Fixed Points of \(n\)-Valued Multimaps of the Circle

by

Robert F. Brown

Presented by Czesław BESSAGA

Summary. A multifunction \(\phi: X \rightarrow Y\) is \(n\)-valued if \(\phi(x)\) is an unordered subset of \(n\) points of \(Y\) for each \(x \in X\). The (continuous) \(n\)-valued multimaps \(\phi: S^1 \rightarrow S^1\) are classified up to homotopy by an integer-valued degree. In the Nielsen fixed point theory of such multimaps, due to Schirmer, the Nielsen number \(N(\phi)\) of an \(n\)-valued \(\phi: S^1 \rightarrow S^1\) of degree \(d\) equals \(|n - d|\) and \(\phi\) is homotopic to an \(n\)-valued power map that has exactly \(|n - d|\) fixed points. Thus the Wecken property, that Schirmer established for manifolds of dimension at least three, also holds for the circle. An \(n\)-valued multimap \(\phi: S^1 \rightarrow S^1\) of degree \(d\) splits into \(n\) selfmaps of \(S^1\) if and only if \(d\) is a multiple of \(n\).

1. Introduction. A multifunction \(\phi: X \rightarrow Y\) is a function such that \(\phi(x)\) is a subset of \(Y\) for each \(x \in X\). For \(S\) a subset of \(Y\), the set \(\phi^{-1}(S)\) consists of the points \(x \in X\) such that \(\phi(x) \subseteq S\), and the set \(\phi_+^{-1}(S)\) consists of the points \(x \in X\) such that \(\phi(x) \cap S \neq \emptyset\). A multifunction \(\phi\) is said to be upper semicontinuous (usc) if \(U\) open in \(Y\) implies \(\phi^{-1}(U)\) is open in \(X\). It is lower semicontinuous (lsc) if \(U\) open in \(Y\) implies \(\phi_+^{-1}(U)\) is open in \(X\). A multifunction that is both upper semicontinuous and lower semicontinuous is said to be continuous. Although the term multimap is sometimes used for a more general concept, in this paper it will mean a continuous multifunction. An \(n\)-valued multifunction \(\phi: X \rightarrow Y\) is a function that assigns to each \(x \in X\) an unordered subset of exactly \(n\) points of \(Y\). Thus an \(n\)-valued multimap is a continuous \(n\)-valued multifunction.

O’Neill [6] proved a version of the Lefschetz fixed point theorem for a large class of multimaps \(\phi: X \rightarrow X\) of finite polyhedra that includes the \(n\)-valued multimaps. Multimaps in this class induce a vector space of endomorphisms of the homology of \(X\). He proved that if any endomorphism

2000 Mathematics Subject Classification: Primary 55M20; Secondary 54C60, 55M25.

Key words and phrases: continuous multifunction, Wecken property, degree of \(n\)-valued map.
has a nonzero Lefschetz number, then \( \phi \) has a fixed point, that is, \( x \in \phi(x) \) for some \( x \in X \).

The Nielsen fixed point theory of \( n \)-valued multimaps was developed by Schirmer in a series of papers [7]-[9]. For \( \phi: X \to X \) an \( n \)-valued multimap of a finite polyhedron, the Nielsen number \( N(\phi) \) has the property that for any \( n \)-valued continuous homotopy \( \Delta: X \times I \to X \) with \( \Delta(x,0) = \phi(x) \), the multimap \( \psi: X \to X \) defined by \( \psi(x) = \Delta(x,1) \) has at least \( N(\phi) \) fixed points.

The main result of [9] extended a celebrated theorem of Wecken [10] in the following way. If \( \phi: X \to X \) is an \( n \)-valued multimap where \( X \) is a compact triangulable manifold, with or without boundary, of dimension at least three, then there is an \( n \)-valued multimap \( \psi: X \to X \) homotopic to \( \phi \) such that \( \psi \) has exactly \( N(\phi) \) fixed points. As in the single-valued theory, we will refer to this property as the Wecken property for \( n \)-valued multimaps.

