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FUNCTIONAL ANALYSIS

Isomorphisms of Cartesian Products of ℓ-Power Series Spaces

by

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Summary. Let ℓ be a Banach sequence space with a monotone norm $\|\cdot\|_\ell$, in which the canonical system (e_i) is a normalized symmetric basis. We give a complete isomorphic classification of Cartesian products $E_0^\ell(a) \times E_\infty^\ell(b)$ where $E_0^\ell(a) = K^\ell(\exp(-p^{-1}a_i))$ and $E_\infty^\ell(b) = K^\ell(\exp(pa_i))$ are finite and infinite ℓ-power series spaces, respectively. This classification is the generalization of the results by Chalov *et al.* [Studia Math. 137 (1999)] and Djakov *et al.* [Michigan Math. J. 43 (1996)] by using the method of compound linear topological invariants developed by the third author.

1. Introduction. Let ℓ be a Banach sequence space in which $\{e_i = (\delta_{i,j})_{j\in\mathbb{N}} : i \in \mathbb{N}\}$ forms an unconditional basis. The norm $\|\cdot\|_\ell$ is called *monotone* [4] if $\|x\|_\ell \le \|y\|_\ell$ whenever $x = (\xi_i)$, $y = (\eta_i)$, $|\xi_i| \le |\eta_i|$, $i \in \mathbb{N}$. We denote by Λ the set of all such spaces ℓ with monotone norm, and by $\Lambda^{(\mathrm{s})}$ the class of those of them with symmetric canonical basis $\{e_i\}$. For a given $\ell \in \Lambda$ and a Köthe matrix $A = (a_{i,n})_{i,n\in\mathbb{N}}$ we define the *ℓ-Köthe space* $X = K^\ell(A)$ as a Fréchet space of scalar sequences $x = (\xi_i)$ such that $(\xi_i a_{i,n}) \in \ell$, for each n, with the topology generated by the system of seminorms $\{|(\xi_i)|_n := \|(\xi_i a_{i,n})\|_\ell : n \in \mathbb{N}\}$.

We generalize some results from [2], [3] (see Theorems 10 and 8 below) by considering ℓ-Köthe spaces instead of usual Köthe spaces with $\ell = l^p$. Here we use a certain version of compound linear topological invariants developed in [8]–[10]. For the sake of transparency we simplify, as compared with [2], [3], the principal part of the proof (Lemma 5 is important to this end), omitting some elementary but long computations. The more general situation calls for

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some revision of the compound invariant method, such as using S. Krein's interpolation method of analytic scales (see Lemmas 6, 7); we also prefer to use the modified basic characteristic $\widetilde{\beta}(V,U)$ (see Section 3 below).

2. Preliminaries. Set $\mathcal{P} := \{a = (a_i)_{i\in\mathbb{N}} : a_i \geq 1, \forall i\}$. For $a \in \mathcal{P}$ we introduce the weighted ℓ-space as $\ell(a) := \{x = (\xi_i) : \|x\|_{\ell(a)} := \|(\xi_i\, a_i)\|_\ell < \infty\}$. For $a \in \mathcal{P}$ we consider its *counting function* ([6], [7]):

$$\mu_a(\tau,t) := |\{n \in \mathbb{N} : \tau < a_n < t\}|, \quad 0 < \tau < t < \infty,$$

where $|S|$ is the number of elements in S if it is finite, and ∞ otherwise.

PROPOSITION 1 (see [10]). *Let $a = (a_i), b = (b_i) \in \mathcal{P}$ and*

$$\mu_a(\tau,t) \leq \mu_b(\tau/\Delta, \Delta t), \quad 1 \leq \tau < t < \infty, \tag{1}$$

for some constant $\Delta > 1$. Then there is an injection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $a_i \leq \Delta^2 b_{\sigma(i)}$ and $b_{\sigma(i)} \leq \Delta^2 a_i$ for $i \in \mathbb{N}$.

