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TOPOLOGY

Fixed Points of n-Valued Multimaps of the Circle

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Summary. A multifunction $\phi\colon X \multimap Y$ is n-valued if $\phi(x)$ is an unordered subset of n points of Y for each $x \in X$. The (continuous) n-valued multimaps $\phi\colon S^1 \multimap S^1$ are classified up to homotopy by an integer-valued degree. In the Nielsen fixed point theory of such multimaps, due to Schirmer, the Nielsen number $N(\phi)$ of an n-valued $\phi\colon S^1 \multimap S^1$ of degree d equals $|n-d|$ and ϕ is homotopic to an n-valued power map that has exactly $|n-d|$ fixed points. Thus the Wecken property, that Schirmer established for manifolds of dimension at least three, also holds for the circle. An n-valued multimap $\phi\colon S^1 \multimap S^1$ of degree d splits into n selfmaps of S^1 if and only if d is a multiple of n.

1. Introduction. A *multifunction* $\phi\colon X \multimap Y$ is a function such that $\phi(x)$ is a subset of Y for each $x \in X$. For S a subset of Y, the set $\phi^{-1}(S)$ consists of the points $x \in X$ such that $\phi(x) \subseteq S$, and the set $\phi_+^{-1}(S)$ consists of the points $x \in X$ such that $\phi(x) \cap S \neq \emptyset$. A multifunction ϕ is said to be *upper semicontinuous* (*usc*) if U open in Y implies $\phi^{-1}(U)$ is open in X. It is *lower semicontinuous* (*lsc*) if U open in Y implies $\phi_+^{-1}(U)$ is open in X. A multifunction that is both upper semicontinuous and lower semicontinuous is said to be *continuous*. Although the term *multimap* is sometimes used for a more general concept, in this paper it will mean a continuous multifunction. An *n-valued multifunction* $\phi\colon X \multimap Y$ is a function that assigns to each $x \in X$ an unordered subset of exactly n points of Y. Thus an *n-valued multimap* is a continuous n-valued multifunction.

O'Neill [6] proved a version of the Lefschetz fixed point theorem for a large class of multimaps $\phi\colon X \multimap X$ of finite polyhedra that includes the n-valued multimaps. Multimaps in this class induce a vector space of endomorphisms of the homology of X. He proved that if any endomorphism

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has a nonzero Lefschetz number, then ϕ has a fixed point, that is, $x \in \phi(x)$ for some $x \in X$.

The Nielsen fixed point theory of n-valued multimaps was developed by Schirmer in a series of papers [7]–[9]. For $\phi\colon X \multimap X$ an n-valued multimap of a finite polyhedron, the Nielsen number $N(\phi)$ has the property that for any n-valued continuous homotopy $\varDelta\colon X \times I \multimap X$ with $\varDelta(x,0) = \phi(x)$, the multimap $\psi\colon X \multimap X$ defined by $\psi(x) = \varDelta(x,1)$ has at least $N(\phi)$ fixed points.

The main result of [9] extended a celebrated theorem of Wecken [10] in the following way. If $\phi\colon X \multimap X$ is an n-valued multimap where X is a compact triangulable manifold, with or without boundary, of dimension at least three, then there is an n-valued multimap $\psi\colon X \multimap X$ homotopic to ϕ such that ψ has exactly $N(\phi)$ fixed points. As in the single-valued theory, we will refer to this property as the *Wecken property* for n-valued multimaps.

If $f_0, f_1, \ldots, f_{n-1}\colon X \to X$ are n maps such that $j \neq k$ implies $f_j(x) \neq f_k(x)$ for all $x \in X$ then

$$\phi(x) = \{f_0(x), f_1(x), \ldots, f_{n-1}(x)\}$$

defines an n-valued multimap $\phi\colon X \multimap X$ that is called *split* in [8]. Only two examples of nonsplit n-valued multimaps are included in Schirmer's papers; see page 75 of [7] and page 219 of [8]. The examples are of n-valued multimaps on the unit circle S^1 and thus the Wecken theorem of [9] does not apply to them. In both cases, the number of fixed points of the map ϕ that Schirmer defines is precisely $N(\phi)$, but there is no general such result about n-valued multimaps of the circle.