If \( f_0, f_1, \ldots, f_{n-1}: X \to X \) are \( n \) maps such that \( j \neq k \) implies \( f_j(x) \neq f_k(x) \) for all \( x \in X \) then

\[
\phi(x) = \{f_0(x), f_1(x), \ldots, f_{n-1}(x)\}
\]

defines an \( n \)-valued multimap \( \phi: X \to X \) that is called split in [8]. Only two examples of nonsplit \( n \)-valued multimaps are included in Schirmer’s papers; see page 75 of [7] and page 219 of [8]. The examples are of \( n \)-valued multimaps on the unit circle \( S^1 \) and thus the Wecken theorem of [9] does not apply to them. In both cases, the number of fixed points of the map \( \phi \) that Schirmer defines is precisely \( N(\phi) \), but there is no general such result about \( n \)-valued multimaps of the circle.

We recall that, in the single-valued case, among the manifolds only surfaces can fail to have the Wecken property that a selfmap \( f: X \to X \) is homotopic to a map with exactly \( N(f) \) fixed points [4], [5]. With regard to the 1-dimensional manifolds, the Wecken property holds for maps of the interval because they are all homotopic to a constant map. For \( X = S^1 \), there is the following well known argument that establishes the Wecken property for single-valued maps. By the classification theorem ([2, p. 39]), if \( f: S^1 \to S^1 \) is of degree \( d \), then \( f \) is homotopic to the power map \( \phi_d \) defined by viewing \( S^1 \) as the unit circle in the complex plane and setting \( \phi_d(z) = z^d \). Thus \( N(f) = N(\phi_d) \). It has long been known that \( N(\phi_d) = |1 - d| \) and clearly \( \phi_d \) has \( |1 - d| \) fixed points except in the case \( d = 1 \). Since \( \phi_1 \), the identity map, is homotopic to a fixed point free map, every selfmap \( f \) on the circle is homotopic to a map with \( N(f) \) fixed points.

The Wecken property is easily seen to hold for \( n \)-valued multimaps of the interval \( I \), as follows. Let \( \phi: I \to I \) be a multimap. Define \( \Delta: I \times I \to I \) by \( \Delta(s,t) = \phi(st) \); then \( \Delta \) is continuous by Theorems 1 and 1’ on page 113 of [2]. Thus \( \phi \) is homotopic to the constant \( n \)-valued multimap \( \kappa: I \to I \)
defined by \( \kappa(t) = \phi(0) \), which has \( n \) fixed points, whereas \( N(\kappa) = n \) by Corollary 7.3 of [8].

The purpose of this paper is to prove that the circle also has the Wecken property for \( n \)-valued multimaps. In outline, the argument follows that of the single-valued setting, but there are several significant issues that must be addressed in the \( n \)-valued case. In Section 2, we extend the definition of the degree of a selfmap of the circle to define the degree of an \( n \)-valued multimap of the circle and we discuss its properties. Section 3 introduces a collection of \( n \)-valued multimaps we call \( n \)-valued power maps \( \phi_{n,d} : S^1 \to S^1 \) and we extend the classification theorem by proving that an \( n \)-valued multimap \( \phi : S^1 \to S^1 \) of degree \( d \) is homotopic to \( \phi_{n,d} \). We prove in Section 4 that \( \phi_{n,d} \) has \( |n - d| \) fixed points if \( n \neq d \) and then that \( N(\phi_{n,d}) = |n - d| \) for all \( n \) and \( d \). In Section 5, the Wecken property for \( n \)-valued multimaps of the circle is easily seen to follow from the previous results. Moreover, we characterize the split \( n \)-valued multimaps of the circle: an \( n \)-valued multimap is split if and only if its degree is a multiple of \( n \).

2. The degree of an \( n \)-valued multimap of the circle. We begin with some general properties of \( n \)-valued multimaps. The following result is a special case of a theorem of O’Neill [6] but, according to [9], it was essentially known much earlier [1].

**Lemma 2.1 (Splitting Lemma).** Let \( \phi : X \to Y \) be an \( n \)-valued multimap and let

\[
\Gamma_\phi = \{(x, y) \in X \times Y : y \in \phi(x)\}
\]

be the graph of \( \phi \). The map \( p_1 : \Gamma_\phi \to X \) defined by \( p_1(x, y) = x \) is a covering space. It follows that if \( X \) is simply connected, then any \( n \)-valued multimap \( \phi : X \to Y \) is split.

