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PROBABILITY THEORY AND STOCHASTIC PROCESSES

The Probabilistic Solution of the Dirichlet Problem for Degenerate Elliptic Equations

by

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Summary. We study the Dirichlet problem for degenerate elliptic equations, and show that the probabilistic solution is a unique viscosity solution.

1. Introduction. We deal with the Dirichlet problem for a linear degenerate elliptic equation in a bounded domain G of $\mathbb{R}^N$ with boundary ∂G:

$$\begin{cases} H_\alpha(x, u(x), Du(x), D^2u(x)) := \alpha u - \mathcal{L}u - g = 0 & \text{in } G, \\ u = h & \text{on } \partial G, \end{cases} \tag{1}$$

for $\alpha > 0$. Here we are given

$$g : \text{bounded continuous on } \overline{G}, \qquad h \in C(\partial G), \tag{2}$$

$$b : \mathbb{R}^N \to \mathbb{R}^N,\ \sigma : \mathbb{R}^N \to \mathbb{R}^{N\times M},\ \text{Lipschitz on } \overline{G}, \tag{3}$$

and $\mathcal{L}$ denotes the second-order differential operator defined by

$$\mathcal{L}u = \frac{1}{2}\operatorname{tr}(\sigma(x)\sigma^\top(x)D^2u) + (b(x), Du).$$

Let $\{X_t\}$ be a solution to the stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x, \quad t \geq 0, \tag{4}$$

on a complete probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ carrying an M-dimensional standard Brownian motion $\{B_t\}$, where $\mathcal{F}_t$ denotes the σ-algebra generated by B_s, $s \leq t$.

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The purpose of this paper is to show that

$$u(x) = E\Big[\int_0^\tau e^{-\alpha t} g(X_t)\,dt + e^{-\alpha\tau} h(X_\tau)\Big] \tag{5}$$

is a unique viscosity solution of the degenerate elliptic equation (1), where τ is the time of first exit from G defined by $\tau = \inf\{t \geq 0 : X_t \notin G\}$. This kind of probabilistic solution has been studied by many authors [1], [5], [7], [9], [12] when $\mathcal{L}$ is uniformly elliptic and a regularity condition on ∂G is satisfied. In the present paper, we assume the following mild condition on the boundary: there exist a non-negative function $\zeta \in C(\overline{G}) \cap C^2(G)$ and $\lambda > 0$ such that

$$\begin{cases} -\alpha\zeta + \mathcal{L}\zeta + \lambda \leq 0 & \text{in } G, \\ \zeta = 0 & \text{on } \partial G. \end{cases} \tag{6}$$

By (6) and the Markov property of the stopped process $Z_t = X_{t\wedge\tau}$ at τ, we show that

$$v(x) = E\Big[\int_0^\tau e^{-(\alpha+1/\varepsilon)t}\Big\{\frac{1}{\varepsilon}\, v + g\Big\}(X_t)\,dt + e^{-(\alpha+1/\varepsilon)\tau} h(X_\tau)\Big] \tag{7}$$

admits a unique solution v, independent of $\varepsilon > 0$, which coincides with u. Furthermore, by passage to the limit as $\alpha \to 0$, we give a viscosity solution w of

$$\begin{cases} H_0(x, w(x), Dw(x), D^2w(x)) := -\mathcal{L}w - g = 0 & \text{in } G, \\ w = h & \text{on } \partial G. \end{cases} \tag{8}$$

2. Main results. Following [3], we define the notion of viscosity solution of (1).

DEFINITION 2.1. $k \in C(\overline{G})$ is called a *viscosity solution* of (1) if

$$k(x) = h(x) \quad \text{for all } x \in \partial G, \tag{9}$$

and for any $\varphi \in C^2(G)$ and any local maximum (resp., minimum) point $z \in G$ of $k - \varphi$, we have

$$H_\alpha(z, k(z), D\varphi(z), D^2\varphi(z)) \leq 0 \quad (\text{resp., } H_\alpha(z, k(z), D\varphi(z), D^2\varphi(z)) \geq 0).$$

Concerning (9), we mention the works [4], [11] for the existence of viscosity solutions of semilinear elliptic PDE's in terms of backward stochastic differential equations under the generalized boundary conditions.

