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DIFFERENTIAL GEOMETRY

A Useful Characterization of Some Real Hypersurfaces in a Nonflat Complex Space Form

by

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Summary. We characterize totally η-umbilic real hypersurfaces in a nonflat complex space form $\widetilde{M}_n(c)$ ($=\mathbb{C}P^n(c)$ or $\mathbb{C}H^n(c)$) and a real hypersurface of type (A_2) of radius $\pi/(2\sqrt{c})$ in $\mathbb{C}P^n(c)$ by observing the shape of *some* geodesics on those real hypersurfaces as curves in the ambient manifolds (Theorems 1 and 2).

1. Introduction. A curve $\gamma=\gamma(s)$ (parametrized by its arclength s) on a Riemannian manifold M is called a *plane curve* if it is locally contained in some real 2-dimensional totally geodesic submanifold of M.

In some cases, it is possible to deduce the geometric properties of a submanifold by observing the shape of geodesics on it (for example, see [FS, M, MO, S]). It is known that a hypersurface M^n isometrically immersed into Euclidean space $\mathbb{R}^{n+1}$ is locally a standard sphere if and only if every geodesic of M is mapped to a plane curve of positive curvature in $\mathbb{R}^{n+1}$. Such a characterization is quite natural, but it requires a very large amount of information because of the condition on every geodesic of M (cf. [OT]). In this context, we are interested in a useful criterion for a hypersurface M^n of $\mathbb{R}^{n+1}$ to be a standard sphere. For example, we can see that a hypersurface M^n in $\mathbb{R}^{n+1}$ is a standard sphere if and only if at each point x of M^n there exists an orthonormal basis $v_1,\ldots,v_n$ of $T_x(M^n)$ such that all geodesics of M through x in direction v_i+v_j $(1\leq i\leq j\leq n)$ are mapped to

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plane curves of positive curvature in the ambient space $\mathbb{R}^{n+1}$ (see Proposition 1).

On the other hand, in a nonflat complex space form $\widetilde{M}_n(c)$, $c \neq 0$, which is either a complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature $c > 0$ or a complex hyperbolic space $\mathbb{C}H^n(c)$ of constant holomorphic sectional curvature $c < 0$, there does not exist a real hypersurface all of whose geodesics are mapped to plane curves in the ambient space. This comes from the fact that a nonflat complex space form does not admit totally umbilic real hypersurfaces. However, there exist real hypersurfaces M^{2n-1}'s all of whose geodesics orthogonal to the characteristic vector field ξ of M are mapped to plane curves in $\widetilde{M}_n(c)$. In fact, every totally η-umbilic real hypersurface has this property (see Section 2 for the definition of totally η-umbilic real hypersurfaces).

The main purpose of this paper is to provide a useful characterization of totally η-umbilic real hypersurfaces of $\widetilde{M}_n(c)$, $c \neq 0$, in the above sense (Theorem 1).

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2. Preliminaries. Let M^{2n-1} be a real hypersurface (with unit normal vector field $\mathcal{N}$) of a nonflat n-dimensional complex space form $\widetilde{M}_n(c)$ ($= \mathbb{C}P^n(c)$ or $\mathbb{C}H^n(c)$) of constant holomorphic sectional curvature c. The Riemannian connections $\widetilde{\nabla}$ of $\widetilde{M}_n(c)$ and ∇ of M are related by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \langle AX, Y\rangle \mathcal{N} \quad \text{and} \quad \widetilde{\nabla}_X \mathcal{N} = -AX, \tag{2.1}$$

for vector fields X and Y tangent to M, where $\langle\ ,\ \rangle$ denotes the standard Riemannian metric of $\widetilde{M}_n(c)$ and A is the shape operator of M in $\widetilde{M}_n(c)$. It is known that M admits an almost contact metric structure $(\phi, \xi, \eta, \langle\ ,\ \rangle)$ induced from the Kähler structure J of $\widetilde{M}_n(c)$. The characteristic vector field ξ of M is defined as $\xi = -J\mathcal{N}$ and this structure satisfies

$$\phi^2 = -I + \eta \otimes \xi,\ \eta(\xi) = 1 \quad \text{and} \quad \langle \phi X, \phi Y\rangle = \langle X, Y\rangle - \eta(X)\eta(Y),$$

where I denotes the identity map of the tangent bundle TM of M. It follows from (2.1) that

$$\nabla_X \xi = \phi AX. \tag{2.2}$$

The eigenvalues and eigenvectors of the shape operator A are called *principal curvatures* and *principal curvature vectors*, respectively. In the following, we denote by V_λ the eigenspace associated to the principal curvature λ, that is, $V_\lambda = \{v \in TM \mid Av = \lambda v\}$.