We recall that, in the single-valued case, among the manifolds only surfaces can fail to have the Wecken property that a selfmap $f\colon X \to X$ is homotopic to a map with exactly $N(f)$ fixed points [4], [5]. With regard to the 1-dimensional manifolds, the Wecken property holds for maps of the interval because they are all homotopic to a constant map. For $X = S^1$, there is the following well known argument that establishes the Wecken property for single-valued maps. By the classification theorem ([3, p. 39]), if $f\colon S^1 \to S^1$ is of degree d, then f is homotopic to the *power map* ϕ_d defined by viewing S^1 as the unit circle in the complex plane and setting $\phi_d(z) = z^d$. Thus $N(f) = N(\phi_d)$. It has long been known that $N(\phi_d) = |1-d|$ and clearly ϕ_d has $|1-d|$ fixed points except in the case $d = 1$. Since ϕ_1, the identity map, is homotopic to a fixed point free map, every selfmap f on the circle is homotopic to a map with $N(f)$ fixed points.

The Wecken property is easily seen to hold for n-valued multimaps of the interval I, as follows. Let $\phi\colon I \multimap I$ be a multimap. Define $\varDelta\colon I \times I \multimap I$ by $\varDelta(s,t) = \phi(st)$; then $\varDelta$ is continuous by Theorems 1 and 1′ on page 113 of [2]. Thus ϕ is homotopic to the constant n-valued multimap $\kappa\colon I \multimap I$

defined by $\kappa(t) = \phi(0)$, which has n fixed points, whereas $N(\kappa) = n$ by Corollary 7.3 of [8].

The purpose of this paper is to prove that the circle also has the Wecken property for n-valued multimaps. In outline, the argument follows that of the single-valued setting, but there are several significant issues that must be addressed in the n-valued case. In Section 2, we extend the definition of the degree of a selfmap of the circle to define the degree of an n-valued multimap of the circle and we discuss its properties. Section 3 introduces a collection of n-valued multimaps we call *n-valued power maps* $\phi_{n,d}\colon S^1 \multimap S^1$ and we extend the classification theorem by proving that an n-valued multimap $\phi\colon S^1 \multimap S^1$ of degree d is homotopic to $\phi_{n,d}$. We prove in Section 4 that $\phi_{n,d}$ has $|n-d|$ fixed points if $n \neq d$ and then that $N(\phi_{n,d}) = |n-d|$ for all n and d. In Section 5, the Wecken property for n-valued multimaps of the circle is easily seen to follow from the previous results. Moreover, we characterize the split n-valued multimaps of the circle: an n-valued multimap is split if and only if its degree is a multiple of n.

2. The degree of an n-valued multimap of the circle. We begin with some general properties of n-valued multimaps. The following result is a special case of a theorem of O'Neill [6] but, according to [9], it was essentially known much earlier [1].

LEMMA 2.1 (Splitting Lemma). *Let $\phi\colon X \multimap Y$ be an n-valued multimap and let*

$$\Gamma_\phi = \{(x,y) \in X \times Y : y \in \phi(x)\}$$

be the graph of ϕ. The map $p_1\colon \Gamma_\phi \to X$ defined by $p_1(x,y) = x$ is a covering space. It follows that if X is simply connected, then any n-valued multimap $\phi\colon X \multimap Y$ is split.