**Theorem 2.1.** Let \( \Delta : X \times I \to Y \) be an \( n \)-valued homotopy; write \( \Delta = \{\delta^t : X \to Y\} \). If \( \delta^0 \) is split, so also is \( \Delta \). Thus an \( n \)-valued multimap homotopic to a split \( n \)-valued multimap is also split.

**Proof.** Write \( \delta^0 = \{f^0_0, f^0_1, \ldots, f^0_{n-1}\} \) where \( f^0_j : X \to Y \). Define

\[
\hat{f}^0_j : X \times \{0\} \to \Gamma_\Delta \subseteq (X \times I) \times Y
\]

by \( \hat{f}^0_j(x, 0) = ((x, 0), f^0_j(x)) \). Since \( p_1 : \Gamma_\Delta \to X \times I \) is a covering space by Lemma 2.1, by the covering homotopy property there is a map \( \hat{f}_0 : X \times I \to \Gamma_\Delta \) such that \( p_1 \hat{f}_0 \) is the identity map of \( X \times I \). Let \( p_2 : \Gamma_\Delta \to Y \) be projection. Then \( p_2 \hat{f}_0(x, t) \in \delta^t \) so \( p_2 \hat{f}_0 \) is a selection for \( \Delta \) and we can write \( \Delta = \{p_2 \hat{f}_0, \Delta'\} \) where \( \Delta' : X \to Y \) is an \( (n - 1) \)-valued homotopy \( \Delta' = \{\delta'^0\} \) with \( \delta'^0 = \{f^1_0, \ldots, f^1_{n-1}\} \). Repeated application of the covering homotopy property produces a splitting \( \Delta = \{p_2 \hat{f}_0, p_2 \hat{f}_1, \ldots, p_2 \hat{f}_{n-1}\} \). If an
$n$-valued multimap $\psi: X \to Y$ is homotopic to a split $n$-valued multimap $\phi = \{f_0, \ldots, f_{n-1}\}$ by a homotopy $\Delta$ with $\delta^0 = \phi$ and $\delta^1 = \psi$, then $\psi = \{f_0^1, \ldots, f_{n-1}^1\}$ where $f_j^1(x) = p_2\tilde{f}_j(x, 1)$.

Now we turn our attention to the circle and let $p: \mathbb{R} \to S^1$ be the universal covering space where $p(t) = e^{i2\pi t}$. We will denote points of the circle by $p(t)$ for $0 \leq t < 1$. Let $\phi: S^1 \to S^1$ be an $n$-valued multimap. Then the $n$-valued function $\phi_p: I \to S^1$ is continuous by Theorems 1 and 1' on page 133 of [2]. Therefore $\phi p$ is split and, using the ordering on $S^1$ imposed by $p$ from the ordering of $\mathbb{R}$, we write $\phi_p = \{f_0, f_1, \ldots, f_{n-1}\}$ where the maps $f_j: I \to S^1$ have the property $f_j(0) = p(t_j)$ for $0 \leq t_0 < t_1 < \cdots < t_{n-1} < 1$. Let $\tilde{f}_j: I \to \mathbb{R}$ be the lift of $f_j$ such that $\tilde{f}_j(0) = t_j$. We note that if $0 \leq j < k \leq n - 1$, then $\tilde{f}_j(t) < \tilde{f}_k(t)$ for all $t \in I$ because $f_j(p(t)) \neq f_k(p(t))$.

Since $\phi$ is well defined, the sets $\phi_0(0)$ and $\phi_1(1)$ must be identical. Consequently, $\tilde{f}_0(1) = v + t_j$ for some integers $v, J$ where $0 \leq J \leq n - 1$. We define $\text{Deg}(\phi)$, the degree of the $n$-valued multimap $\phi: S^1 \to S^1$, by

$$\text{Deg}(\phi) = nv + J.$$

The degree can be defined just in terms of $\tilde{f}_0(1)$ because that value determines $\tilde{f}_1(1)$ for all $j$, as the next result demonstrates.