Let $A := (a_{i,n})_{i,n\in\mathbb{N}}$, $B := (b_{j,n})_{j,n\in\mathbb{N}}$ be Köthe matrices and $\ell \in \Lambda^{(\mathrm{s})}$. Then the Cartesian product of the ℓ-Köthe spaces $K^\ell(A)$ and $K^\ell(B)$ is naturally isomorphic to the space $K^\ell(C)$ where $C = (c_{k,n})_{k,n\in\mathbb{N}}$ is such that $c_{k,n}$ equals $a_{i,n}$ if $k = 2i - 1$, and $b_{i,n}$ if $k = 2i$. For $a \in \mathcal{P}$ and $\lambda_n \nearrow \alpha$, $-\infty < \alpha \leq \infty$, we call the ℓ-Köthe space $E^\ell_\alpha(a) := K^\ell(\exp(\lambda_n a_i))$ an *ℓ-power series space of finite* (respectively, *infinite*) *type* if $\alpha < \infty$ (respectively, $\alpha = \infty$).

Let $X = K^\ell(A)$ and $\widetilde{X} = K^\ell(\widetilde{A})$ be ℓ-Köthe spaces. An operator $T : X \to \widetilde{X}$ is called *quasidiagonal* if there exists an injection $\varphi : \mathbb{N} \to \mathbb{N}$ and constants t_i, $i \in \mathbb{N}$, such that $Te_i := t_i e_{\varphi(i)}$, $i \in \mathbb{N}$. We write $X \overset{\mathrm{qd}}{\simeq} \widetilde{X}$ ($X \overset{\mathrm{qd}}{\hookrightarrow} \widetilde{X}$) if there is a quasidiagonal isomorphism (respectively, a quasidiagonal imbedding) $T : X \to \widetilde{X}$.

LEMMA 2 (cf. [10], [7]). *Let X and $\widetilde{X}$ be ℓ-Köthe spaces with $X \overset{\mathrm{qd}}{\hookrightarrow} \widetilde{X}$ and $\widetilde{X} \overset{\mathrm{qd}}{\hookrightarrow} X$. Then $X \overset{\mathrm{qd}}{\simeq} \widetilde{X}$.*

3. Geometric invariant characteristics. Let $\mathcal{X}$ be a class of locally convex spaces and Γ be a set with an equivalence relation $\sim$. We say that $\gamma : \mathcal{X} \to \Gamma$ is a *linear topological invariant* if $X \simeq \widetilde{X}$ implies $\gamma(X) \sim \gamma(\widetilde{X})$, $X, \widetilde{X} \in \mathcal{X}$. For more details about linear topological invariants we refer to [10].

Suppose E is a vector space, U and V are absolutely convex sets in E, and $\mathcal{E}_V$ is the set of all finite-dimensional subspaces of E that are spanned by elements of V. Set $\mathcal{L}(V,U) := \{L \in \mathcal{E}_V : \exists q := q(L) < 1,\, L \cap U \subset qV\}$. Dealing with Banach sequence spaces with monotone norm, it is convenient to consider the characteristic

$$\widetilde{\beta}(V,U) := \sup\{\dim L : L \in \mathcal{L}(V,U)\},$$

which is a modification of the characteristic $\beta(V,U)$ (see, e.g., [10], [2], [3]). We shall use the following obvious properties of this characteristic: (a) if $V_1 \subset V$, $U \subset U_1$ then $\widetilde{\beta}(V_1,U_1) \le \widetilde{\beta}(V,U)$; (b) $\widetilde{\beta}(CV,U) = \widetilde{\beta}(V,C^{-1}U)$ for any constant $C > 0$; (c) if T is a linear injection on E then $\widetilde{\beta}(T(V),T(U)) = \widetilde{\beta}(V,U)$; (d) $\widetilde{\beta}(V\cap F, U\cap F) \le \widetilde{\beta}(V,U)$ if F is a subspace of E.

Let E be a vector sequence space containing the system $\{e_i\}_{i\in\mathbb{N}}$. Given $a \in \mathcal{P}$, we define the weighted ball $B^\ell(a) = \{x \in E \cap \ell(a) : \|x\|_{\ell(a)} \le 1\}$. For any $a = (a_i)$, $b = (b_i) \in \mathcal{P}$ we set $a \wedge b := (\min\{a_i,b_i\})$ and $a \vee b := (\max\{a_i,b_i\})$.

LEMMA 3 (cf. [10], [2], [3]). *Let $a,b \in \mathcal{P}$. Then*

(i) $\frac{1}{2}B^\ell(a\wedge b) \subset \mathrm{conv}(B^\ell(a)\cup B^\ell(b)) \subset B^\ell(a\wedge b)$;
(ii) $B^\ell(a\vee b) \subset B^\ell(a)\cap B^\ell(b) \subset 2B^\ell(a\vee b)$.