THEOREM 2.1. *Let $\Delta\colon X \times I \multimap Y$ be an n-valued homotopy; write $\Delta = \{\delta^t\colon X \multimap Y\}$. If δ^0 is split, so also is Δ. Thus an n-valued multimap homotopic to a split n-valued multimap is also split.*

Proof. Write $\delta^0 = \{f_0^0, f_1^0, \ldots, f_{n-1}^0\}$ where $f_j^0\colon X \to Y$. Define

$$\widehat{f}_0^0\colon X \times \{0\} \to \Gamma_\Delta \subseteq (X \times I) \times Y$$

by $\widehat{f}_0^0(x,0) = ((x,0), f_0^0(x))$. Since $p_1\colon \Gamma_\Delta \to X \times I$ is a covering space by Lemma 2.1, by the covering homotopy property there is a map $\widehat{f}_0\colon X \times I \to \Gamma_\Delta$ such that $p_1\widehat{f}_0$ is the identity map of $X \times I$. Let $p_2\colon \Gamma_\Delta \to Y$ be projection. Then $p_2\widehat{f}_0(x,t) \in \delta^t(x)$ so $p_2\widehat{f}_0$ is a selection for Δ and we can write $\Delta = \{p_2\widehat{f}_0, \Delta'\}$ where $\Delta'\colon X \multimap Y$ is an $(n-1)$-valued homotopy $\Delta' = \{\delta'^t\}$ with $\delta'^0 = \{f_1^0, \ldots, f_{n-1}^0\}$. Repeated application of the covering homotopy property produces a splitting $\Delta = \{p_2\widehat{f}_0, p_2\widehat{f}_1, \ldots, p_2\widehat{f}_{n-1}\}$. If an

n-valued multimap $\psi\colon X \multimap Y$ is homotopic to a split n-valued multimap $\phi = \{f_0, \dots, f_{n-1}\}$ by a homotopy $\varDelta$ with $\delta^0 = \phi$ and $\delta^1 = \psi$, then $\psi = \{f^1_0, \dots, f^1_{n-1}\}$ where $f^1_j(x) = p_2\widehat{f}_j(x,1)$. ■

Now we turn our attention to the circle and let $p\colon \mathbb{R} \to S^1$ be the universal covering space where $p(t) = e^{i2\pi t}$. We will denote points of the circle by $p(t)$ for $0 \le t < 1$. Let $\phi\colon S^1 \multimap S^1$ be an n-valued multimap. Then the n-valued function $\phi p\colon I \multimap S^1$ is continuous by Theorems 1 and 1′ on page 133 of [2]. Therefore ϕp is split and, using the ordering on S^1 imposed by p from the ordering of $\mathbb{R}$, we write $\phi p = \{f_0, f_1, \dots, f_{n-1}\}$ where the maps $f_j\colon I \to S^1$ have the property $f_j(0) = p(t_j)$ for $0 \le t_0 < t_1 < \dots < t_{n-1} < 1$. Let $\widetilde{f}_j\colon I \to \mathbb{R}$ be the lift of f_j such that $\widetilde{f}_j(0) = t_j$. We note that if $0 \le j < k \le n-1$, then $\widetilde{f}_j(t) < \widetilde{f}_k(t)$ for all $t \in I$ because $f_j(p(t)) \ne f_k(p(t))$.

Since ϕ is well defined, the sets $\phi p(0)$ and $\phi p(1)$ must be identical. Consequently, $\widetilde{f}_0(1) = v + t_J$ for some integers v, J where $0 \le J \le n-1$. We define $\mathrm{Deg}(\phi)$, the *degree* of the n-valued multimap $\phi\colon S^1 \multimap S^1$, by

$$\mathrm{Deg}(\phi) = nv + J.$$

The degree can be defined just in terms of $\widetilde{f}_0(1)$ because that value determines $\widetilde{f}_j(1)$ for all j, as the next result demonstrates.

