{B}$ is a $\mathcal{B}$-ultrafilter.

(ii)→(i). Let $(\mathcal{B}, \oplus, \odot)$ be a Boolean algebra of size $\leq |\mathbb{R}|$. We shall show that there exists a non-trivial homomorphism $g : \mathcal{B} \to 2$. Then $g^{-1}(1)$ will be the required ultrafilter. Let $\mathcal{A} = \{A_i : i \in I \subset \mathbb{R}\}$ be the set of all finite subalgebras of $\mathcal{B}$. Without loss of generality we may assume that $I = \mathbb{R}$. Identify $\mathcal{B}$ and $\mathcal{A}$ with $\mathbb{R}$ and let

$$\begin{aligned}\mathcal{G} = \{[p] : \exists K \in [\mathbb{R}]^{<\omega}, \mathrm{Dom}(p) = \bigcup\{\{i\} \times A_i : i \in K\},\\ \forall i \in K,\ p(i, \cdot) : A_i \to 2 = \{0, 1\} \text{ is a non-trivial homomorphism,}\\ \text{and } \forall i, j \in K, \text{ if } A_i \subset A_j \text{ then } p(i, \cdot) \subset p(j, \cdot)\} \cup \{\emptyset\}.\end{aligned}$$

Clearly, $\mathcal{G}$ is a family of clopen subsets of $2^{\mathbb{R}\times\mathbb{R}}$ closed under finite intersections. By hypothesis, let $\mathcal{F}$ be a $\mathcal{G}$-ultrafilter.

CLAIM 1. *For every $i \in \mathbb{R}$ there is a $[p] \in \mathcal{F}$ with $\{i\} \times A_i \subset \mathrm{Dom}(p)$.*

Proof of Claim 1. Assume that for some $i \in \mathbb{R}$ and every $[p] \in \mathcal{F}$, $\{i\} \times A_i \not\subset \mathrm{Dom}(p)$. Then for every non-trivial homomorphism $q : A_i \to 2$ (A_i is a finite subalgebra and therefore such a q exists), $[\{i\} \times q] \in \mathcal{G}$ meets non-trivially each member of $\mathcal{F}$. Since $\mathcal{F}$ is maximal it follows that $[\{i\} \times q] \in \mathcal{F}$. This is a contradiction.

Define $g : \mathcal{B} \to 2$ by setting $g(b) = p((i, b))$, where $i \in \mathbb{R}$ is such that A_i is the subalgebra of $\mathcal{B}$ generated by $\{b\}$ and $[p] \in \mathcal{F}$ is such that $\{i\} \times A_i \subset \mathrm{Dom}(p)$.

CLAIM 2. *g is well-defined.*

Proof of Claim 2. Let $[p]$ and $[q]$ in $\mathcal{F}$ be such that $\{i\} \times A_i$ is included in both $\mathrm{Dom}(p)$ and $\mathrm{Dom}(q)$. Since $\mathcal{F}$ is a filter, p and q are compatible, hence $p((i, b)) = q((i, b))$. Thus, g is well-defined as required.

Since for every $a, b \in \mathcal{B}$, the Boolean subalgebra $\mathcal{B}(a,b)$ generated by a and b is finite, it follows that $\mathcal{B}(a,b) = A_i$ for some $i \in \mathbb{R}$. Fix $[p] \in \mathcal{F}$ with $\{i\} \times A_i \subset \mathrm{Dom}(p)$. Since $p(i,\cdot) : A_i \to 2$ is a homomorphism, it follows that $g(a \oplus b) = p(i, a \oplus b) = p(i,a) + p(i,b) = g(a) + g(b)$ and $g(a \odot b) = p(i, a \odot b) = p(i,a)p(i,b) = g(a)g(b)$. Thus, $g : \mathcal{B} \to 2$ is a (non-trivial) homomorphism as required. ■

4. Questions

1. Does $\mathrm{TPC}(2^{\mathbb{R}})$ imply $\mathrm{TP}(2^{\mathbb{R}})$ in ZF?
2. Is $\mathrm{TPC}(2^{\mathbb{R}})$ provable in $\mathrm{ZF} + \mathrm{CAC}(\mathbb{R})$?

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