**Lemma 2.2.** Let $\phi: S^1 \to S^1$ be an $n$-valued multimap of degree $\text{Deg}(\phi) = nv + J$. For $\phi_p = \{f_0, f_1, \ldots, f_{n-1}\}$ where the maps $f_j: I \to S^1$ have the property $f_j(0) = p(t_j)$ with $0 \leq t_0 < t_1 < \cdots < t_{n-1} < 1$ and $\tilde{f}_j$ the lift of $f_j$ such that $f_j(0) = t_j$, we have $f_{n-1}(1) - \tilde{f}_0(1) < 1$. Therefore, $\tilde{f}_j(1) = v + t_{j+j}$ for $j = 0, \ldots, (n - 1) - J$ and, if $J \geq 1$, then $\tilde{f}_j(1) = v + 1 + t_{j-(n-J)}$ for $j = n - J, \ldots, n - 1$.

**Proof.** Define $F: I \to \mathbb{R}$ by $F(t) = \tilde{f}_{n-1}(t) - \tilde{f}_0(t)$. Then $F(0) = t_{n-1} - t_0 < 1$. If $F(1) > 1$, then $F(t^*) = 1$ for some $t^* \in (0, 1)$ and thus $\tilde{f}_{n-1}(t^*) = f_0(t^*) + 1$. But $\tilde{f}_j$ is a lift of $f_j$ so we would have

$$pf_{n-1}(t^*) = f_{n-1}(p(t^*)) = p(\tilde{f}_0(t^*) + 1) = p(\tilde{f}_0(t^*)) = f_0(p(t^*))$$

contrary to the definition of a splitting. The formulas for the $\tilde{f}_j(1)$ then follow because $\tilde{f}_0(t) < \tilde{f}_1(t) < \cdots < \tilde{f}_{n-1}(t)$ for all $t \in I$.

The fact that this definition of degree agrees with the classical definition when $n = 1$ is a special case of the following result.

**Theorem 2.2.** If $\phi: S^1 \to S^1$ is a split $n$-valued multimap, then $\text{Deg}(\phi)$ equals $n$ times the classical degree of the maps in the splitting.

**Proof.** Write $\phi = \{f_0, f_1, \ldots, f_{n-1}\}$ where $f_j(p(0)) = p(t_j)$ and $0 \leq t_0 < t_1 < \cdots < t_{n-1} < 1$. Let $\tilde{f}_j: I \to \mathbb{R}$ be the lift of $fjp: I \to S^1$ such that
\[
\tilde{f}_j(0) = t_j. \text{ Since } f_0 : S^1 \to S^1, \text{ we have } \tilde{f}_0(1) = v + \tilde{f}_0(0) = v + t_0 \text{ for some integer } v \text{ and thus } \deg(\phi) = nv. \text{ Moreover, Lemma 2.2 implies that } \\
\tilde{f}_j(1) = v + t_j \text{ for } j = 0, \ldots, n - 1. \text{ On the other hand, by the argument on page 39 of [3], each map } f_j \text{ is homotopic to the power map } \phi_v : S^1 \to S^1 \text{ and therefore it is of classical degree } \deg(f_j) = v, \text{ so } \deg(\phi) = n \deg(f_j). \]

**Theorem 2.3.** If \( n \)-valued multmaps \( \phi, \psi : S^1 \to S^1 \) are homotopic, then \( \deg(\phi) = \deg(\psi) \).