We usually call M a *Hopf hypersurface* if the characteristic vector ξ is a principal curvature vector. The following is useful ([NR]):

LEMMA 1. *For a Hopf hypersurface M with $A\xi = \delta\xi$ in a nonflat complex space form $\widetilde{M}_n(c)$ the following hold.*

(1) *δ is locally constant.*
(2) *If $Av = \lambda v$ for $v \perp \xi$, then $(2\lambda - \delta)A\phi v = (\delta\lambda + c/2)\phi v$. In particular,*

$$A\phi v = \frac{\delta\lambda + c/2}{2\lambda - \delta}\,\phi v \quad \text{when } c > 0.$$

It is known that every tube (of sufficiently small constant radius) around each Kähler submanifold of $\widetilde{M}_n(c)$, $c \neq 0$, is a Hopf hypersurface. This fact tells us that the notion of Hopf hypersurface is natural in the theory of real hypersurfaces in a nonflat complex space form (see [NR]).

In the following, we consider Hopf hypersurfaces with *constant principal curvatures*. These hypersurfaces are completely classified ([NR]). A Hopf hypersurface in $\mathbb{C}P^n(c)$ $(n \geq 2)$ with constant principal curvatures is locally congruent to one of the following:

(A_1) a geodesic sphere of radius r, where $0 < r < \pi/\sqrt{c}$;
(A_2) a tube of radius r around a totally geodesic $\mathbb{C}P^k(c)$ $(1 \leq k \leq n-2)$, where $0 < r < \pi/\sqrt{c}$;
(B) a tube of radius r around a complex hyperquadric $\mathbb{C}Q^{n-1}$, where $0 < r < \pi/(2\sqrt{c})$,
(C) a tube of radius r around $\mathbb{C}P^1(c) \times \mathbb{C}P^{(n-1)/2}(c)$, where $0 < r < \pi/(2\sqrt{c})$ and $n \geq 5$ is odd;
(D) a tube of radius r around a complex Grassmannian $\mathbb{C}G_{2,5}$, where $0 < r < \pi/(2\sqrt{c})$ and $n = 9$;
(E) a tube of radius r around the Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \pi/(2\sqrt{c})$ and $n = 15$.

These real hypersurfaces are said to be of type (A_1), (A_2), (B), (C), (D) and (E). Real hypersurfaces of type (A_1) or (A_2) are jointly called real hypersurfaces of type (A). The numbers of distinct principal curvatures of these real hypersurfaces are $2, 3, 3, 5, 5, 5$, respectively. One should notice that a geodesic sphere of radius r $(0 < r < \pi/\sqrt{c})$ in $\mathbb{C}P^n(c)$ is congruent to a tube of radius $\pi/\sqrt{c} - r$ over a totally geodesic hyperplane $\mathbb{C}P^{n-1}(c)$.

A Hopf hypersurface M in $\mathbb{C}H^n(c)$ $(n \geq 2)$ with constant principal curvatures is locally congruent to one of the following ([NR]):

(A_0) a horosphere in $\mathbb{C}H^n(c)$;
($A_{1,0}$) a geodesic sphere of radius r $(0 < r < \infty)$;
($A_{1,1}$) a tube of radius r around a totally geodesic $\mathbb{C}H^{n-1}(c)$, where $0 < r < \infty$;
(A_2) a tube of radius r around a totally geodesic $\mathbb{C}H^k$ $(1 \leq k \leq n-2)$, where $0 < r < \infty$;

(B) a tube of radius r around a totally real totally geodesic $\mathbb{R}H^n(c/4)$, where $0 < r < \infty$.

These real hypersurfaces are said to be of type (A_0), (A_1), (A_2) and (B). Here, type (A_1) means either ($\mathrm{A}_{1,0}$) or ($\mathrm{A}_{1,1}$), and real hypersurfaces of type (A_0), (A_1) or (A_2) are jointly called real hypersurfaces of type (A). A real hypersurface of type (B) with radius $r = (1/\sqrt{|c|})\ln(2+\sqrt{3})$ has two distinct constant principal curvatures. Except this real hypersurface of type (B) with radius $r = (1/\sqrt{|c|})\ln(2+\sqrt{3})$, the numbers of distinct principal curvatures of these real hypersurfaces are $2, 2, 2, 3, 3$, respectively.

