Closure Theorem for Partially Semialgebraic Sets

by

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Summary. In 1988 it was proved by the first author that the closure of a partially semialgebraic set is partially semialgebraic. The essential tool used in that proof was the regular separation property. Here we give another proof without using this tool, based on the semianalytic L-cone theorem (Theorem 2), a semianalytic analog of the Cartan–Remmert–Stein lemma with parameters.

The authors began work on this joint paper in September 2002, but in November 2002 a heart attack put an end to Stanisław Łojasiewicz's life. The second author wishes to honour Professor Stanisław Łojasiewicz with this paper.

1. Let M be an analytic manifold. If X is a finite-dimensional real vector space of dimension n, then an analytic function $f : M \times X \to \mathbb{R}$ is said to be an X-*polynomial* if for an open neighborhood U of any point of M, in some (and then any) linear coordinate system $X \to \mathbb{R}^n_t$ the restriction $f_{U \times X}$ is a polynomial in t with coefficients analytic on U.

A subset $E \subset M \times X$ is said to be X-*semialgebraic* if every point $a \in M$ has an open neighborhood U such that E_U is described in $U \times X$ by X-polynomials. This definition implies that finite unions and finite intersections of X-semialgebraic sets are X-semialgebraic.

When Y is another finite-dimensional real vector space, an analytic mapping $M \times X \to Y$ is said to be an X-*polynomial mapping* if in some (and then any) linear coordinate system in Y its components are X-polynomials.

If N is a real analytic manifold, Y a finite-dimensional real vector space, $f : M \to N$ an analytic mapping and $g : M \times X \to Y$ an X-polynomial mapping, then the inverse image under the mapping $(x, v) \mapsto (f(x), g(x, v))$ of a Y-algebraic subset of $N \times Y$ is an X-semialgebraic subset of $M \times X$.

2. Let X be a topological space and let $\mathcal{A}$ be an algebra of continuous real functions on it. We say that a subset E of X is *described by* $\mathcal{A}$ if

$$E = \bigcup_{i=1}^{s} \bigcap_{j=1}^{r} E_{ij}$$

where E_{ij} is of the form $\{f_{ij} > 0\}$ or $\{f_{ij} < 0\}$ or $\{f_{ij} = 0\}$ with $f_{ij} \in \mathcal{A}$.

Let $S(\mathcal{A})$ be the class of subsets of X which are described by $\mathcal{A}$. Note that $\mathcal{A}[t]$ is an algebra of continuous real functions on $X \times \mathbb{R}$. Let $\pi : X \times \mathbb{R} \to X$ be the natural projection.

The following facts have been proved in [2, pp. 105–110].

PROPOSITION 1. *The connected components theorem for $S(\mathcal{A})$ implies*:

(1) *The connected components theorem for $S(\mathcal{A}[t])$.*
(2) $E \in S(\mathcal{A}[t]) \Rightarrow \pi(E) \in S(\mathcal{A})$.

Let $\mathcal{A}_U$ be the ring of analytic functions on a relatively compact neighborhood U of a given point $a \in M$ which have analytic extensions onto neighborhoods of $\overline{U}$. Then, by the connected components theorem, each set in $S(\mathcal{A}_U)$ has a finite number of connected components. Hence, by induction, we obtain the following statement:

(1) Each set in $S(\mathcal{A}_U[t_1, \ldots, t_m])$ has a finite number of components.
(2) If $E \in S(\mathcal{A}_U[t_1, \ldots, t_m])$, then $\pi(E)$ is semianalytic in M, where $\pi : M \times \mathbb{R}^k \to M$ is the natural projection for $k = 1, \ldots, m$.

As consequences, using linear coordinate systems, we obtain the following connected components theorem and generalized Tarski–Seidenberg theorem.

Let X be a finite-dimensional real vector space. We say a family $\mathcal{F}$ of subsets of $M \times X$ is *locally X-finite* if every point of M has a neighborhood U such that the family of non-empty traces of the sets of $\mathcal{F}$ on $U \times X$ is finite. Observe that the union of any locally X-finite family of X-semialgebraic subsets of $M \times X$ is X-semialgebraic.

CONNECTED COMPONENTS THEOREM. *All connected components of an X-semialgebraic set $E \subset M \times X$ are X-semialgebraic and their family is locally X-finite. Any open-closed subset of E is X-semialgebraic.*

Now, let Y be a finite-dimensional real vector space and let $\pi : M \times X \times Y \to M \times X$ be the natural projection.