**Proof.** Let \( \Delta = \{ \delta^t \} : S^1 \to S^1 \) be an \( n \)-valued homotopy with \( \phi = \delta^0 \) and \( \psi = \delta^1 \). We will show that there exists \( \varepsilon > 0 \) such that if \( |t - t'| < \varepsilon \), then \( \deg(\delta^t) = \deg(\delta^{t'}) \), that is, the degree is locally constant. Since the degree is integer-valued, that will imply that it is constant and therefore \( \deg(\phi) = \deg(\psi) \). Write \( \delta^t = \{ f^t_0, f^t_1, \ldots, f^t_{n-1} \} \) where \( f^t_j(0) = p(t_j) \) for \( 0 \leq t_0 < t_1 < \cdots < t_{n-1} < 1 \). Let \( \tilde{f}_j^t : I \to \mathbb{R} \) be the lift of \( f^t_j \) such that \( \tilde{f}_j^t(0) = t_j \). We use the corresponding notation for \( \delta^{t'} \). If \( \tilde{f}_j^t(1) = v + t_j \) where \( t_j > 0 \) then, by the continuity of \( \Delta \), if \( \varepsilon > 0 \) is small enough, \( |t - t'| < \varepsilon \) implies that \( \tilde{f}_j^{t'}(1) = v + t_j' \) where \( t_j' > 0 \) and therefore

\[
\deg(\delta^t) = \deg(\delta^{t'}) = nv + J.
\]

If \( \tilde{f}_0^t(1) = v = v + 0 \), that means \( t_0 = 0 \) so

\[
\tilde{f}_j^t(1) = v + t_j = v + \tilde{f}_j^0(0)
\]

for all \( j \) by Lemma 2.2. Therefore, the \( f^t_j : S^1 \to S^1 \) defined by \( f^t_j(p(s)) = p\tilde{f}_j^t(s) \) splits \( \delta^t \) and thus \( \deg(\delta^t) = n \cdot \deg(f^t_0) \) by Theorem 2.2. Since \( \delta^{t'} \) is homotopic to \( \delta^t \), Theorem 2.1 shows that \( \delta^{t'} \) is also split and \( f^t_0 \) is homotopic to \( f^t_0 \), so, for the classical degrees, \( \deg(f^t_0) = \deg(f^t_0) \) and thus \( \deg(\delta^t) = \deg(\delta^{t'}) \). \( \blacksquare \)

**3. The classification theorem.** For integers \( d \) and \( n \geq 1 \), we define the \( n \)-valued multimap we call the \( n \)-valued power map \( \phi_{n,d} : S^1 \to S^1 \) by

\[
\phi_{n,d}(p(t)) = \left\{ p\left(\frac{d}{n}t\right), p\left(\frac{d}{n}t + \frac{1}{n}\right), \ldots, p\left(\frac{d}{n}t + \frac{n - 1}{n}\right) \right\}.
\]

Since

\[
\phi_{1,d}(p(t)) = p(dt) = e^{itd} = (e^{it})^d = (p(t))^d,
\]

we see that \( \phi_{1,d} = \phi_d \). The example on page 75 of [7] is \( \phi_{2,1} \) and the example on page 219 of [8] is \( \phi_{2,-1} \).

**Lemma 3.1.** The degree of \( \phi_{n,d} \) is \( d \).
Proof. We see that \( \phi_{n,d}p = (p\tilde{f}_0, \ldots, p\tilde{f}_{n-1}) \) where \( \tilde{f}_j(t) = dt/n + j/n \) so \( \tilde{f}_j(0) = j/n = t_j \). Write \( d = nv + J \) where \( 0 \leq J \leq n - 1 \). Then

\[
\tilde{f}_0(1) = \frac{d}{n} = v + \frac{J}{n} = v + \tilde{f}_J(0) = v + t_J
\]

so, from the definition, \( \text{Deg}(\phi_{n,d}) = nv + J = d. \)

**Theorem 3.1** (Classification Theorem). If \( \phi: S^1 \to S^1 \) is an \( n \)-valued multimap of degree \( d \), then \( \phi \) is homotopic to \( \phi_{n,d} \).