A real hypersurface M of $\widetilde{M}_n(c)$ ($n \geq 2$) is called *totally η-umbilic* if its shape operator A is of the form $A = \alpha I + \beta\eta \otimes \xi$ for some smooth functions α and β on M. This definition can be easily rewritten as $AX = kX$ for each vector X on M which is orthogonal to the characteristic vector ξ of M, where k is a smooth function on M. It is known that every totally η-umbilic hypersurface is a Hopf hypersurface with constant principal curvatures. The following classification theorem of totally η-umbilic real hypersurfaces M shows that these two functions α and β are automatically constant on M (see [NR]):

THEOREM A. *Let M^{2n-1} be a totally η-umbilic real hypersurface of a nonflat complex space form $\widetilde{M}_n(c)$ ($n \geq 2$) (with shape operator $A = \alpha I + \beta\eta \otimes \xi$). Then M is locally congruent to one of the following*:

(P) *a geodesic sphere of radius r ($0 < r < \pi/\sqrt{c}$) in $\mathbb{C}P^n(c)$, where $\alpha = (\sqrt{c}/2)\cot(\sqrt{c}r/2)$ and $\beta = -1/\alpha$;*

(H) (i) *a horosphere in $\mathbb{C}H^n(c)$, where $\alpha = \beta = \sqrt{|c|}/2$;*

(ii) *a geodesic sphere of radius r ($0 < r < \infty$) in $\mathbb{C}H^n(c)$, where $\alpha = (\sqrt{|c|}/2)\coth(\sqrt{|c|}r/2)$ and $\beta = 1/\alpha$;*

(iii) *a tube of radius r ($0 < r < \infty$) around a totally geodesic complex hyperplane $\mathbb{C}H^{n-1}(c)$ in $\mathbb{C}H^n(c)$, where $\alpha = (\sqrt{|c|}/2)\cdot\tanh(\sqrt{|c|}r/2)$ and $\beta = 1/\alpha$.*

It is known that every totally η-umbilic real hypersurface M has two distinct constant principal curvatures. For later use we prepare the following lemma (see [NR]).

LEMMA 2. *Let M be a real hypersurface in a nonflat complex space form $\widetilde{M}_n(c)$ ($n \geq 2$). Then the following are equivalent.*

(1) *M is of type (A).*

(2) *$\phi A = A\phi$.*

(3) *$\langle(\nabla_X A)Y, Z\rangle = (c/4)(-\eta(Y)\langle\phi X, Z\rangle - \eta(Z)\langle\phi X, Y\rangle)$ for arbitrary vectors X, Y and Z on M.*

Next we recall ruled real hypersurfaces in a nonflat complex space form, which are typical examples of non-Hopf hypersurfaces. A real hypersurface M is called a *ruled real hypersurface* in a nonflat complex space form $\widetilde{M}_n(c)$ $(n \geq 2)$ if the holomorphic distribution T^0 defined by $T^0(x) = \{X \in T_xM \mid X \perp \xi\}$ for $x \in M$ is integrable and each of its integral manifolds is a totally geodesic complex hypersurface $M_{n-1}(c)$ of $\widetilde{M}_n(c)$. A ruled real hypersurface is constructed in the following manner. Given an arbitrary regular curve γ defined on an interval I in $\widetilde{M}_n(c)$ we have at each point $\gamma(t)$ $(t \in I)$ a totally geodesic complex hypersurface $M^{(t)}_{n-1}(c)$ that is orthogonal to the plane spanned by $\{\dot\gamma(t), J\dot\gamma(t)\}$. Then we see that $M = \bigcup_{t \in I} M^{(t)}_{n-1}(c)$ is a ruled real hypersurface in $\widetilde{M}_n(c)$. The following gives a characterization of ruled real hypersurfaces in terms of the shape operator A (see [NR]).

Lemma 3. *For a real hypersurface M in a nonflat complex space form $\widetilde{M}_n(c)$ $(n \geq 2)$, the following conditions are equivalent.*

(1) *M is a ruled real hypersurface.*

(2) *The shape operator A of M satisfies the following equalities on the open dense subset $M_1 = \{x \in M \mid \nu(x) \neq 0\}$ with a unit vector field U orthogonal to ξ:*

$$A\xi = \mu\xi + \nu U, \quad AU = \nu\xi, \quad AX = 0$$

for an arbitrary tangent vector X orthogonal to ξ and U. Here μ, ν are differentiable functions on M defined by $\mu = \langle A\xi, \xi\rangle$ and $\nu = \|A\xi - \mu\xi\|$.

(3) *The shape operator A of M satisfies $\langle Av, w\rangle = 0$ for arbitrary tangent vectors $v, w \in T_xM$ orthogonal to ξ_x at each point $x \in M$.*

We treat ruled real hypersurfaces locally, because generally such hypersurfaces have self-intersections and singularities. When we study ruled real hypersurfaces, we usually omit points where ξ is principal and suppose that ν does not vanish everywhere, that is, a ruled hypersurface M is usually supposed to have $M_1 = M$.