Now the main results of this paper are the following.

THEOREM 2.2. *Assume* (2), (3), (6). *Then there exists a unique solution* v *of* (7), *for sufficiently small* $\varepsilon > 0$, *in the Banach space* $\mathcal{C}$ *of all bounded uniformly continuous functions* f *on* $\mathbb{R}^N$ *with norm* $\|f\| = \sup_x |f(x)|$.

THEOREM 2.3. *Assume* (2), (3), (6). *Then* v *is a unique viscosity solution of* (1), *independent of* $\varepsilon > 0$, *and*

$$v = u. \tag{10}$$

For the uniqueness of (8), we assume that there exists a constant $\nu > 0$ such that

$$\frac{1}{2}\nu\,\mathrm{tr}(\sigma(x)\sigma^\top(x)I_1) + (b(x), e_1) > 0 \quad \text{for all } x \in G, \tag{11}$$

where $e_1 = (1, 0, \dots, 0)^\top \in \mathbb{R}^N$ and $I_1 = (e_1, 0e_1, \dots, 0e_1) \in \mathbb{R}^N \otimes \mathbb{R}^N$.

THEOREM 2.4. *Under the assumptions of Theorem* 2.3, *suppose that* (11) *holds and*

$$E[\tau] < \infty \quad \text{or} \quad g = 0. \tag{12}$$

Then

$$w(x) = E\Big[\int_0^\tau g(X_t)\,dt + h(X_\tau)\Big]$$

is a unique viscosity solution of (8).

3. Proof of Theorem 2.2

LEMMA 3.1. *Let* G_γ *be the potential operator of* $\{X_t\}$, *i.e.*,

$$G_\gamma f(x) = E\Big[\int_0^\infty e^{-\gamma t} f(X_t)\,dt\Big], \qquad \gamma > 0.$$

Then, under (3), *the class* $\mathcal{D} := \{\gamma G_\gamma f : f \in \mathcal{C},\ \gamma > 0\}$ *is dense in* $\mathcal{C}$.

Proof. Let $\{X_t'\}$ be the solution of (4) with $X_0' = x'$. Since

$$|b(x) - b(x')| + \|\sigma(x) - \sigma(x')\| \le L|x - x'| \quad \text{for some } L > 0,$$

we have

$$E[|X_t - X_t'|^2] \le |x - x'|^2 e^{(2L+L^2)t}.$$

Let $f \in \mathcal{C}$. It is clear that

$$\begin{aligned}|\gamma G_\gamma f(x) - f(x)| &\le E\Big[\int_0^\infty \gamma e^{-\gamma t}|f(X_t) - f(x)|\,dt\Big] \\ &\le E\Big[\int_0^\infty e^{-t}|f(X_{t/\gamma}) - f(x)|\,dt\Big] \to 0 \quad \text{as } \gamma \to \infty.\end{aligned}$$

Note that for any $\varrho > 0$ there exists $C_\varrho > 0$ such that

$$|f(x) - f(y)| \le C_\varrho|x - y| + \varrho, \quad x, y \in \mathbb{R}^N. \tag{13}$$

Hence, taking sufficiently large $\gamma_0 > 0$, we have

$$|\gamma G_\gamma f(x) - \gamma G_\gamma f(x')| \le \int_0^\infty e^{-t} C_\varrho |x-x'| e^{(L+L^2/2)t/\gamma}\,dt + \varrho \le C_\varrho \left(1 - \frac{L+L^2/2}{\gamma_0}\right)^{-1} |x-x'| + \varrho$$

for all $\gamma \ge \gamma_0$. Also, f can be extended over the compactification $\overline{\mathbb{R}^N}$, embedded in $[0,1]^\infty$, of $\mathbb{R}^N$. Therefore

$$\gamma G_\gamma f(x) \to f(x) \quad \text{as } \gamma \to \infty,\ x \in \overline{\mathbb{R}^N}.$$