LEMMA 2.2. *Let $\phi\colon S^1 \multimap S^1$ be an n-valued multimap of degree* $\mathrm{Deg}(\phi) = nv + J$. *For $\phi p = \{f_0, f_1, \dots, f_{n-1}\}$ where the maps $f_j\colon I \to S^1$ have the property $f_j(0) = p(t_j)$ with $0 \le t_0 < t_1 < \dots < t_{n-1} < 1$ and $\widetilde{f}_j$ the lift of f_j such that $\widetilde{f}_j(0) = t_j$, we have $\widetilde{f}_{n-1}(1) - \widetilde{f}_0(1) < 1$. Therefore, $\widetilde{f}_j(1) = v + t_{J+j}$ for $j = 0, \dots, (n-1) - J$ and, if $J \ge 1$, then $\widetilde{f}_j(1) = v + 1 + t_{j-(n-J)}$ for $j = n-J, \dots, n-1$.*

Proof. Define $F\colon I \to \mathbb{R}$ by $F(t) = \widetilde{f}_{n-1}(t) - \widetilde{f}_0(t)$. Then $F(0) = t_{n-1} - t_0 < 1$. If $F(1) > 1$, then $F(t^*) = 1$ for some $t^* \in (0,1)$ and thus $\widetilde{f}_{n-1}(t^*) = \widetilde{f}_0(t^*) + 1$. But $\widetilde{f}_j$ is a lift of f_j so we would have

$$p\widetilde{f}_{n-1}(t^*) = f_{n-1}(p(t^*)) = p(\widetilde{f}_0(t^*) + 1) = p(\widetilde{f}_0(t^*)) = f_0(p(t^*))$$

contrary to the definition of a splitting. The formulas for the $\widetilde{f}_j(1)$ then follow because $\widetilde{f}_0(t) < \widetilde{f}_1(t) < \dots < \widetilde{f}_{n-1}(t)$ for all $t \in I$. ■

The fact that this definition of degree agrees with the classical definition when $n = 1$ is a special case of the following result.

THEOREM 2.2. *If $\phi\colon S^1 \multimap S^1$ is a split n-valued multimap, then* $\mathrm{Deg}(\phi)$ *equals n times the classical degree of the maps in the splitting.*

Proof. Write $\phi = \{f_0, f_1, \dots, f_{n-1}\}$ where $f_j(p(0)) = p(t_j)$ and $0 \le t_0 < t_1 < \dots < t_{n-1} < 1$. Let $\widetilde{f}_j\colon I \to \mathbb{R}$ be the lift of $f_jp\colon I \to S^1$ such that

$\widetilde{f}_j(0) = t_j$. Since $f_0\colon S^1 \to S^1$, we have $\widetilde{f}_0(1) = v + \widetilde{f}_0(0) = v + t_0$ for some integer v and thus $\mathrm{Deg}(\phi) = nv$. Moreover, Lemma 2.2 implies that $\widetilde{f}_j(1) = v + t_j$ for $j = 0, \ldots, n-1$. On the other hand, by the argument on page 39 of [3], each map f_j is homotopic to the power map $\phi_v\colon S^1 \to S^1$ and therefore it is of classical degree $\deg(f_j) = v$, so $\mathrm{Deg}(\phi) = n \deg(f_j)$. ■

THEOREM 2.3. *If n-valued multimaps $\phi, \psi\colon S^1 \multimap S^1$ are homotopic, then* $\mathrm{Deg}(\phi) = \mathrm{Deg}(\psi)$.

Proof. Let $\Delta = \{\delta^t\}\colon S^1 \multimap S^1$ be an n-valued homotopy with $\phi = \delta^0$ and $\psi = \delta^1$. We will show that there exists $\varepsilon > 0$ such that if $|t - t'| < \varepsilon$, then $\mathrm{Deg}(\delta^t) = \mathrm{Deg}(\delta^{t'})$, that is, the degree is locally constant. Since the degree is integer-valued, that will imply that it is constant and therefore $\mathrm{Deg}(\phi) = \mathrm{Deg}(\psi)$. Write $\delta^t p = \{f_0^t, f_1^t, \ldots, f_{n-1}^t\}$ where $f_j^t(0) = p(t_j)$ for $0 \le t_0 < t_1 < \cdots < t_{n-1} < 1$. Let $\widetilde{f}_j^t\colon I \to \mathbb{R}$ be the lift of f_j^t such that $\widetilde{f}_j^t(0) = t_j$. We use the corresponding notation for $\delta^{t'}$. If $\widetilde{f}_j^t(1) = v + t_J$ where $t_J > 0$ then, by the continuity of Δ, if $\varepsilon > 0$ is small enough, $|t - t'| < \varepsilon$ implies that $\widetilde{f}_j^{t'}(1) = v + t'_J$ where $t'_J > 0$ and therefore