We review the notion of Frenet curves of order 2. A smooth curve $\gamma = \gamma(s)$ in a Riemannian manifold M parametrized by its arclength s is called a *Frenet curve of proper order* 2 if there exist a field of orthonormal frames $\{\dot\gamma(s), Y_s\}$ along γ and a positive smooth function $\kappa(s)$ satisfying the following system of ordinary differential equations:

$$\nabla_{\dot\gamma}\dot\gamma = \kappa(s)Y_s \quad \text{and} \quad \nabla_{\dot\gamma}Y_s = -\kappa(s)\dot\gamma. \tag{2.3}$$

The function κ is called the *curvature* of the Frenet curve γ of proper order 2. Here we note that we do *not* allow the curvature $\kappa(s)$ to vanish at any point. Therefore curves with inflection points, such as $y = x^3$ on a Euclidean xy-plane, are not Frenet curves of proper order 2. A curve is called a Frenet curve *of order* 2 if it is either a Frenet curve of proper order 2 or a geodesic. When the curvature κ is a constant function along γ, say k, the curve satisfying (2.3) is called a *circle* of curvature k on M. Needless to say, a geodesic is regarded as a circle of null curvature.

For a Frenet curve γ of proper order 2 in a Kähler manifold M_n (with Riemannian connection ∇ and complex structure J), we define a *complex torsion* τ_γ by $\tau_\gamma = \langle \dot\gamma(s), JY_s\rangle$. Of course we have $-1 \le \tau_\gamma \le 1$. Note that the complex torsion τ_γ is automatically constant. In fact, we can see that

$$\begin{aligned}\nabla_{\dot\gamma}\langle \dot\gamma(s), JY_s\rangle &= \langle \nabla_{\dot\gamma}\dot\gamma(s), JY_s\rangle + \langle \dot\gamma(s), J\nabla_{\dot\gamma}Y_s\rangle\\ &= \kappa\langle Y_s, JY_s\rangle - \kappa\langle \dot\gamma(s), J\dot\gamma(s)\rangle = 0.\end{aligned}$$

We know that a Frenet curve γ of proper order 2 in a nonflat complex space form $\widetilde{M}_n(c)$ $(n \ge 2)$ is a plane curve (with positive curvature function) if and only if $\tau_\gamma = \pm 1, 0$. When $\tau_\gamma = \pm 1$, this curve γ lies on $\mathbb{C}P^1(c)$ or $\mathbb{C}H^1(c)$, which are complex lines of $\widetilde{M}_n(c)$. Also, when $\tau = 0$, this curve γ lies on $\mathbb{R}P^2(c/4)$ or $\mathbb{R}H^2(c/4)$, which are real parts of totally geodesic Kähler surfaces $M_2(c)$ in the ambient spaces $\widetilde{M}_n(c)$.

3. Results. The main purpose of this paper is to prove the following:

THEOREM 1. *Let M be a connected real hypersurface of a nonflat complex space form $\widetilde{M}_n(c)$ $(n \ge 2)$. Then the following are equivalent.*

(1) *M is totally η-umbilic in $\widetilde{M}_n(c)$.*
(2) *At each $x \in M$ there exist orthonormal vectors $v_1, \dots, v_{2n-2}$ orthogonal to ξ such that all geodesics of M through x in direction $v_i + v_j$ $(1 \le i \le j \le 2n-2)$ are mapped to Frenet curves of proper order* 2 *in $\widetilde{M}_n(c)$.*
(3) *At each $x \in M$ there exist orthonormal vectors $v_1, \dots, v_{2n-2}$ orthogonal to ξ such that all geodesics of M through x in direction $v_i + v_j$ $(1 \le i \le j \le 2n-2)$ are mapped to plane curves of positive curvature in $\widetilde{M}_n(c)$.*
(4) *At each $x \in M$ there exist orthonormal vectors $v_1, \dots, v_{2n-2}$ orthogonal to ξ such that all geodesics of M through x in direction $v_i + v_j$ $(1 \le i \le j \le 2n-2)$ are mapped to circles of positive curvature in $\widetilde{M}_n(c)$.*

Proof. (1) $\Rightarrow$ (2), (3), (4). Let M be a totally η-umbilic real hypersurface in $\widetilde{M}_n(c)$. We take an arbitrary point x of M and any unit vector $v \in T_x(M)$ which is orthogonal to the characteristic vector ξ_x. Let $\gamma = \gamma(s)$ be a geodesic on M with $\gamma(0) = x$ and $\dot\gamma(0) = v$. Note that $Av = \alpha v$ (see Theorem A). Then, from (2.2) and Lemma 2(2) we have

$$\nabla_{\dot\gamma}\langle \dot\gamma, \xi\rangle = \langle \dot\gamma, \nabla_{\dot\gamma}\xi\rangle = \langle \dot\gamma, \phi A\dot\gamma\rangle = \langle \dot\gamma, A\phi\dot\gamma\rangle = -\langle \phi A\dot\gamma, \dot\gamma\rangle = 0.$$