By the Riesz representation theorem, for every Λ in the dual space $\mathcal{C}^*$, there exists a measure μ on $\overline{\mathbb{R}^N}$ such that $\Lambda(f) = \int f\,d\mu$ for all $f \in \mathcal{C}$. So, if $\Lambda(\widehat{f}\,) = 0$ for all $\widehat{f} \in \mathcal{D}$, then $\Lambda(f) = \lim_{\gamma\to\infty} \Lambda(\gamma G_\gamma f) = 0$. By the Hahn–Banach theorem, the closure $\overline{\mathcal{D}}$ coincides with $\mathcal{C}$.

Proof of Theorem 2.2. For any $f \in \mathcal{C}$, we define the operator $\mathcal{T} = \mathcal{T}_\varepsilon$ by

$$\mathcal{T} f(x) = E\left[\int_0^\tau e^{-\alpha_\varepsilon t}\left\{\frac{1}{\varepsilon} f + g\right\}(X_t)\,dt + e^{-\alpha_\varepsilon \tau} h(X_\tau)\right],$$

where $\alpha_\varepsilon = \alpha + 1/\varepsilon$. Clearly, $\|\mathcal{T} f\| < \infty$.

We shall show that

(14) $\mathcal{T} f(x)$ is uniformly continuous on $\mathbb{R}^N$ if $h \in \mathcal{D}$.

Let $h = \gamma G_\gamma \eta \in \mathcal{D}$ for some $\eta \in \mathcal{C}$. Then, by the resolvent equation for G_γ,

$$h = G_{\alpha_\varepsilon} \widetilde{\eta}, \qquad \widetilde{\eta} := \gamma\{\eta - (\gamma - \alpha_\varepsilon) G_\gamma \eta\}.$$

By the Markov property,

$$\mathcal{T} f(x) = E\left[\int_0^\tau e^{-\alpha_\varepsilon t}\left\{\frac{1}{\varepsilon} f + g\right\}(X_t)\,dt + \int_\tau^\infty e^{-\alpha_\varepsilon t} \widetilde{\eta}(X_t)\,dt\right].$$

We choose $\varepsilon < 1/(L + L^2/2)$. Then, taking into account (13), we see that

$$E\Big[\int_0^{\tau\wedge\tau'} e^{-\alpha_\varepsilon t} |f(X_t) - f(X'_t)|\,dt\Big] \le E\Big[\int_0^\infty e^{-\alpha_\varepsilon t} |f(X_t) - f(X'_t)|\,dt\Big] \le \frac{C_\varrho |x-x'|}{1/\varepsilon - (L+L^2/2)} + \frac{\varrho}{\alpha_\varepsilon},$$

where τ' is the time of first exit for $\{X'_t\}$. Also, by (6) and Ito's formula,

$$\begin{aligned} E\Big[\int_{\tau\wedge\tau'}^{\tau} e^{-\alpha_\varepsilon t}\lambda\,dt\Big] &\le E[e^{-\alpha_\varepsilon(\tau\wedge\tau')}\zeta(X_{\tau\wedge\tau'}) - e^{-\alpha_\varepsilon\tau}\zeta(X_\tau)] \\ &= E[\{e^{-\alpha_\varepsilon\tau'}\zeta(X_{\tau'}) - e^{-\alpha_\varepsilon\tau}\zeta(X_\tau)\}1_{\{\tau'<\tau\}}] \\ &\le E[\{e^{-\alpha_\varepsilon\tau'}\zeta(X_{\tau'}) - e^{-\alpha_\varepsilon\tau'}\zeta(X'_{\tau'})\}1_{\{\tau'<\tau\}}] \\ &\le E[e^{-\alpha_\varepsilon\tau'}|\zeta(X_{\tau'}) - \zeta(X'_{\tau'})|1_{\{\tau'<\tau\}}]. \end{aligned}$$