Proof. (i) Let $I := \{i \in \mathbb{N} : a_i \le b_i\}$, $J := \mathbb{N}\setminus I$ and $x = (\xi_i)_{i\in\mathbb{N}} \in B^\ell(a\wedge b)$. Define $u = (u_i)$ so that $u_i = \xi_i$ if $i \in I$ and 0 otherwise; set $v := x - u$. Then, by the monotonicity of the norm, we have

$$\|u\|_{\ell(a)} = \|u\|_{\ell(a\wedge b)} \le \|x\|_{\ell(a\wedge b)}, \qquad \|v\|_{\ell(b)} = \|v\|_{\ell(a\wedge b)} \le \|x\|_{\ell(a\wedge b)}.$$

Hence $u,v \in B^\ell(a)\cup B^\ell(b)$ and $\frac{1}{2}x = \frac{1}{2}u + \frac{1}{2}v \in \mathrm{conv}(B^\ell(a)\cup B^\ell(b))$.

For the second inclusion, take $x = \sum_{i=1}^n \lambda_i u_i \in \mathrm{conv}(B^\ell(a)\cup B^\ell(b))$, where $u_i \in B^\ell(a)\cup B^\ell(b)$ and $\sum_{i=1}^n \lambda_i = 1$. By the monotonicity of the norm, we have either $\|u_i\|_{\ell(a\wedge b)} \le \|u_i\|_{\ell(a)} \le 1$ or $\|u_i\|_{\ell(a\wedge b)} \le \|u_i\|_{\ell(b)} \le 1$. Hence, in both cases

$$\|x\|_{\ell(a\wedge b)} = \Big\|\sum_{i=1}^n \lambda_i u_i\Big\|_{\ell(a\wedge b)} \le \sum_{i=1}^n \lambda_i \|u_i\|_{\ell(a\wedge b)} \le \sum_{i=1}^n \lambda_i = 1,$$

that is, $x \in B^\ell(a\wedge b)$.

(ii) Let $x \in B^\ell(a\vee b)$. By the monotonicity of the norm, we have $\|x\|_{\ell(a)} \le \|x\|_{\ell(a\vee b)} \le 1$ and $\|x\|_{\ell(b)} \le \|x\|_{\ell(a\vee b)} \le 1$. Hence $x \in B^\ell(a)\cap B^\ell(b)$. For the second inclusion, take $x \in B^\ell(a)\cap B^\ell(b)$, that is, $\|x\|_{\ell(a)} \le 1$ and $\|x\|_{\ell(b)} \le 1$. Then the monotonicity of the norm yields

$$\|(\xi_i \max\{a_i,b_i\})\|_\ell \le \|(\xi_i a_i + \xi_i b_i)\|_\ell \le \|x\|_{\ell(a)} + \|x\|_{\ell(b)}.$$

Thus we get $\|x\|_{\ell(a\vee b)} \le 2$, hence $x \in 2B^\ell(a\vee b)$. ∎

LEMMA 4 (cf. [10], [2], [3]). $\widetilde{\beta}(B^\ell(a), B^\ell(b)) = |\{i : a_i/b_i < 1\}|$.

Proof. Let $I = \{i : a_i < b_i\}$ and M be the linear span of the set $\{e_i : i \in I\}$. Define a projection $P : E \to M$ such that $Px := \sum_{i\in I} \xi_i e_i$ where $x = (\xi_i) \in E$. Take $x = (\xi_i) \in M \cap B^\ell(b)$. If $\dim M = \infty$, then $\sup\{\dim L : L \in \mathcal{L}(V,U)\} = \infty$. So, trivially we have $\widetilde{\beta}(B^\ell(a), B^\ell(b)) = \dim M$. If M is finite-dimensional, then $\|x\|_{\ell(a)} < \|x\|_{\ell(b)} \le 1$. So there exists $q = q(M) < 1$ such that $\|x\|_{\ell(a)} < q$ and $M \cap B^\ell(b) \subset qB^\ell(a)$, that is, $\widetilde{\beta}(B^\ell(a), B^\ell(b)) \ge \dim M = |I|$.