**Proof.** We again write \( \phi p = \{f_0, f_1, \ldots, f_{n-1}\} : I \to S^1 \) and lift \( f_j \) to \( \tilde{f}_j : I \to \mathbb{R} \) such that \( \tilde{f}_j(0) = t_j \) where \( f_j(0) = p(t_j) \) and \( 0 \leq t_0 < t_1 < \cdots < t_{n-1} < 1 \). Define maps \( \tilde{h}^s_j : I \times I \to \mathbb{R} \) by

\[
\tilde{h}^s_j(t) = s\left(\frac{d}{n}t + j\right) + (1-s)\tilde{f}_j(t).
\]

Then it is clear that \( j < k \) implies \( \tilde{h}^s_j(t) < \tilde{h}^s_k(t) \) for all \( s, t \in I \). Write \( \text{Deg}(\phi) = d = nv + J \) where \( 0 \leq J \leq n - 1 \). Suppose \( 0 \leq j \leq (n-1) - J \). Then, by Lemma 2.2, we have \( \tilde{h}^s_j(1) - \tilde{h}^s_{j+1}(0) = v \). For \( J \geq 1 \) and \( n - J \leq j \leq n - 1 \), Lemma 2.2 implies that \( \tilde{h}^s_j(1) = \tilde{h}^s_{j-(n-J)}(0) = v + 1 \). Thus, for all \( s \in I \), the sets \( \{\tilde{h}_j^s(0)\} \) and \( \{\tilde{h}_j^s(1)\} \) are identical. Therefore, setting

\[
\Delta(p(t), s) = \{\tilde{h}_0^s(t), \tilde{h}_1^s(t), \ldots, \tilde{h}_{n-1}^s(t)\}
\]

we obtain a homotopy \( \Delta : S^1 \times I \to S^1 \) between \( \phi \) and \( \phi_{n,d} \).

**4. Properties of the \( n \)-valued power maps**

**Theorem 4.1.** If \( n \neq d \), then the \( n \)-valued power map \( \phi_{n,d} \) has \( |n - d| \) fixed points, each of nonzero index, and no two fixed points are in the same fixed point class, therefore \( N(\phi_{n,d}) = |n - d| \).

**Proof.** If \( p(t) \in \phi_{n,d}(p(t)) \) for some \( t \) such that \( 0 \leq t < 1 \) then, for some \( j = 0, 1, \ldots, n-1 \), we have

\[
p\left(\frac{d}{n}t + \frac{j}{n}\right) = p(t)
\]

and therefore

\[
\frac{d}{n}t + \frac{j}{n} - t = \frac{(d - n)t}{n} + \frac{j}{n} = r
\]

for some integer \( r \). Since \( n \neq d \), the possible solutions are of the form

\[
t = \frac{nr - j}{d - n}
\]

where \( r \) and \( j \) are integers and \( 0 \leq j \leq n - 1 \). We require that \( 0 \leq t < 1 \) so if \( d - n > 0 \), then \( 0 \leq nr - j < d - n \), whereas if \( d - n < 0 \), then \( 0 \geq nr - j > d - n \). In either case, there are \( |d - n| \) such integers and we
conclude that $\phi_{n,d}$ has $|d - n|$ fixed points. Each of the $|n - d|$ fixed points of $\phi_{n,d}$ is transversal and therefore of index $\pm 1$ (see page 210 of [8]).

It remains to prove that no two of the fixed points of $\phi_{n,d}$ are equivalent in the sense of [8]. Noting that the fixed points are of the form $p\left(\frac{nr - j}{d - n}\right)$, we will make use of the fact that

$$\frac{d}{n} \left( \frac{nr - j}{d - n} \right) + \frac{j}{n} = r + \frac{nr - j}{d - n}.$$ 

For $k = 0, 1$, let

$$x_k = p\left( \frac{nr_k - j_k}{d - n} \right) = p(\tilde{x}_k)$$

be two fixed points of $\phi_{n,d}$ and let $a: I \to S^1$ be a path such that $a(k) = x_k$. Let $\tilde{a}: I \to \mathbb{R}$ be the lift of $a$ such that $\tilde{a}(0) = \tilde{x}_0 \in [0, 1)$. Since $a = \rho \tilde{a}$, we can write

$$\phi_{n,d}a(t) = \phi_{n,d}p(\tilde{a}(t)) = \left\{ p\left( \frac{d}{n} \tilde{a}(t) \right), p\left( \frac{d}{n} \tilde{a}(t) + \frac{1}{n} \right), \ldots, p\left( \frac{d}{n} \tilde{a}(t) + \frac{n - 1}{n} \right) \right\}$$

a split multimap. The fixed points $x_0$ and $x_1$ are in the same fixed point class if there exists a path $a$ connecting them and some $j^*$ with $0 \leq j^* \leq n - 1$ such that $g_{j^*}(x_k) = x_k$ for $k = 0, 1$ and the paths $a, g_{j^*}: I \to S^1$ are homotopic relative to the endpoints (see [8, p. 214]).