Since for any $\varrho > 0$ there exists $C_\varrho > 0$ such that

$$|\zeta(x) - \zeta(x')| \le C_\varrho|x - x'| + \varrho, \qquad x, x' \in \overline{G},$$

we have

$$\begin{aligned} E\Big[\int_{\tau\wedge\tau'}^{\tau} e^{-\alpha_\varepsilon t}\,dt\Big] &\le C_\varrho E[e^{-\alpha_\varepsilon\tau'}|X_{\tau'} - X'_{\tau'}|] + \varrho \\ &\le \lim_{T\to\infty} C_\varrho E[e^{-2\alpha_\varepsilon(\tau'\wedge T)}|X_{\tau'\wedge T} - X'_{\tau'\wedge T}|^2]^{1/2} + \varrho \\ &\le C_\varrho|x - x'| + \varrho. \end{aligned}$$

Combining these facts, we get

$$|\mathcal{T}f(x) - \mathcal{T}f(x')| \le C_\varrho|x - x'| + \varrho, \quad x, x' \in \mathbb{R}^N,$$

which implies (14).

Now, we have

$$\begin{aligned} |\mathcal{T}f_1(x) - \mathcal{T}f_2(x)| &\le \frac{1}{\varepsilon} E\Big[\int_0^\infty e^{-\alpha_\varepsilon t}|f_1(X_t) - f_2(X_t)|\,dt\Big] \\ &\le \frac{1}{\varepsilon\alpha + 1}\,\|f_1 - f_2\|, \qquad f_1, f_2 \in \mathcal{C}. \end{aligned}$$

Thus the map $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ is a contraction, and $\mathcal{T}$ has a fixed point $v \in \mathcal{C}$ for sufficiently small $\varepsilon > 0$.

In the general case, by Lemma 3.1, there exists $h_\gamma \in \mathcal{D}$ such that $h_\gamma \to h \in \mathcal{C}$ as $\gamma \to \infty$. Let $v^{(\gamma)}$ be the solution of (7) for h_γ. Then

$$v^{(\gamma)}(x) = E\Big[\int_0^\tau e^{-\alpha_\varepsilon t}\Big\{\frac{1}{\varepsilon}\,v^{(\gamma)} + g\Big\}(X_t)\,dt + e^{-\alpha_\varepsilon\tau}h_\gamma(X_\tau)\Big],$$

and hence

$$\|v^{(\gamma)} - v^{(\gamma')}\| \le \frac{\varepsilon\alpha + 1}{\varepsilon\alpha}\,\|h_\gamma - h_{\gamma'}\| \to 0 \qquad \text{as } \gamma, \gamma' \to \infty.$$

Therefore $v^{(\gamma)}$ converges to $v \in \mathcal{C}$, which satisfies (7). The uniqueness is immediate.

4. Proof of Theorem 2.3

LEMMA 4.1. *Under the assumptions of Theorem* 2.3, *the solution* $v \in \mathcal{C}$ *of* (7) *satisfies*

$$v(x) = E\left[\int_0^{\theta\wedge\tau} e^{-\alpha_\varepsilon s}\left\{\frac{1}{\varepsilon}v + g\right\}(X_s)\,ds + e^{-\alpha_\varepsilon(\theta\wedge\tau)}v(X_{\theta\wedge\tau})\right] \tag{15}$$

for any bounded $\{\mathcal{F}_t\}$*-stopping time* θ.

Proof. We notice that (15) corresponds to the dynamic programming principle [6]. We shall deduce (15) stepwise from the Markov property of the stopped process $Z_t = X_{t\wedge\tau}$.