TARSKI–SEIDENBERG THEOREM. *If $E \subset M \times X \times Y$ is $(X \times Y)$-semialgebraic, then $\pi(E)$ is X-semialgebraic (in $M \times X$).*

3. Let M be a real analytic manifold and let L be a finite-dimensional real vector space. We say that a semianalytic set $E \subset M \times L$ is an L-*cone* if each fiber E_x is a cone, for $x \in M$. This is equivalent to saying that $c_\lambda(E) = E$ for $\lambda > 0$, where $c_\lambda = \mathrm{id}_M \times \lambda \, \mathrm{id}_L : M \times L \to M \times L$.

We have the following straightforward properties:

(1) The closure of an L-cone is an L-cone.
(2) Finite unions, intersections and differences of L-cones are L-cones.
(3) The subset of points of dimension k of a semianalytic L-cone is a semianalytic L-cone.
(4) The set of smooth points of dimension k of an L-cone is a semianalytic L-cone.

We define an L-*conical germ* to be any set germ A at $(a, 0) \in M \times L$ satisfying $c_\lambda(A) = A$ for any $\lambda \in \mathbb{R}$, $\lambda > 0$. Clearly, the germ at $(a, 0)$, with $a \in M$, of any L-cone C in $M \times L$ is L-conical, and we have $\dim_{(a,0)} C = \dim C_{(a,0)}$. Observe that for L-cones C, D, we have

$$C \subset D \iff C_{(a,0)} \subset D_{(a,0)} \text{ for all } a \in M$$

and also

$$C = D \iff C_{(a,0)} = D_{(a,0)} \text{ for all } a \in M.$$

LEMMA 1. *For any L-conical semianalytic germ A at $(a, 0) \in M \times L$ there is a semianalytic L-cone C in $U \times L$, where U is an open neighborhood of a such that $C_{(a,0)} = A$. If A is analytic, so is C.*

Proof. Let E be a semianalytic representative of A in a neighborhood $U \times \Omega$ of $(a, 0)$. For any $u \in U$ and $v \in L$ with $|v| = 1$ there is a maximal segment $(0, r(u, v))v$ which is contained in E_u or disjoint from E_u (^{1}). As A is L-conical, we must have $r = \inf\{r(u, v) : u \in W, |v| = 1\} > 0$, with a relatively compact semianalytic neighborhood $W \subset U$ of a. Then, if $0 < s < r$, the set

$$D = E \cap (W \times \{|x| < s\}) \subset M \times L$$

is a union of some segments $u \times (0, s)v$ or $u \times [0, s)v$, with $|v| = 1$. It follows that $C = \bigcup_{\lambda > 0} c_\lambda(D)$ is a semianalytic L-cone (^{2}) such that $C_{(a,0)} = A$.

We will need the following fact:

PROPOSITION 2. *Any decreasing sequence E_ν of analytic subsets of M is locally stationary, i.e. each point $a \in M$ has a neighborhood U such that the sequence $E_\nu \cap U$ is stationary.*

(^{1}) Since $E_u \cap \mathbb{R}v$ is semianalytic.

(^{2}) Its germ at any point is the germ of some $c_\lambda(D)$ (at this point).

Proof. Since at any point $c \in M$ the ring $\mathcal{A}_c$ of analytic germs at c is noetherian, the sequence of germs $(E_\nu)_0$ is stationary, i.e. for some p we have $(E_\nu)_0 = (E_p)_0$ for all $\nu \geq p$. Taking a stratification at c compatible with E_p (^{3}), i.e. such that $E_p \cap U$ is the union of some of its leaves, and using a sort of the identity principle: *if an open non-empty subset of a connected analytic leaf Λ is contained in E_p, then $\Lambda \subset E_p$*, we get the result.

We have the following real analog of the Cartan–Remmert–Stein lemma with parameters.

We say that a subset E of $M \times L$ is *L-algebraic* if each $a \in M$ has an open neighborhood U such that E is defined (by equalities) by polynomials in $x \in L$ with coefficients analytic in U (it can always be defined by one function).

THEOREM 1. *Each analytic L-cone in $M \times L$ is L-algebraic.*

Proof. Fix any $a \in M$. Our L-cone E is defined in some open neighborhood $W = U \times V$ of $(a, 0)$ by an analytic function $f = \sum_{i=0}^{\infty} h_i$, where h_i is a form of degree i with coefficients analytic in U. For any $(x, v) \in U \times V$ we have the equivalence

$$f(x, v) = 0 \iff h_i(x, v) = 0, \ i = 0, 1, 2, \ldots.$$

In fact, if $f(x, v) = 0$, then, as E is an L-cone, we have $0 = f(x, tv) = \sum_{i=0}^{\infty} h_i(x, v)t^i$ for $t > 0$ small enough, hence $h_i(x, v) = 0$, $i = 0, 1, \ldots$. The sequence of analytic sets $E_k = \{(x, v) \in U \times V : h_i(x, v) = 0, i = 1, \ldots, k\}$ is decreasing, hence by Proposition 2, it must be locally stationary, so, diminishing W, we have $\{f = 0\} \cap W = E_k$ for some k. Thus the sets E and E_k are equal in W, and so, being L-cones, they must be equal in $U \times L$.