We claim that the condition $g_{j^*}(x_0) = x_0$ implies that $j^* = j_0$. To prove it, we note that since $a(0) = \tilde{x}_0$, it follows that

$$p\left( \frac{d}{n} \left( \frac{nr_0 - j_0}{d - n} \right) + \frac{j^*}{n} \right) = p\left( \frac{nr_0 - j_0}{d - n} \right)$$

and therefore

$$\frac{d}{n} \left( \frac{nr_0 - j_0}{d - n} \right) + \frac{j^*}{n} = \frac{nr_0 - j_0}{d - n} + m$$

for some integer $m$, which implies

$$r_0 + \frac{nr_0 - j_0}{d - n} + \frac{j^* - j_0}{n} = \frac{nr_0 - j_0}{d - n} + m,$$

so

$$\frac{j^* - j_0}{n} = m - r_0,$$

an integer. But $0 \leq j^*, j_0 \leq n - 1$ and therefore $j^* = j_0$. This establishes the claim and we write $g = g_{j^*} = g_{j_0}: I \to S^1$ as the path from $x_0$ to $x_1$ that is homotopic to $a$ relative to the endpoints.

Let $\tilde{g}: I \to \mathbb{R}$ be the lift of $g$ defined by

$$\tilde{g}(t) = \frac{d}{n} \tilde{a}(t) + \frac{j_0}{n} - r_0.$$
Then \( \tilde{g}(0) = \tilde{x}_0 = \tilde{a}(0) \). Since \( ag^{-1} \) is a contractible loop, its lift \( \tilde{a}\tilde{g}^{-1} \) is also a loop and thus \( \tilde{g}(1) = \tilde{a}(1) = \tilde{x}_1 + q \) for some integer \( q \). Now

\[
\tilde{g}(1) = \frac{d}{n} \left( \frac{nr_1 - j_1}{d - n} + q \right) + \frac{j_0}{n} - r_0
\]

\[
= r_1 + \frac{nr_1 - j_1}{d - n} + \frac{j_0 - j_1}{n} + \frac{d}{n} q - r_0,
\]

which implies that

\[
q = r_1 - r_0 + \frac{j_0 - j_1}{n} + \frac{d}{n} q
\]

and thus that

\[
q = \frac{nr_1 - j_1}{d - n} - \frac{nr_0 - j_0}{d - n} = \tilde{x}_1 - \tilde{x}_0.
\]

Then \( 0 \leq \tilde{x}_0, \tilde{x}_1 < 1 \) implies that \( q = 0 \) so \( \tilde{x}_0 = \tilde{x}_1 \) and therefore \( x_0 = x_1 \).

We conclude that no two distinct fixed points of \( \phi_{n,d} \) are in the same fixed point class. ■

5. The Wecken property and split multimaps

**Theorem 5.1** (The Wecken Property). The circle has the Wecken property for \( n \)-valued multimaps because, if \( \phi : S^1 \to S^1 \) is an \( n \)-valued multimap of degree \( d \), then \( N(\phi) = |n - d| \) and there is an \( n \)-valued multimap homotopic to \( \phi \) that has exactly \( |n - d| \) fixed points.

**Proof.** By Theorem 3.1, \( \phi \) is homotopic to \( \phi_{n,d} \) so \( N(\phi) = N(\phi_{n,d}) \) by Theorem 6.5 of [8]. If \( d = n \), then \( \phi \) is homotopic to \( \phi_{n,n} \). Choose \( 0 < \varepsilon < 1/n \) and define \( \Delta : S^1 \times I \to S^1 \) by

\[
\Delta(p(t), s) = \left\{ p(t + s\varepsilon), p\left( t + s\varepsilon + \frac{1}{n} \right), \ldots, p\left( t + s\varepsilon + \frac{n-1}{n} \right) \right\}.
\]