STEP 1. Let X_t^x denote the solution to (4) with $X_0 = x$. It is well known that $\psi(r) := E[\sup_{0\le t\le r}|X_t^x - X_t^y|^2]$ is continuous. Also, by Gronwall's inequality, we observe that

$$\psi(r) \le C_r|x-y|^2, \quad x, y \in \mathbb{R}^N,$$

for some constant $C_r > 0$ depending on $r > 0$. Hence, if $x_n \to x$ in $\mathbb{R}^N$, then $X_t^{x_n} \to X_t^x$ in probability. Let f be a bounded continuous function on $\mathbb{R}^N$. Then

$$E[f(X_t^{x_n})] \to E[f(X_t^x)],$$

which implies that the map $x \mapsto E[f(X_t^x)]$ is continuous. Let A be an open subset of $\mathbb{R}^N$. We set

$$f_k(x) = \frac{kd(x, A^c)}{1 + kd(x, A^c)}.$$

It is clear that $f_k(x) \nearrow 1_A(x)$ as $k \to \infty$, and so the map $x \mapsto P(X_t^x \in A)$ is Borel measurable. By the monotone class theorem, this map is Borel measurable for all Borel sets A. Therefore we conclude that the map $x \mapsto P(X_{t\wedge\tau}^x \in A)$ is Borel measurable.

STEP 2. Let $\{x_t\}$ be the solution of

$$dx_t = b(x_t)dt + \sigma(x_t)d\beta_t, \quad x_0 = x,$$

for a Brownian motion $\{\beta_t\}$ on the canonical probability space $(W, \mathcal{B}, P, \{\mathcal{B}_t\})$ [9], and consider the stopped process $z_t = x_{t\wedge\widetilde{\tau}}$, where $\widetilde{\tau}$ is the time of first exit from G. Then

$$dz_t = b(z_t)1_{\{t<\widetilde{\tau}\}}dt + \sigma(z_t)1_{\{t<\widetilde{\tau}\}}d\beta_t, \quad z_0 = x. \tag{16}$$

We note that the pathwise uniqueness holds for (16), and τ and $\widetilde{\tau}$ have the same law. In view of the Yamada–Watanabe theorem [9], we can show that the uniqueness in law holds for (16). Thus the two processes $\{Z_t, B_t\}$ and $\{z_t, \beta_t\}$ have the same law.

STEP 3. Fix $T>0$ and let $Q(\omega,\cdot)$ be the regular conditional probability for $E[\cdot\,|\,\mathcal{B}_T](\omega)$. We define the system $(W,\mathcal{B},\widetilde{P},\{\widetilde{\mathcal{B}}_t\};\widetilde{z},\widetilde{\beta})$ by

$$\widetilde{\beta}_t=\beta_{t+T}-\beta_T(\omega_0),\qquad \widetilde{z}_t=z_{t+T},\qquad \widetilde{\mathcal{B}}_t=\mathcal{B}_{t+T},\qquad \widetilde{P}(\cdot)=Q(\omega_0,\cdot),$$

for fixed $\omega_0\in\Omega_0$ with $P(\Omega_0)=1$. Let $A\in\widetilde{\mathcal{B}}_t,\ \xi\in\mathbb{R}^N$ and $u\geq t$. It is easy to see that $\{\beta_{u+T}-\beta_{t+T}\}$ is a Brownian motion under P, and thus

$$\begin{aligned}E^{\widetilde{P}}[e^{i(\xi,\widetilde{\beta}_u-\widetilde{\beta}_t)}1_A]&=E[e^{i(\xi,\widetilde{\beta}_u-\widetilde{\beta}_t)}1_A\,|\,\mathcal{B}_T]\\&=E[E[e^{i(\xi,\beta_{u+T}-\beta_{t+T})}\,|\,\mathcal{B}_{t+T}]1_A\,|\,\mathcal{B}_T]\\&=e^{-|\xi|^2(u-t)/2}\widetilde{P}(A),\end{aligned}$$