PROPOSITION 3. *For any semianalytic germ S at a point $a \in M$, there exists the smallest analytic germ at a which contains S. Moreover, it has the same dimension as S.*

Proof. Let $\mathcal{A}$ be the ring of germs of analytic functions at a and $\mathcal{I}$ the ideal of germs which vanish on S. As $\mathcal{A}$ is noetherian, the ideal $\mathcal{I}$ is generated by germs at a of some analytic functions $f_1, \ldots, f_s$ in a neighborhood of a. Then the analytic germ

$$\{f_1 = \cdots = f_s = 0\}_a$$

is as required. The second part follows by taking a normal stratification compatible with a representative of S.

PROPOSITION 4. *Any semianalytic L-cone is contained in an algebraic L-cone of the same dimension.*

(^{3}) E.g. a normal one (see [4]).

Proof. Let E be our semianalytic L-cone. Let $a \in M$. By Proposition 3 we take the smallest analytic germ S which contains $E_{(a,0)}$. It is L-conical since also $c_\lambda(S)$ is such a germ (in view of $c_\lambda(E_{(a,0)}) = E_{(a,0)}$). By Lemma 1 there is an analytic L-cone C such that $C_{(a,0)} = S$. It has the same dimension as E and it is algebraic, by Theorem 1.

4. We will now need some other facts from semianalytic geometry. For any subset E of a topological space X, we define a decreasing sequence of closed sets $V_i = V_i(E)$, $i = 1, 2, \ldots$, as follows. We define by recursion $E_0 = E$ and $E_{i+1} = \overline{E}_i \setminus E_i$, and we put $V_i = V_i(E) = \overline{E}_i$. In particular, $V_0 = \overline{E}$ and $V_0 \supset V_1 \supset \cdots$.

LEMMA 2. *If $V_{2r} = \emptyset$, then $E = (V_0 \setminus V_1) \cup \cdots \cup (V_{2r-2} \setminus V_{2r-1})$.*

Proof. In fact, $E_i = \overline{E}_i \setminus E_{i+1}$, and also

$$(V_i \setminus V_{i+1}) \cup (V_{i+1} \setminus E_{i+1}) = \overline{E}_i \setminus \overline{E}_{i+1}.$$

Hence, for p even, we have

$$E_{2r-p} = (V_{2r-p} \setminus V_{2r-p+1}) \cup (V_{2r-p+2} \setminus V_{2r-p+3}) \cup \cdots \cup (V_{2r-2} \setminus V_{2r-1}),$$

and we get the result for $p = 2r$.

PROPOSITION 5. *A set $E \subset M$ is semianalytic if and only if the sets $V_i = V_i(E)$ are closed semianalytic and $V_s = \emptyset$ for some s. Then V_{i+1} is nowhere dense in V_i, $i = 0, 1, \ldots$, so $\dim V_i \leq \dim E$, and we have $V_i = \emptyset$ for $i > n = \dim M$. If $2r > n$, then*

$$(*) \qquad\qquad E = (V_0 \setminus V_1) \cup \cdots \cup (V_{2r-2} \setminus V_{2r-1}).$$

Observe that if E is a cone, so are the V_i's.

Proof. Indeed, by Lemma 2, the condition is sufficient. Now suppose that the set E is semianalytic. Then the sets V_i are semianalytic, V_{i+1} is nowhere dense in V_i, $i = 0, 1, \ldots$, and $V_i = \emptyset$ for $i > n$ (see [4, II.5, II.7]). Applying Lemma 2 we get $(*)$. The last statement follows from the definition of $V(E)$.

PROPOSITION 6. *If E is semianalytic in M, then the set*

$$\{x \in M : \dim_x E = k\}$$

is semianalytic.

Proof. In fact, it is sufficient to take a distinguished (or normal) stratification of an interval Q, compatible with E (see [1]). Then the points of our set in Q are precisely those which belong to closures (in Q) of leaves of dimension k and do not belong to the closure of any leaf of dimension $> k$.

REMARK 1. If a semianalytic set is of constant dimension k, then so is its closure.