$$\mathrm{Deg}(\delta^t) = \mathrm{Deg}(\delta^{t'}) = nv + J.$$

If $\widetilde{f}_0^t(1) = v = v + 0$, that means $t_0 = 0$ so

$$\widetilde{f}_j^t(1) = v + t_j = v + \widetilde{f}_j^t(0)$$

for all j by Lemma 2.2. Therefore, the $f_j^t\colon S^1 \to S^1$ defined by $f_j^t(p(s)) = p\widetilde{f}_j^t(s)$ splits δ^t and thus $\mathrm{Deg}(\delta^t) = n \cdot \deg(f_0^t)$ by Theorem 2.2. Since $\delta^{t'}$ is homotopic to δ^t, Theorem 2.1 shows that $\delta^{t'}$ is also split and $f_0^{t'}$ is homotopic to f_0^t so, for the classical degrees, $\deg(f_0^t) = \deg(f_0^{t'})$ and thus $\mathrm{Deg}(\delta^t) = \mathrm{Deg}(\delta^{t'})$. ■

3. The classification theorem. For integers d and $n \ge 1$, we define the n-valued multimap we call the *n-valued power map* $\phi_{n,d}\colon S^1 \multimap S^1$ by

$$\phi_{n,d}(p(t)) = \left\{p\left(\frac{d}{n}\,t\right), p\left(\frac{d}{n}\,t + \frac{1}{n}\right), \ldots, p\left(\frac{d}{n}\,t + \frac{n-1}{n}\right)\right\}.$$

Since

$$\phi_{1,d}(p(t)) = p(dt) = e^{i2\pi dt} = (e^{i2\pi t})^d = (p(t))^d,$$

we see that $\phi_{1,d} = \phi_d$. The example on page 75 of [7] is $\phi_{2,1}$ and the example on page 219 of [8] is $\phi_{2,-1}$.

LEMMA 3.1. *The degree of $\phi_{n,d}$ is d.*

Proof. We see that $\phi_{n,d}p = (p\widetilde{f}_0, \ldots, p\widetilde{f}_{n-1})$ where $\widetilde{f}_j(t) = dt/n + j/n$ so $\widetilde{f}_j(0) = j/n = t_j$. Write $d = nv + J$ where $0 \le J \le n-1$. Then

$$\widetilde{f}_0(1) = \frac{d}{n} = v + \frac{J}{n} = v + \widetilde{f}_J(0) = v + t_J$$

so, from the definition, $\mathrm{Deg}(\phi_{n,d}) = nv + J = d$. ∎

THEOREM 3.1 (Classification Theorem). *If $\phi\colon S^1 \multimap S^1$ is an n-valued multimap of degree d, then ϕ is homotopic to $\phi_{n,d}$.*

Proof. We again write $\phi p = \{f_0, f_1, \ldots, f_{n-1}\} : I \multimap S^1$ and lift f_j to $\widetilde{f}_j\colon I \to \mathbb{R}$ such that $\widetilde{f}_j(0) = t_j$ where $f_j(0) = p(t_j)$ and $0 \le t_0 < t_1 < \cdots < t_{n-1} < 1$. Define maps $\widetilde{h}_j^s\colon I \times I \to \mathbb{R}$ by

$$\widetilde{h}_j^s(t) = s\left(\frac{d}{n}t + j\right) + (1-s)\widetilde{f}_j(t).$$