To obtain $\widetilde{\beta}(B^\ell(a), B^\ell(b)) \le |I|$ we assume the contrary. Then there exists L such that $|I| < \dim L$. Hence we can find $x = \sum_{i=1}^\infty \xi_i e_i \in L$, $x \ne 0$, such that $Px = 0$. But then $\xi_i = 0$ for $i \in I$ and $\xi_i \ne 0$ for some $i \notin I$. Since $a_i \ge b_i$ for $i \notin I$, and the norm is monotone, we obtain $\|(\xi_i a_i)\|_\ell \ge \|(\xi_i b_i)\|_\ell$, that is, $\|x\|_{\ell(a)} \ge \|x\|_{\ell(b)}$. On the other hand, since $x \in L$, we get $\|x\|_{\ell(a)} \le q\|x\|_{\ell(b)}$ with $q = q(L) < 1$, which implies $\|x\|_{\ell(a)} < \|x\|_{\ell(b)}$. This contradiction completes the proof. ∎

LEMMA 5 (cf. [1], [2], [10]). *Let $a^{(j)} = (a_{ij}) \in \mathcal{P}$, $j = 1,2,3,4$. Then*

$$\widetilde{\beta}\Big(B^\ell(a^{(4)}) \cap B^\ell(a^{(3)}), \operatorname{conv}\Big(\frac{1}{2}\,(B^\ell(a^{(3)}) \cup B^\ell(a^{(2)}) \cup B^\ell(a^{(1)}))\Big)\Big) \ge \left|\left\{i : \frac{a_{i4}}{a_{i3}} \le 1,\ \frac{a_{i3}}{a_{i2}} \le 1,\ \frac{a_{i3}}{a_{i1}} \le 1\right\}\right|, \tag{2}$$

$$\widetilde{\beta}\Big(B^\ell(a^{(4)}) \cap B^\ell(a^{(3)}) \cap B^\ell(a^{(2)}), \operatorname{conv}\Big(\frac{1}{2}\,(B^\ell(a^{(2)}) \cup B^\ell(a^{(1)}))\Big)\Big) \ge \left|\left\{i : \frac{a_{i4}}{a_{i2}} \le 1,\ \frac{a_{i3}}{a_{i2}} \le 1,\ \frac{a_{i2}}{a_{i1}} \le 1\right\}\right|. \tag{3}$$

Proof. By Lemmas 3 and 4,

$$\begin{aligned}
&\widetilde{\beta}\Big(B^\ell(a^{(4)}) \cap B^\ell(a^{(3)}), \operatorname{conv}\Big(\frac{1}{2}\,(B^\ell(a^{(3)}) \cup B^\ell(a^{(2)}) \cup B^\ell(a^{(1)}))\Big)\Big) \\
&\quad \ge \widetilde{\beta}\Big(B^\ell(a^{(4)} \vee a^{(3)}), \frac{1}{2}\,B^\ell(a^{(3)} \wedge a^{(2)} \wedge a^{(1)})\Big) \\
&\quad = \left|\left\{i : \frac{\max\{a_{i4}, a_{i3}\}}{\min\{a_{i3}, a_{i2}, a_{i1}\}} < 2\right\}\right| \ge \left|\left\{i : \frac{\max\{a_{i4}, a_{i3}\}}{\min\{a_{i3}, a_{i2}, a_{i1}\}} \le 1\right\}\right| \\
&\quad = \left|\left\{i : \frac{a_{i3}}{a_{i3}} \le 1,\ \frac{a_{i4}}{a_{i1}} \le 1,\ \frac{a_{i4}}{a_{i2}} \le 1,\ \frac{a_{i4}}{a_{i3}} \le 1,\ \frac{a_{i3}}{a_{i2}} \le 1,\ \frac{a_{i3}}{a_{i1}} \le 1\right\}\right|.
\end{aligned}$$

The first three inequalities in the last braces can be omitted, since the first one is always true, while the second and third are consequences of the others, as $a_{i4}/a_{i1} = (a_{i4}/a_{i3})(a_{i3}/a_{i1})$ and $a_{i4}/a_{i2} = (a_{i4}/a_{i3})(a_{i3}/a_{i2})$. Hence, (2) is proved.