This, together with $\langle\dot\gamma(0),\xi\rangle = \langle v,\xi\rangle = 0$, shows that $\dot\gamma(s)$ is perpendicular to $\xi_{\gamma(s)}$, so that

$$A\dot\gamma(s) = \alpha\dot\gamma(s) \quad \text{for each } s. \tag{3.1}$$

Therefore, we see from (2.1) and (3.1) that

$$\widetilde\nabla_{\dot\gamma}\dot\gamma = \langle A\dot\gamma,\dot\gamma\rangle\mathcal{N} = \alpha\mathcal{N} \quad \text{and} \quad \widetilde\nabla_{\dot\gamma}\mathcal{N} = -A\dot\gamma = -\alpha\dot\gamma.$$

Moreover, $\tau_\gamma = \langle\dot\gamma, J\mathcal{N}\rangle = -\langle\dot\gamma,\xi\rangle = 0$. Therefore when $c > 0$ (resp. $c < 0$) the geodesic γ is a circle of positive curvature $|\alpha|$ on $\mathbb{R}P^2(c/4)$ (resp. $\mathbb{R}H^2(c/4)$).

By the definitions we have the following inclusions: {the plane curves of positive curvature} $\subset$ {the Frenet curves of proper order 2} and {the circles of positive curvature} $\subset$ {the Frenet curves of proper order 2}. Hence in the rest of the proof, it suffices to verify that (2) implies (1).

Let $\gamma_i = \gamma_i(s)$ $(1 \le i \le 2n-2)$ be geodesics of M with $\gamma_i(0) = x$ and $\dot\gamma_i(0) = v_i$. Then by assumption we have

$$\widetilde\nabla_{\dot\gamma_i}\dot\gamma_i = \kappa_i(s)Y_i(s) \quad \text{and} \quad \widetilde\nabla_{\dot\gamma_i}Y_i(s) = -\kappa_i(s)\dot\gamma_i$$

for some positive smooth functions κ_i. Hence

$$\widetilde\nabla_{\dot\gamma_i}(\widetilde\nabla_{\dot\gamma_i}\dot\gamma_i) = (\kappa_i(s))'Y_i(s) - (\kappa_i(s))^2\dot\gamma_i. \tag{3.2}$$

From the first equality in (2.1) we note that

$$\kappa_i(s)Y_i(s) = \langle A\dot\gamma_i(s),\dot\gamma_i(s)\rangle\mathcal{N}_{\gamma_i(s)}. \tag{3.3}$$

On the other hand, from (2.1) we get

$$\widetilde\nabla_{\dot\gamma_i}(\widetilde\nabla_{\dot\gamma_i}\dot\gamma_i) = (\nabla_{\dot\gamma_i}\langle A\dot\gamma_i,\dot\gamma_i\rangle)\mathcal{N} - \langle A\dot\gamma_i,\dot\gamma_i\rangle A\dot\gamma_i. \tag{3.4}$$

Comparing the tangential components of (3.2) and (3.4), from (3.3) we obtain $\langle A\dot\gamma_i,\dot\gamma_i\rangle A\dot\gamma_i = \kappa_i^2\dot\gamma_i$, so that at $s = 0$ we have

$$\langle Av_i,v_i\rangle Av_i = (\kappa_i(0))^2 v_i \quad \text{for all } i \in \{1,\dots,2n-2\}. \tag{3.5}$$

Since $\kappa_i(0) \ne 0$, this tells us that

$$Av_i = \kappa_i(0)v_i \quad \text{or} \quad Av_i = -\kappa_i(0)v_i \quad \text{for all } i \in \{1,\dots,2n-2\}. \tag{3.6}$$

Let $\gamma_{ij} = \gamma_{ij}(s)$ $(1 \le i < j \le 2n-2)$ be geodesics of M with $\gamma_{ij}(0) = x$ and $\dot\gamma_{ij}(0) = (v_i + v_j)/\sqrt{2}$. Then by a similar computation we see that

$$\langle A(v_i+v_j), v_i+v_j\rangle A(v_i+v_j) = 2(\kappa_{ij}(0))^2(v_i+v_j) \tag{3.7}$$

for some positive $\kappa_{ij}(0)$. Taking the inner product of (3.7) and the vector $v_i - v_j$, we have

$$\langle Av_i,v_i\rangle = \langle Av_j,v_j\rangle \quad \text{for any distinct } i,j \in \{1,\dots,2n-2\}. \tag{3.8}$$

It follows from (3.6) and (3.8) that $AX = kX$ at x for all X orthogonal to ξ_x and for some k. Hence M is totally η-umbilic in $\widetilde{M}_n(c)$, since x is arbitrary. ∎

The proof of Theorem 1 yields the following proposition.