Then it is clear that $j < k$ implies $\widetilde{h}_j^s(t) < \widetilde{h}_k^s(t)$ for all $s, t \in I$. Write $\mathrm{Deg}(\phi) = d = nv + J$ where $0 \le J \le n-1$. Suppose $0 \le j \le (n-1)-J$. Then, by Lemma 2.2, we have $\widetilde{h}_j^s(1) - \widetilde{h}_{J+j}^s(0) = v$. For $J \ge 1$ and $n-J \le j \le n-1$, Lemma 2.2 implies that $\widetilde{h}_j^s(1) - \widetilde{h}_{j-(n-J)}^s(0) = v+1$. Thus, for all $s \in I$, the sets $\{p\widetilde{h}_j^s(0)\}$ and $\{p\widetilde{h}_j^s(1)\}$ are identical. Therefore, setting

$$\Delta(p(t), s) = \{p\widetilde{h}_0^s(t), p\widetilde{h}_1^s(t), \ldots, p\widetilde{h}_{n-1}^s(t)\}$$

we obtain a homotopy $\Delta\colon S^1 \times I \multimap S^1$ between ϕ and $\phi_{n,d}$. ∎

4. Properties of the n-valued power maps

THEOREM 4.1. *If $n \ne d$, then the n-valued power map $\phi_{n,d}$ has $|n-d|$ fixed points, each of nonzero index, and no two fixed points are in the same fixed point class, therefore $N(\phi_{n,d}) = |n-d|$.*

Proof. If $p(t) \in \phi_{n,d}(p(t))$ for some t such that $0 \le t < 1$ then, for some $j = 0, 1, \ldots, n-1$, we have

$$p\left(\frac{d}{n}t + \frac{j}{n}\right) = p(t)$$

and therefore

$$\frac{d}{n}t + \frac{j}{n} - t = \frac{(d-n)t}{n} + \frac{j}{n} = r$$

for some integer r. Since $n \ne d$, the possible solutions are of the form

$$t = \frac{nr - j}{d - n}$$

where r and j are integers and $0 \le j \le n-1$. We require that $0 \le t < 1$ so if $d - n > 0$, then $0 \le nr - j < d - n$, whereas if $d - n < 0$, then $0 \ge nr - j > d - n$. In either case, there are $|d-n|$ such integers and we conclude that $\phi_{n,d}$ has $|d-n|$ fixed points. Each of the $|n-d|$ fixed points of $\phi_{n,d}$ is transversal and therefore of index ± 1 (see page 210 of [8]).

It remains to prove that no two of the fixed points of $\phi_{n,d}$ are equivalent in the sense of [8]. Noting that the fixed points are of the form $p\big(\frac{nr-j}{d-n}\big)$, we will make use of the fact that

$$\frac{d}{n}\left(\frac{nr-j}{d-n}\right)+\frac{j}{n}=r+\frac{nr-j}{d-n}.$$

For $k=0,1$, let

$$x_k=p\left(\frac{nr_k-j_k}{d-n}\right)=p(\widetilde{x}_k)$$

be two fixed points of $\phi_{n,d}$ and let $a\colon I\to S^1$ be a path such that $a(k)=x_k$. Let $\widetilde{a}\colon I\to\mathbb{R}$ be the lift of a such that $\widetilde{a}(0)=\widetilde{x}_0\in[0,1)$. Since $a=p\widetilde{a}$, we can write

$$\begin{aligned}\phi_{n,d}a(t)&=\phi_{n,d}p(\widetilde{a}(t))\\&=\left\{p\left(\frac{d}{n}\,\widetilde{a}(t)\right),p\left(\frac{d}{n}\,\widetilde{a}(t)+\frac{1}{n}\right),\dots,p\left(\frac{d}{n}\,\widetilde{a}(t)+\frac{n-1}{n}\right)\right\}\\&=\{g_0(t),g_1(t),\dots,g_{n-1}(t)\},\end{aligned}$$

a split multimap. The fixed points x_0 and x_1 are in the same fixed point class if there exists a path a connecting them and some j^* with $0\le j^*\le n-1$ such that $g_{j^*}(x_k)=x_k$ for $k=0,1$ and the paths $a,g_{j^*}\colon I\to S^1$ are homotopic relative to the endpoints (see [8, p. 214]).