Analogously we have

$$\widetilde{\beta}\left(B^{\ell}(a^{(4)}) \cap B^{\ell}(a^{(3)}) \cap B^{\ell}(a^{(2)}), \operatorname{conv}\left(\frac{1}{2}\,(B^{\ell}(a^{(2)}) \cup B^{\ell}(a^{(1)}))\right)\right)$$
$$\geq \left|\left\{i : \frac{a_{i2}}{a_{i2}} \leq 1,\ \frac{a_{i3}}{a_{i1}} \leq 1,\ \frac{a_{i4}}{a_{i1}} \leq 1,\ \frac{a_{i3}}{a_{i2}} \leq 1,\ \frac{a_{i4}}{a_{i2}} \leq 1,\ \frac{a_{i2}}{a_{i1}} \leq 1\right\}\right|.$$

Again removing unimportant inequalities in the last braces, we obtain (3). ∎

Now we construct an *analytic scale of Banach spaces* ([5, IV.1]) connecting the spaces $\ell(a)$ and $\ell(b)$.

LEMMA 6. *Let $\ell \in \Lambda$ and $a, b \in \mathcal{P}$. Then $E_\alpha = \ell(a^{1-\alpha}b^{\alpha})$ is an analytic scale such that $E_0 = \ell(a)$ and $E_1 = \ell(b)$.*

Proof. Consider the normed linear space $M := \{x = (\xi_k) \in \ell(a) : \exists k_0 = k_0(x),\ \xi_k = 0 \text{ for } k \geq k_0\}$, which is a dense subspace of $\ell(a)$. We define an operator $T(z) : M \to M$ by $T(z)x := (\xi_k(b_k/a_k)^z)$ where $x = (\xi_k)$. Clearly conditions 1°–5° in the definition of the analytic scale ([5, IV,1.9]) are satisfied. By the monotonicity of the norm,

$$\|x\|_\alpha := \sup_{-\infty<\tau<\infty} \|T(\alpha + i\tau)x\|_{\ell(a)} = \sup_{-\infty<\tau<\infty} \left\|\left(\xi_k\left(\frac{b_k}{a_k}\right)^{\alpha+i\tau} a_k\right)\right\|_{\ell}$$
$$= \sup_{-\infty<\tau<\infty} \left\|\left(\xi_k\left(\frac{b_k}{a_k}\right)^{i\tau} (a_k)^{1-\alpha}(b_k)^{\alpha}\right)\right\|_{\ell} = \|x\|_{\ell(a^{1-\alpha}b^{\alpha})}.$$

Hence, $E_\alpha := \ell(a^{1-\alpha}b^{\alpha})$. ∎

Applying the interpolation theorem for analytic scales ([5, IV, Theorem 1.10]) to the above scale we obtain the following

LEMMA 7 (cf. [10], [2], [3]). *Suppose E and $\widetilde{E}$ are ℓ-Köthe spaces, (e_i) and $(\widetilde{e}_i)$ are their canonical bases, and $T : E \to \widetilde{E}$ is a linear operator. If $a, \widetilde{a}, b, \widetilde{b} \in \mathcal{P}$ and $T(B^{\ell}(a)) \subset B^{\ell}(\widetilde{a})$, $T(B^{\ell}(b)) \subset B^{\ell}(\widetilde{b})$ then for any $\alpha \in (0,1)$ we have*

$$T((B^{\ell}(a))^{1-\alpha}(B^{\ell}(b))^{\alpha}) \subset (B^{\ell}(\widetilde{a}))^{1-\alpha}(B^{\ell}(\widetilde{b}))^{\alpha},$$

where $(B^{\ell}(a_m))^{1-\alpha}(B^{\ell}(a_n))^{\alpha} = B^{\ell}(a^{1-\alpha}b^{\alpha})$.

4. Imbedding of ℓ-power series spaces

THEOREM 8. *Let $\ell \in \Lambda^{(s)}$ and $a, \widetilde{a} \in \mathcal{P}$. Then the following statements are equivalent:*

(i) $E^{\ell}_{\nu}(a) \hookrightarrow E^{\ell}_{\nu}(\widetilde{a})$, $\nu = 0, \infty$;

(ii) *there exist $\Delta > 0$ and an injection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $a_i \leq \Delta \widetilde{a}_{\sigma(i)}$ and $\widetilde{a}_{\sigma(i)} \leq \Delta a_i$;*

(iii) $E^{\ell}_{\nu}(a) \overset{\mathrm{qd}}{\hookrightarrow} E^{\ell}_{\nu}(\widetilde{a})$, $\nu = 0, \infty$.