Proposition 1. *Let M^n be a hypersurface of a Riemannian manifold $\widetilde{M}^{n+1}$. Then M^n is totally umbilic but not totally geodesic in $\widetilde{M}^{n+1}$ if and only if at each $x \in M$ there exist orthonormal vectors $v_1, \dots, v_n \in T_xM$ such that all geodesics of M through x in direction $v_i + v_j$ $(1 \leq i \leq j \leq n)$ are mapped to Frenet curves of proper order 2 in the ambient space $\widetilde{M}^{n+1}$.*

Motivated by Theorem 1, we establish the following:

Theorem 2. *Let M be a real hypersurface of a nonflat complex space form $\widetilde{M}_n(c)$ $(n \geq 2)$. Then the following are equivalent.*

(1) *M is locally either a totally η-umbilic real hypersurface in $\widetilde{M}_n(c)$ or a real hypersurface of type (A_2) with radius $\pi/(2\sqrt{c})$ in $\mathbb{C}P^n(c)$, that is, a tube over a totally geodesic $\mathbb{C}P^k(c)$ $(1 \leq k \leq n-2)$ of radius $\pi/(2\sqrt{c})$ in $\mathbb{C}P^n(c)$.*
(2) *At each $x \in M$ there exist orthonormal vectors $v_1, \dots, v_{2n-2}$ orthogonal to ξ such that all geodesics of M through x in direction v_i $(1 \leq i \leq 2n-2)$ are mapped to Frenet curves of proper order 2 with the same curvature in $\widetilde{M}_n(c)$.*
(3) *At each $x \in M$ there exist orthonormal vectors $v_1, \dots, v_{2n-2}$ orthogonal to ξ such that all geodesics of M through x in direction v_i $(1 \leq i \leq 2n-2)$ are mapped to plane curves of the same positive curvature in $\widetilde{M}_n(c)$.*
(4) *At each $x \in M$ there exist orthonormal vectors $v_1, \dots, v_{2n-2}$ orthogonal to ξ such that all geodesics of M through x in direction v_i $(1 \leq i \leq 2n-2)$ are mapped to circles of the same positive curvature in $\widetilde{M}_n(c)$.*

Proof. (1) $\Rightarrow$ (2), (3), (4). We only have to consider a real hypersurface M of type (A_2) with radius $\pi/(2\sqrt{c})$ in $\mathbb{C}P^n(c)$. Then the tangent bundle TM of M is decomposed as (see [NR])

$$TM = V_{\sqrt{c}/2} \oplus V_{-\sqrt{c}/2} \oplus \{\xi\}_{\mathbb{R}}.$$

Here, $A\xi = 0$, $V_{\sqrt{c}/2} = \{X \in TM \mid AX = (\sqrt{c}/2)X\}$, $V_{-\sqrt{c}/2} = \{X \in TM \mid AX = -(\sqrt{c}/2)X\}$, $\dim V_{\sqrt{c}/2} = 2k$ and $\dim V_{-\sqrt{c}/2} = 2n-2-2k$. We take an orthonormal basis $v_1, \dots, v_{2n-2}$ orthogonal to ξ_x in such a way that $\{v_1, \dots, v_{2k}\}$ (resp. $\{v_{2k+1}, \dots, v_{2n-2}\}$) is an orthonormal basis of $V_{\sqrt{c}/2}$ (resp. $V_{-\sqrt{c}/2}$).

Let $\gamma_i = \gamma_i(s)$ $(1 \leq i \leq 2k)$ be geodesics of M with $\gamma_i(0) = x$ and $\dot{\gamma}_i(0) = v_i$. Then, as in the proof of Theorem 1, we find that the vector $\dot{\gamma}_i(s)$ is perpendicular to the characteristic vector $\xi_{\gamma_i(s)}$ for every s. This, combined with Lemma 2(3), yields

$$\nabla_{\dot\gamma_i}\left\|A\dot\gamma_i - \frac{\sqrt{c}}{2}\,\dot\gamma_i\right\|^2 = 2\left\langle (\nabla_{\dot\gamma_i}A)\dot\gamma_i, A\dot\gamma_i - \frac{\sqrt{c}}{2}\,\dot\gamma_i\right\rangle$$
$$= 2\langle (\nabla_{\dot\gamma_i}A)\dot\gamma_i, A\dot\gamma_i\rangle - \sqrt{c}\langle (\nabla_{\dot\gamma_i}A)\dot\gamma_i, \dot\gamma_i\rangle = 0.$$

Since $A\dot\gamma_i(0) - (\sqrt{c}/2)\dot\gamma_i(0) = Av_i - (\sqrt{c}/2)v_i = 0$ $(1 \le i \le 2k)$, we see that

$$A\dot\gamma_i(s) = \frac{\sqrt{c}}{2}\dot\gamma_i(s) \quad (1 \le i \le 2k) \text{ for every } s. \tag{3.9}$$