where $E^{\widetilde{P}}$ denotes the expectation with respect to $\widetilde{P}$. Hence $(W,\mathcal{B},\widetilde{P},\widetilde{\mathcal{B}}_t;\widetilde{\beta})$ is a Brownian motion. Further, taking the difference and using the change of variables, we have

$$\begin{aligned}\widetilde{z}_t&=x+\int_0^{t+T}b(z_u)1_{\{u<\widetilde{\tau}\}}\,du+\int_0^{t+T}\sigma(z_u)1_{\{u<\widetilde{\tau}\}}\,d\beta_u\\&=z_T+\int_T^{t+T}b(z_u)1_{\{u<\widetilde{\tau}\}}\,du+\int_T^{t+T}\sigma(z_u)1_{\{u<\widetilde{\tau}\}}\,d\beta_u\\&=z_T+\int_0^t b(\widetilde{z}_r)1_{\{r<\widetilde{\tau}\}}\,dr+\int_0^t\sigma(\widetilde{z}_r)1_{\{r<\widetilde{\tau}\}}\,d\widetilde{\beta}_r,\qquad P\text{-a.s.}\end{aligned}$$

Therefore

$$\widetilde{z}_t=z_T(\omega_0)+\int_0^t b(\widetilde{z}_r)1_{\{r<\widetilde{\tau}\}}\,dr+\int_0^t\sigma(\widetilde{z}_r)1_{\{r<\widetilde{\tau}\}}\,d\widetilde{\beta}_r,\qquad \widetilde{P}\text{-a.s.},$$

and

$$z_t=z_T(\omega_0)+\int_0^t b(z_r)1_{\{r<\widetilde{\tau}\}}\,dr+\int_0^t\sigma(z_r)1_{\{r<\widetilde{\tau}\}}\,d\beta_r,\qquad P_{z_T(\omega_0)}\text{-a.s.},$$

where P_x denotes the probability measure induced by $\{z_t\}$ with $z_0=x$. Since $\{\widetilde{z}_t,\widetilde{P}\}$ and $\{z_t,P_{z_T(\omega_0)}\}$ have the same law,

$$E^{\widetilde{P}}[f(\widetilde{z}_t)]=E_{z_T(\omega_0)}[f(z_t)]$$

for any bounded Borel function f. On the other hand,

$$E_x[f(z_{t+T})\,|\,\mathcal{B}_T](\omega_0)=E^{Q(\omega_0,\cdot)}[f(z_{t+T})]=E^{\widetilde{P}}[f(\widetilde{z}_t)].$$

This implies that $\{z_t,P_x\}$ has the Markov property:

$$E_x[f(z_{t+T})\,|\,\mathcal{B}_T]=E_{z_T}[f(z_t)],\quad \text{a.s.}$$

STEP 4. By Step 2 and Theorem 2.2, we have

$$v(x) = \mathcal{T}v(x) = E_x\Big[\int_0^\infty e^{-\alpha_\varepsilon t}\ \xi(z_t)\,dt\Big],$$

where

$$\xi(x) := \begin{cases} \frac{1}{\varepsilon}v(x) + g(x) & \text{if } x \in G, \\ \alpha_\varepsilon h(x) & \text{if } x \in \partial G. \end{cases}$$

By the Markov property,

$$\begin{aligned} e^{\alpha_\varepsilon t} E_x\Big[\int_t^\infty e^{-\alpha_\varepsilon s}\xi(z_s)\,ds\,\Big|\,\mathcal{B}_t\Big] &= E_x\Big[\int_0^\infty e^{-\alpha_\varepsilon s}\xi(z_{s+t})\,ds\,\Big|\,\mathcal{B}_t\Big] \\ &= E_{z_t}\Big[\int_0^\infty e^{-\alpha_\varepsilon s}\xi(z_s)\,ds\Big] \\ &= v(z_t), \qquad \text{a.s.} \end{aligned}$$