We claim that the condition $g_{j^*}(x_0)=x_0$ implies that $j^*=j_0$. To prove it, we note that since $a(0)=\widetilde{x}_0$, it follows that

$$p\left(\frac{d}{n}\left(\frac{nr_0-j_0}{d-n}\right)+\frac{j^*}{n}\right)=p\left(\frac{nr_0-j_0}{d-n}\right)$$

and therefore

$$\frac{d}{n}\left(\frac{nr_0-j_0}{d-n}\right)+\frac{j^*}{n}=\frac{nr_0-j_0}{d-n}+m$$

for some integer m, which implies

$$r_0+\frac{nr_0-j_0}{d-n}+\frac{j^*-j_0}{n}=\frac{nr_0-j_0}{d-n}+m,$$

so

$$\frac{j^*-j_0}{n}=m-r_0,$$

an integer. But $0\le j^*,j_0\le n-1$ and therefore $j^*=j_0$. This establishes the claim and we write $g=g_{j^*}=g_{j_0}\colon I\to S^1$ as the path from x_0 to x_1 that is homotopic to a relative to the endpoints.

Let $\widetilde{g}\colon I\to\mathbb{R}$ be the lift of g defined by

$$\widetilde{g}(t)=\frac{d}{n}\,\widetilde{a}(t)+\frac{j_0}{n}-r_0.$$

Then $\widetilde{g}(0) = \widetilde{x}_0 = \widetilde{a}(0)$. Since ag^{-1} is a contractible loop, its lift $\widetilde{a}\widetilde{g}^{-1}$ is also a loop and thus $\widetilde{g}(1) = \widetilde{a}(1) = \widetilde{x}_1 + q$ for some integer q. Now

$$\begin{aligned}\widetilde{g}(1) &= \frac{d}{n}\left(\frac{nr_1 - j_1}{d-n} + q\right) + \frac{j_0}{n} - r_0 \\ &= r_1 + \frac{nr_1 - j_1}{d-n} + \frac{j_0 - j_1}{n} + \frac{d}{n}\,q - r_0,\end{aligned}$$

which implies that

$$q = r_1 - r_0 + \frac{j_0 - j_1}{n} + \frac{d}{n}\,q$$

and thus that

$$q = \frac{nr_1 - j_1}{d-n} - \frac{nr_0 - j_0}{d-n} = \widetilde{x}_1 - \widetilde{x}_0.$$

Then $0 \le \widetilde{x}_0, \widetilde{x}_1 < 1$ implies that $q = 0$ so $\widetilde{x}_0 = \widetilde{x}_1$ and therefore $x_0 = x_1$. We conclude that no two distinct fixed points of $\phi_{n,d}$ are in the same fixed point class. ∎

5. The Wecken property and split multimaps

THEOREM 5.1 (The Wecken Property). *The circle has the Wecken property for n-valued multimaps because, if $\phi\colon S^1 \multimap S^1$ is an n-valued multimap of degree d, then $N(\phi) = |n-d|$ and there is an n-valued multimap homotopic to ϕ that has exactly $|n-d|$ fixed points.*

Proof. By Theorem 3.1, ϕ is homotopic to $\phi_{n,d}$ so $N(\phi) = N(\phi_{n,d})$ by Theorem 6.5 of [8]. If $d = n$, then ϕ is homotopic to $\phi_{n,n}$. Choose $0 < \varepsilon < 1/n$ and define $\Delta\colon S^1 \times I \multimap S^1$ by

$$\Delta(p(t), s) = \left\{p(t + s\varepsilon), p\left(t + s\varepsilon + \frac{1}{n}\right), \ldots, p\left(t + s\varepsilon + \frac{n-1}{n}\right)\right\}.$$