It follows from (2.1) and (3.9) that

$$\widetilde\nabla_{\dot\gamma_i}\dot\gamma_i = \langle A\dot\gamma_i, \dot\gamma_i\rangle = \frac{\sqrt{c}}{2}\mathcal{N} \quad \text{and} \quad \widetilde\nabla_{\dot\gamma_i}\mathcal{N} = -A\dot\gamma_i = -\frac{\sqrt{c}}{2}\,\dot\gamma_i.$$

This, together with $\tau_\gamma = \langle \dot\gamma_i, J\mathcal{N}\rangle = -\langle \dot\gamma_i, \xi\rangle = 0$, implies that the curve γ_i is a circle of positive curvature $\sqrt{c}/2$ on $\mathbb{R}P^2(c/4)$.

Similarly we can verify that the geodesics γ_i $(2k+1 \le i \le 2n-2)$ of M with $\gamma_i(0) = x$ and $\dot\gamma_i(0) = v_i$ satisfy

$$\widetilde\nabla_{\dot\gamma_i}\dot\gamma_i = \langle A\dot\gamma_i, \dot\gamma_i\rangle = \frac{\sqrt{c}}{2}\,(-\mathcal{N}) \quad \text{and} \quad \widetilde\nabla_{\dot\gamma_i}(-\mathcal{N}) = A\dot\gamma_i = -\frac{\sqrt{c}}{2}\,\dot\gamma_i.$$

So we find that these curves are circles of the same curvature $\sqrt{c}/2$ on $\mathbb{R}P^2(c/4)$.

(2), (3), (4) $\Rightarrow$ (1). We only have to prove that (2) implies (1). By the same argument as in the proof of Theorem 1, (3.6) gives

$$Av_i = kv_i \quad \text{or} \quad Av_i = -kv_i \quad \text{for } 1 \le i \le 2n-2, \tag{3.10}$$

where k is a positive number. Note that our real hypersurface M is Hopf. Indeed, $\langle A\xi, v_i\rangle = \langle \xi, Av_i\rangle = 0$ for $1 \le i \le 2n-2$. Moreover, M has at most three distinct principal curvatures $k, -k$ and $\delta = \langle A\xi, \xi\rangle$ at each of its points. Note that Lemma 1(1) shows that δ is locally constant on M. Moreover, when $c > 0$, it follows from Lemma 1(2) that

$$k = \frac{\delta k + c/2}{2k - \delta} \quad \text{or} \quad -k = \frac{\delta k + c/2}{2k - \delta}. \tag{3.11}$$

But the latter case does not hold, because $c > 0$. Hence the real hypersurface M is either of type (A_1), that is, M is totally η-umbilic, or of type (A_2) (see [NR]). But the shape operator of a real hypersurface of type (A_2) with radius r $(\neq \pi/(2\sqrt{c}))$ does not satisfy (3.10). In fact, a real hypersurface of type (A_2) of radius r $(0 < r < \pi/\sqrt{c})$ has principal curvatures

$$\lambda_1 = \frac{\sqrt{c}}{2}\cot\frac{\sqrt{c}r}{2}, \qquad \lambda_2 = -\frac{\sqrt{c}}{2}\tan\frac{\sqrt{c}r}{2}, \qquad \delta = \sqrt{c}\cot(\sqrt{c}r).$$

Note that $|\lambda_1| \neq |\lambda_2|$ for each $r \neq \pi/(2\sqrt{c})$. Thus we obtain the desired statement (1) in Theorem 2 when $c > 0$.

Next, we consider the case of $c < 0$. Suppose that $2k - \delta \neq 0$ on some open neighborhood $\mathcal{U}_x$ of x. Then (3.11) asserts that k is constant on $\mathcal{U}_x$. This, together with the continuity of the principal curvature function k on M, implies that if $2k - \delta = 0$ at some $y \in M$, then there exists an open neighborhood $\mathcal{U}_y$ of y such that $2k - \delta$ is identically zero on $\mathcal{U}_y$. Thus M is a Hopf hypersurface with at most three constant principal curvatures $k, -k$ and $\delta = \langle A\xi, \xi\rangle$. Hence M is either of type (A_0), type (A_1), that is, M is totally η-umbilic, of type (A_2) or of type (B). But the shape operator of no real hypersurface of type (A_2) or of type (B) satisfies (3.10). Indeed, a real hypersurface of type (A_2) of radius r $(0 < r < \infty)$ has principal curvatures

$$\lambda_1 = \frac{\sqrt{|c|}}{2}\coth\frac{\sqrt{|c|}r}{2}, \qquad \lambda_2 = \frac{\sqrt{|c|}}{2}\tanh\frac{\sqrt{|c|}r}{2}, \qquad \delta = \sqrt{|c|}\coth(\sqrt{|c|}r),$$

and a real hypersurface of type B of radius r $(0 < r < \infty)$ has principal curvatures

$$\lambda_1 = \frac{\sqrt{|c|}}{2}\coth\frac{\sqrt{|c|}r}{2}, \qquad \lambda_2 = \frac{\sqrt{|c|}}{2}\tanh\frac{\sqrt{|c|}r}{2}, \qquad \delta = \sqrt{|c|}\tanh(\sqrt{|c|}r). \quad \blacksquare$$

REMARKS.