Proof. The implications (ii)⇒(iii) and (iii)⇒(i) are obvious. Due to Proposition 1, it remains to prove that (i) implies the estimate (1).

Because of similarity, we only consider the case $\nu = \infty$. Suppose that $T : E^\ell_\nu(a) \to E^\ell_\nu(\widetilde{a})$ is an embedding. Set $U_p := B^\ell(\alpha_p)$, $V_p := B^\ell(\widetilde{\alpha}_p)$, $\alpha_p := (\exp(pa_i))_{i\in\mathbb{N}}$ and $\widetilde{\alpha}_p := (\exp(p\widetilde{a}_i))_{i\in\mathbb{N}}$. So, $(p^{-1}U_p)$ and $(p^{-1}V_p)$ are bases of neighborhoods of zero in $E^\ell_\nu(a)$ and $E^\ell_\nu(\widetilde{a})$, respectively. Let $W_p := V_p \cap R(T)$, where $R(T)$ denotes the range of T. Since T is an isomorphism onto its range, we can choose indices

$$p_1 < p < q < q_1 < r_1 < r \tag{4}$$

so that

$$\frac{p}{p_1} W_{p_1} \supset T(U_p) \supset T(U_q) \supset \frac{q}{q_1} W_{q_1} \supset \frac{r}{r_1} W_{r_1} \supset T(U_r) \tag{5}$$

and each number in (4) is twice the previous one. The elementary properties of the characteristic $\widetilde{\beta}$ yield

$$\begin{aligned} \widetilde{\beta}(e^{-\tau}U_p \cap e^t U_r, U_q) &= \widetilde{\beta}(e^{-\tau}T(U_p) \cap e^t T(U_r), T(U_q)) \\ &\le \widetilde{\beta}(K(e^{-\tau}W_{p_1} \cap e^t W_{r_1}), W_{q_1}) \le \widetilde{\beta}(K(e^{-\tau}V_{p_1} \cap e^t V_{r_1}), V_{q_1}) \end{aligned} \tag{6}$$

with $K = r^2$. Taking Lemmas 4 and 3 into account, we estimate the left-hand side of (6) from below and the right-hand side from above; this yields

$$\begin{aligned} &\left|\left\{ i : \frac{\max\{\exp(\tau + pa_i), \exp(-t + ra_i)\}}{\exp(qa_i)} < 1 \right\}\right| \\ &\qquad \le \left|\left\{ i : \frac{\max\{\exp(\tau + p_1\widetilde{a}_i), \exp(-t + r_1\widetilde{a}_i)\}}{\exp(q_1\widetilde{a}_i)} < 2K \right\}\right|, \end{aligned} \tag{7}$$

which is equivalent to

$$\left|\left\{ i : \frac{\tau}{q-p} < a_i < \frac{t}{r-q} \right\}\right| \le \left|\left\{ i : \frac{\tau - \ln(2K)}{q_1 - p_1} < \widetilde{a}_i < \frac{t + \ln(2K)}{r_1 - q_1} \right\}\right|. \tag{8}$$

Changing variables we obtain the estimate (1) with $\Delta = 2r$, which ends the proof. ∎

Corollary 9 (cf. [6], [7], [2]). *Let $\ell \in \Lambda^{(s)}$ and $a, \widetilde{a} \in \mathcal{P}$. Then the following statements are equivalent*:

(i) $E^\ell_\nu(a) \simeq E^\ell_\nu(\widetilde{a})$, $\nu = 0, \infty$;

(ii) *there exist $\Delta > 0$ and a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that*

$$\frac{1}{\Delta} a_i < \widetilde{a}_{\sigma(i)} < \Delta a_i;$$

(iii) $E^\ell_\nu(a) \overset{\text{qd}}{\simeq} E^\ell_\nu(\widetilde{a})$, $\nu = 0, \infty$.