5. We have the following semianalytic analog of the Cartan–Remmert–Stein lemma with parameters.

THEOREM 2. *Any semianalytic L-cone is L-semialgebraic.*

Proof. Let E be a semianalytic L-cone in $M \times L$. We proceed by induction on the dimension k of E. By Proposition 5, we have

$$E = (F_0 \setminus F_1) \cup \cdots \cup (F_{p-1} \setminus F_p),$$

where the F_i are closed semianalytic L-cones of dimension $\leq k$. In view of Proposition 6 and Remark 1, for each $F = F_i$ we have

$$F = S_k \cup \cdots \cup S_0,$$

with S_i a closed semianalytic L-cone of constant dimension i.

By Proposition 4, any L-cone S_i is contained in an L-algebraic L-cone C_i of dimension i. Let S_i^* the set of non-smooth points of S_i. It is a closed semianalytic L-cone of dimension $< i < k$, and it is L-semialgebraic by the induction hypothesis. Then $S_i \setminus S_i^*$ is an analytic submanifold of dimension i; it is dense in S_i (as the latter is of constant dimension i). Next the set C_i^* of points of C_i which are not smooth of dimension i is a closed L-semialgebraic L-cone of dimension $< i$, and $C_i \setminus C_i^*$ is an analytic submanifold.

The set $T = S_i^* \cup C_i^*$ is closed semialgebraic of dimension $< i$. Then $S_i \setminus T$ is closed in $C_i \setminus T$, but it is also open (in the latter set) as both are analytic submanifolds of dimension i. Hence $S_i \setminus T$ is a locally L-finite union of connected components of $C_i \setminus T$, and so it is L-semialgebraic. Since S_i is the closure of $S_i \setminus T$, it must be L-semialgebraic. Consequently, so are the F_i's and E.

6. Let X, Y be finite-dimensional real vector spaces. Consider the hyperplane $H = X \times Y \times 1 \subset X \times Y \times \mathbb{R}$. Let F be a subset of $X \times Y$, and put

$$c(F) = \bigcup_{(x,y) \in F} \{x\} \times \mathbb{R}_+(y, 1),$$

with $\mathbb{R}_+ = (0, \infty)$. Hence, $c(F) \cap H = F \times 1$.

REMARK 2. Let $E \subset X \times Y$ be a Y-cone. Then

$$\widetilde{E} = \{(x, y, z, t) : (x, y) \in E,\ z = ty,\ t > 0\} \subset X \times Y \times Y \times \mathbb{R}$$

is a $(Y \times \mathbb{R})$-cone, and $c(E) = \mu(\widetilde{E})$, where $\mu(x, y, z, t) = (x, z, t)$ for (x, y, z, t) in $X \times Y \times Y \times \mathbb{R}$. Thus, by the Tarski–Seidenberg theorem, if E is a Y-semialgebraic Y-cone then $c(E)$ is a Y-semialgebraic $(Y \times \mathbb{R})$-cone.

REMARK 3. Let $M = X \times Y \times \mathbb{R}_+$ and consider $E \subset X \times Y$. Then $\mathrm{cl}_M(c(E)) = c(\mathrm{cl}_{X \times Y} E)$.

This yields the following equivalence:

PROPOSITION 7. *Let $E \subset X \times Y$ be any Y-cone. Then E is Y-semial-gebraic if and only if $c(E)$ is semianalytic.*

Proof. Consider the $(Y \times \mathbb{R})$-cone $\widetilde{E}$ as in Remark 2. It is $(Y \times Y)$-semialgebraic. By the Tarski–Seidenberg theorem $\mu(\widetilde{E})$ is Y-semialgebraic, hence $c(E)$ is semianalytic. Conversely, since $c(E) \cap H = E \times 1$, the set E is semianalytic. Hence, by Theorem 2, E is Y-semialgebraic.

CLOSURE THEOREM. *The closure of any Y-semialgebraic subset of $X \times Y$ is Y-semialgebraic.*

Proof. Let A be a Y-semialgebraic set. Put $E = c(A)$. Then E is a Y-cone and it is Y-semialgebraic. By Proposition 7, the cone $c(E)$ is semianalytic. Thus $\overline{c(E)}$ is semianalytic (by the closure theorem for semianalytic sets, see [4, II.5, Cor. 5.2]). The set $\overline{E}$ is a Y-cone, and it is also Y-semialgebraic, in view of Remark 3 and Proposition 7. Remark 3 implies that $c(\overline{A})$ is Y-semialgebraic. Hence so is $\overline{A}$, since $\overline{A} \equiv c(\overline{A}) \cap H$.

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