Then \( \phi_{n,n} \) is homotopic by \( \Delta \) to a fixed point free multimap. Furthermore, \( N(\phi) = N(\phi_{n,n}) = 0 \). If \( n \neq d \), then Theorem 4.1 completes the proof because \( N(\phi) = N(\phi_{n,d}) = |n - d| \) and \( \phi_{n,d} \) has \( |n - d| \) fixed points. ■

**Theorem 5.2.** The power map \( \phi_{n,d} \) is split if and only if \( d \) is a multiple of \( n \).

**Proof.** The graph of \( \phi_{n,d} \) is

\[
\Gamma_{\phi_{n,d}} = \left\{ \left( p(t), p\left( \frac{d}{n} t + \frac{j}{n} \right) \right) : t \in \mathbb{R}, j = 0, 1, \ldots, n-1 \right\}.
\]

For \( j \in \{0, 1, \ldots, n-1\} \) define \( \gamma_j : I \to \Gamma_{\phi_{n,d}} \) by

\[
\gamma_j(t) = \left( p(t), p\left( \frac{d}{n} t + \frac{j}{n} \right) \right).
\]
Let $\Gamma^j \subseteq \Gamma_{\phi_{n,d}}$ be the component of the graph containing $(p(0), p(j/n))$. Then $p_{1j}: \Gamma^j \to S^1$, the restriction of $p_1$ to $\Gamma^j$, is a covering space and $\gamma_j$ is a path in $\Gamma^j$ from $(p(0), p(j/n))$ to $(p(0), p(d/n + j/n))$. Write $d = nv + J$ where $0 \leq J \leq n - 1$. Then

$$p \left( \frac{d + j}{n} \right) = p \left( \frac{rn + J + j}{n} \right) = p \left( \frac{J + j}{n} \right)$$

tells us that $p(j/n) = p(d/n + j/n)$ and thus $\gamma_j(0) = \gamma_j(1)$ if, and only if, $J = 0$, that is, if and only if $d$ is a multiple of $n$. If $d$ is not a multiple of $n$, then we have shown that the fiber of every covering space $p_{1j}: \Gamma^j \to S^1$ obtained by restricting $p_1$ to a component of $\Gamma_{\phi_{n,d}}$ contains at least two points. If $\phi_{n,d}$ were split, it would have a selection, that is, there would be a map $f: S^1 \to S^1$ such that $f(p(t)) \in \phi_{n,d}(p(t))$ for each $t \in I$. In particular, $(p(0), f(p(0))) \in \Gamma^j$ for some $j$ and thus $\sigma: S^1 \to \Gamma^j$ defined by $\sigma(p(t)) = (p(t), f(p(t)))$ is a cross-section of the covering space $p_{1j}: \Gamma^j \to S^1$, that is, $p_{1j} \sigma$ is the identity map of $S^1$. Thus $p_{1j} \sigma$ would induce the identity isomorphism on the fundamental group of $S^1$. But that is impossible because the index of the image of the homomorphism induced by $p_{1j}$ in that fundamental group equals the cardinality of the fiber of the covering space, which is greater than one. On the other hand, if $d$ is a multiple of $n$, then $\phi_{n,d}$ splits as $\phi_{n,d} = \{f_0, f_1, \ldots, f_{n-1}\}$ where the map $f_j: S^1 \to S^1$ is defined by $f_j(p(t)) = p(dt/n + j/n)$.

**Corollary 5.1.** If $\phi: S^1 \to S^1$ is an $n$-valued multifmap of degree $d$, then $\phi$ is split if and only if $d$ is a multiple of $n$.

**Proof.** By Theorem 3.1, $\phi$ is homotopic to $\phi_{n,d}$. Therefore, by Theorem 2.1, $\phi$ is split if and only if $\phi_{n,d}$ is split, which, by Theorem 5.2, occurs if and only if $d$ is a multiple of $n$.

**References**


Robert F. Brown
Department of Mathematics
University of California
Los Angeles, CA 90095-1555, U.S.A.
E-mail: rfb@math.ucla.edu

Received September 25, 2006