5. Isomorphisms of Cartesian products of ℓ-power series spaces

THEOREM 10. *Let $\ell \in \Lambda^{(\mathrm{s})}$ and $a, b, \widetilde{a}, \widetilde{b} \in \mathcal{P}$. If $E_0^\ell(a) \times E_\infty^\ell(b) \simeq E_0^\ell(\widetilde{a}) \times E_\infty^\ell(\widetilde{b})$, then there exist $\Delta, \tau_0 > 0$ such that:*

$$|\{k : \tau \le a_k \le t\}| \le |\{k : \tau/\Delta \le \widetilde{a}_k \le \Delta t\}|, \tag{9}$$

$$|\{k : \tau \le b_k \le t\}| \le |\{k : \tau/\Delta \le \widetilde{b}_k \le \Delta t\}|, \tag{10}$$

where $t > \tau \ge \tau_0$.

Proof. The Cartesian products $E_0^\ell(a) \times E_\infty^\ell(b)$ and $E_0^\ell(\widetilde{a}) \times E_\infty^\ell(\widetilde{b})$ are naturally isomorphic to the ℓ-Köthe spaces $X = K^\ell(c_{ip})$ and $\widetilde{X} = K^\ell(d_{ip})$ where

$$c_{ip} = \begin{cases} \exp(-a_k/p) & \text{if } i = 2k-1, \\ \exp(pb_k) & \text{if } i = 2k, \end{cases} \qquad d_{ip} = \begin{cases} \exp(-\widetilde{a}_k/p) & \text{if } i = 2k-1, \\ \exp(p\widetilde{b}_k) & \text{if } i = 2k. \end{cases}$$

Let $T : X \to \widetilde{X}$ be an isomorphism. Set $U_p := B^\ell(\alpha_p)$, $V_p := B^\ell(\widetilde{\alpha}_p)$, $\alpha_p := (c_{ip})_{i\in\mathbb{N}}$ and $\widetilde{\alpha}_p := (d_{ip})_{i\in\mathbb{N}}$. Then $(p^{-1}U_p)$ and $(p^{-1}V_p)$ are bases of neighborhoods of zero in X and $\widetilde{X}$, respectively. Since T is an isomorphism, we can choose indices $p_2 < p < p_1 < q_2 < q < q_1 < r_2 < r < r_1 < s_2 < s < s_1$ so that each of them is twice the previous one and

$$\begin{aligned} &\frac{p}{p_2} V_{p_2} \supset T(U_p) \supset \frac{p}{p_1} V_{p_1}, \qquad \frac{q}{q_2} V_{q_2} \supset T(U_q) \supset \frac{q}{q_1} V_{q_1}, \\ &\frac{r}{r_2} V_{r_2} \supset T(U_r) \supset \tfrac{r}{r_1} V_{r_1}, \qquad \frac{s}{s_2} V_{s_2} \supset T(U_s) \supset \frac{s}{s_1} V_{s_1}. \end{aligned} \tag{11}$$

By properties of $\widetilde{\beta}$, using (11) and Lemma 7, we obtain the estimates

$$\begin{aligned} &\widetilde{\beta}\Big(e^t U_s \cap U_q, \operatorname{conv}\Big(\frac{1}{2}\,(U_q \cup U_p^{1/2} U_r^{1/2} \cup e^\tau U_r)\Big)\Big) \\ &\qquad \le \widetilde{\beta}\Big(K(e^t V_{s_2} \cap V_{q_2}), \operatorname{conv}\Big(\frac{1}{2}\,(V_{q_1} \cup V_{p_1}^{1/2} V_{r_1}^{1/2} \cup e^\tau V_{r_1})\Big)\Big), \end{aligned} \tag{12}$$

$$\begin{aligned} &\widetilde{\beta}\Big(U_p^{1/2} U_r^{1/2} \cap e^t U_r \cap U_q, \operatorname{conv}\Big(\frac{1}{2}\,(U_q \cup e^\tau U_s)\Big)\Big) \\ &\qquad \le \widetilde{\beta}\Big(K(V_{p_2}^{1/2} V_{r_2}^{1/2} \cap e^t V_{r_2} \cap V_{q_2}), \operatorname{conv}\Big(\frac{1}{2}\,(V_{q_1} \cup e^\tau V_{s_1})\Big)\Big) \end{aligned} \tag{13}$$

with $K = s_1^2$. Now we estimate the left-hand side of (12) from below, using Lemma 5, and the right-hand side of (12) from above, applying Lemmas 3 and 4; this results in the following inequality:

$$\begin{aligned} &\left|\left\{i : \frac{c_{iq}}{c_{ip}^{1/2} c_{ir}^{1/2}} \le 1,\ \frac{c_{iq}}{e^{-\tau} c_{ir}} \le 1,\ \frac{e^{-t} c_{is}}{c_{iq}} \le 1\right\}\right| \\ &\qquad \le \left|\left\{i : \frac{d_{iq_2}}{d_{ip_1}^{1/2} d_{ir_1}^{1/2}} \le 16K,\ \frac{d_{iq_2}}{e^{-\tau} d_{ir_1}} \le 16K,\ \frac{e^{-t} d_{is_2}}{d_{iq_1}} \le 16K\right\}\right|. \end{aligned} \tag{14}$$

We examine the first inequality on the left-hand side of (14). For odd indices i we have $(-1/q+1/2p+1/2r)a_k \le 0$, which is impossible because $2p<q$. For even indices i it yields $(2q-p-r)b_k \le 0$, which is trivially true since $2q<r$. As a result the left-hand side of (14) equals

$$\left|\left\{k : \frac{\tau}{r-q} \le b_k \le \frac{t}{s-q}\right\}\right|. \tag{15}$$

In an analogous way, consider the right-hand side of (14). For odd indices i, the first inequality is equivalent to $\widetilde{a}_k \le \ln(16K)/(-1/q_2+1/2p_1+1/2r_1)$ $=: C$. Thus, for $\tau > \tau_1 := C(1/q_2-1/r_1)+\ln(16K)$, the first inequality on the right-hand side of (14) does not hold for odd indices. For even indices i, the first inequality on the right-hand side of (14) is equivalent to $(2q_2 - p_1 - r_1)\widetilde{b}_k \le \ln(16K) =: M$, which is always true since $2q_2<r_1$. Hence, for $\tau>\tau_1$ the right-hand side of (14) is equal to

$$\left|\left\{k : \frac{\tau-M)}{r_1-q_2} \le \widetilde{b}_k \le \frac{t+M}{s_2-q_1}\right\}\right|. \tag{16}$$

Since (15) is less than (16) for $\tau>\tau_1$, we observe that

$$\left|\left\{k : \frac{\tau}{r-q} \le b_k \le \frac{t}{s-q}\right\}\right| \le \left|\left\{k : \frac{\tau-M}{r_1-q_2} \le \widetilde{b}_k \le \frac{t+M}{s_2-q_1}\right\}\right|. \tag{17}$$

Analogously, from (13) we obtain

$$\begin{aligned}\left|\left\{k : \frac{\tau}{1/q-1/s} \le a_k \le \frac{t}{1/q-1/r}\right\}\right| \\ \le \left|\left\{k : \frac{\tau-M}{1/q_2-1/s_1} \le \widetilde{a}_k \le \frac{t+M}{1/q_1-1/r_2}\right\}\right|\end{aligned} \tag{18}$$

for

$$\tau > \tau_2 := \frac{(s_1-q_2)\ln(16K)}{p_2/2+r_2/2-q_1} + M.$$

Changing variables in (17), (18) and setting $\Delta = 2s_1$, one can easily check that the relations (9) and (10) are satisfied for $\tau \ge \tau_0 := 2\max\{\tau_1,\tau_2\}$. ∎

As in [3], we derive the following

COROLLARY 11. *Under the conditions of Theorem* 10 *we have either*

$$E_0^\ell(a) \simeq E_0^\ell(\widetilde{a}) \times F, \qquad E_\infty^\ell(b) \times F \simeq E_\infty^\ell(\widetilde{b}),$$

or

$$E_\infty^\ell(b) \simeq E_\infty^\ell(\widetilde{b}) \times F, \qquad E_0^\ell(a) \times F \simeq E_0^\ell(\widetilde{a}),$$

where $F=\ell$ *or* $F=\mathbb{C}^n$ *with some integer* $n\ge 0$. *In particular, we can take* $F=0$ *if each of the sequences* a, $\widetilde{a}$, b, $\widetilde{b}$ *does not tend to* ∞; *on the other hand, if each of the sequences* a, $\widetilde{a}$, b, $\widetilde{b}$ *tends to* ∞ *then there is an integer* $n\ge 0$ *such that one of the above conditions holds with* $F=\mathbb{C}^n$.

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