(1) In the statements of Theorems 1 and 2, on the real hypersurface M we do not need to take the orthonormal vectors $v_1, \ldots, v_{2n-2}$ orthogonal to ξ_x *continuously* for all $x \in M$.
(2) The following theorem is closely related to Theorems 1 and 2.

THEOREM B ([AKM2]). *Let M^{2n-1} be a real hypersurface of a nonflat complex space form $\widetilde{M}_n(c)$ $(n \geq 2)$. Then M is locally congruent to either a totally η-umbilic real hypersurface or a ruled real hypersurface if and only if every geodesic γ of M whose initial vector $\dot{\gamma}(0)$ is orthogonal to the characteristic vector $\xi_{\gamma(0)}$ of M is mapped to a plane curve in the ambient space $\widetilde{M}_n(c)$.*

Note that in the statement of Theorem B we do *not* suppose that the curvature of the plane curve γ in the ambient space $\widetilde{M}_n(c)$ $(n \geq 2)$, $c \neq 0$, is positive.

We now characterize ruled real hypersurfaces M by using the fact that every geodesic γ of M whose initial vector $\dot{\gamma}(0)$ is orthogonal to the characteristic vector $\xi_{\gamma(0)}$ of M is also a *geodesic* in the ambient space $\widetilde{M}_n(c)$.

PROPOSITION 2. *A real hypersurface M of a nonflat complex space form $\widetilde{M}_n(c)$, $n \geq 2$, is a ruled real hypersurface if and only if at each $x \in M$ there exist orthonormal vectors $v_1, \ldots, v_{2n-2}$ orthogonal to ξ such that all geodesics of M through x in direction $v_i + v_j$ $(1 \leq i \leq j \leq 2n-2)$ are mapped to geodesics in $\widetilde{M}_n(c)$.*

Proof. Suppose that M is ruled. Let γ be a geodesic on M with initial vector $\dot{\gamma}(0)$ perpendicular to $\xi_{\gamma(0)}$ and $M^{(t)}_{n-1}$ (for some t) the integral manifold through the point $x = \gamma(0)$ for the holomorphic distribution T^0M. Since $M^{(t)}_{n-1}$ is totally geodesic in the ambient manifold $\widetilde{M}_n(c)$, we find easily that $M^{(t)}_{n-1}$ is also totally geodesic in the real hypersurface M. As $\dot{\gamma}(0) \in T_x M^{(t)}_{n-1}$, by the uniqueness theorem for geodesics we see that γ lies on $M^{(t)}_{n-1}$, hence is a geodesic as a curve on $\widetilde{M}_n(c)$.

Conversely, it follows from the assumption and the first equality in (2.1) that at each $x \in M$ there exist orthonormal vectors $v_1, \ldots, v_{2n-2}$ orthogonal to ξ such that

$$\langle A(v_i + v_j), v_i + v_j \rangle = 0 \quad \text{for } 1 \leq i \leq j \leq 2n-2.$$

This implies Lemma 3(3), so that M is ruled. ∎

4. Problem. In the previous papers [AKM1, CM], the following characterization of all Hopf hypersurfaces with constant principal curvatures in a nonflat complex space form was given:

THEOREM C. *A real hypersurface M of a nonflat complex space form $\widetilde{M}_n(c)$ $(n \geq 2)$ is locally congruent to a Hopf hypersurface with constant principal curvatures if and only if at each $x \in M$ there exist orthonormal vectors $v_1, \ldots, v_{2n-2}$ orthogonal to ξ such that all geodesics of M through x in direction v_i $(1 \leq i \leq 2n-2)$ are mapped to circles of positive curvature in $\widetilde{M}_n(c)$.*

To end this paper, motivated by Theorem C we pose the following problem:

PROBLEM. *Let M be a real hypersurface of a nonflat complex space form $\widetilde{M}_n(c)$ $(n \geq 2)$. Suppose that at each $x \in M$, there exist orthonormal vectors $v_1, \ldots, v_{2n-2}$ orthogonal to ξ such that all geodesics of M through x in direction v_i $(1 \leq i \leq 2n-2)$ are mapped to Frenet curves of proper order 2 in the ambient space $\widetilde{M}_n(c)$. Is M locally congruent to a Hopf hypersurface with constant principal curvatures?*

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