Then $\phi_{n,n}$ is homotopic by Δ to a fixed point free multimap. Furthermore, $N(\phi) = N(\phi_{n,n}) = 0$. If $n \neq d$, then Theorem 4.1 completes the proof because $N(\phi) = N(\phi_{n,d}) = |n-d|$ and $\phi_{n,d}$ has $|n-d|$ fixed points. ∎

THEOREM 5.2. *The power map $\phi_{n,d}$ is split if and only if d is a multiple of n.*

Proof. The graph of $\phi_{n,d}$ is

$$\Gamma_{\phi_{n,d}} = \left\{\left(p(t), p\left(\frac{d}{n}\,t + \frac{j}{n}\right)\right) : t \in \mathbb{R},\ j = 0, 1, \ldots, n-1\right\}.$$

For $j \in \{0, 1, \ldots, n-1\}$ define $\gamma_j\colon I \to \Gamma_{\phi_{n,d}}$ by

$$\gamma_j(t) = \left(p(t), p\left(\frac{d}{n}\,t + \frac{j}{n}\right)\right).$$

Let $\Gamma^j \subseteq \Gamma_{\phi_{n,d}}$ be the component of the graph containing $(p(0), p(j/n))$. Then $p_{1j}\colon \Gamma^j \to S^1$, the restriction of p_1 to Γ^j, is a covering space and γ_j is a path in Γ^j from $(p(0), p(j/n))$ to $(p(0), p(d/n + j/n))$. Write $d = nv + J$ where $0 \leq J \leq n-1$. Then

$$p\left(\frac{d}{n} + \frac{j}{n}\right) = p\left(\frac{rn + J + j}{n}\right) = p\left(\frac{J+j}{n}\right)$$

tells us that $p(j/n) = p(d/n + j/n)$ and thus $\gamma_j(0) = \gamma_j(1)$ if, and only if, $J = 0$, that is, if and only if d is a multiple of n. If d is not a multiple of n, then we have shown that the fiber of every covering space $p_{1j}\colon \Gamma^j \to S^1$ obtained by restricting p_1 to a component of $\Gamma_{\phi_{n,d}}$ contains at least two points. If $\phi_{n,d}$ were split, it would have a selection, that is, there would be a map $f\colon S^1 \to S^1$ such that $f(p(t)) \in \phi_{n,d}(p(t))$ for each $t \in I$. In particular, $(p(0), f(p(0))) \in \Gamma^j$ for some j and thus $\sigma\colon S^1 \to \Gamma^j$ defined by $\sigma(p(t)) = (p(t), f(p(t))$ is a cross-section of the covering space $p_{1j}\colon \Gamma^j \to S^1$, that is, $p_{1j}\sigma$ is the identity map of S^1. Thus $p_{1j}\sigma$ would induce the identity isomorphism on the fundamental group of S^1. But that is impossible because the index of the image of the homomorphism induced by p_{1j} in that fundamental group equals the cardinality of the fiber of the covering space, which is greater than one. On the other hand, if d is a multiple of n, then $\phi_{n,d}$ splits as $\phi_{n,d} = \{f_0, f_1, \ldots, f_{n-1}\}$ where the map $f_j\colon S^1 \to S^1$ is defined by $f_j(p(t)) = p(dt/n + j/n)$. ■

COROLLARY 5.1. *If $\phi\colon S^1 \multimap S^1$ is an n-valued multimap of degree d, then ϕ is split if and only if d is a multiple of n.*

Proof. By Theorem 3.1, ϕ is homotopic to $\phi_{n,d}$. Therefore, by Theorem 2.1, ϕ is split if and only if $\phi_{n,d}$ is split, which, by Theorem 5.2, occurs if and only if d is a multiple of n. ■

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