This implies that $\{e^{-\alpha_\varepsilon t}v(z_t) + \int_0^t e^{-\alpha_\varepsilon s}\xi(z_s)\,ds\}$ is a martingale and, by Step 2, so is $\{e^{-\alpha_\varepsilon t}v(Z_t) + \int_0^t e^{-\alpha_\varepsilon s}\xi(Z_s)\,ds\}$. Thus, by the optional sampling theorem, we deduce

$$v(x) = E\Big[\int_0^{\theta\wedge\tau} e^{-\alpha_\varepsilon s}\xi(Z_s)\,ds + e^{-\alpha_\varepsilon(\theta\wedge\tau)}v(Z_{\theta\wedge\tau})\Big],$$

which implies (15).

Proof of Theorem 2.3. By (7), it is clear that $v = h$ on ∂G. By Lemma 4.1 and a slight modification of the theory of viscosity solutions [6, Thm. 3.1, p. 220, Cor. 3.1, p. 223], we observe that v is a viscosity solution of

$$\alpha_\varepsilon v - \mathcal{L}v - \Big(\frac{1}{\varepsilon}\,v + g\Big) = 0 \quad \text{in } G. \tag{17}$$

This shows that v is a viscosity solution of (1).

For uniqueness, let v_i, $i = 1, 2$, be two viscosity solutions of (1). We apply Ishii's lemma to

$$\Psi(x, y) := v_1(x) - v_2(y) - \frac{k}{2}\,|x - y|^2$$

at its local maximum point $(x_k, y_k) \in G \times G$ for $k > 0$ to obtain symmetric matrices X, Y such that

$$\begin{aligned} &(k(x_k - y_k), X) \in \overline{J}^{2,+}v_1(x_k), \\ &(k(x_k - y_k), Y) \in \overline{J}^{2,-}v_2(y_k), \\ &\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le 3k\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \qquad I = \text{identity}, \end{aligned}$$

where

$$\overline{J}^{2,+}v_i(x) = \{(p,X) : \exists x_n \to x,\ \exists (p_n, X_n) \in J^{2,+}v_i(x_n),\\ (v_i(x_n), p_n, X_n) \to (v_i(x), p, X)\}, \quad i = 1,2,$$

and $J^{2,+}$ and $J^{2,-}$ are the second-order superjets and subjets. According to [6, Lemma 6.2, p. 240], we have

$$|\mathrm{tr}(\sigma(x)\sigma^\top(x)X) - \mathrm{tr}(\sigma(y)\sigma^\top(y)Y)| \le 3k\|\sigma(x) - \sigma(y)\|^2.$$

Thus, as in [3, Thm. 3.3] or [10, Thm. 3.7], we see that $v_1 = v_2$.

To prove (10), we note that (17) is independent of $\varepsilon > 0$. Let $\widehat{v} = \mathcal{T}_\delta v$ for any $\delta > 0$. Then $\widehat{v}$ is a viscosity solution of

$$\begin{cases} \alpha_\delta \widehat{v} - \mathcal{L}\widehat{v} - \left(\frac{1}{\delta}v + g\right) = 0 & \text{in } G, \\ \widehat{v} = h & \text{on } \partial G. \end{cases}$$

By uniqueness, we have $\widehat{v} = v$, so that v satisfies (7) for all $\varepsilon > 0$, Clearly,

$$E\Big[\int_0^\tau e^{-(\alpha+1/\varepsilon)t} g(X_t)\,dt + e^{-(\alpha+1/\varepsilon)\tau} h(X_\tau)\Big] \to u(x) \quad \text{as } \varepsilon \to \infty,$$

where u is as in (5). Thus, we conclude

$$v(x) = \lim_{\varepsilon\to\infty} E\Big[\int_0^\tau e^{-(\alpha+1/\varepsilon)t}\Big\{\frac{1}{\varepsilon}v + g\Big\}(X_t)\,dt + e^{-(\alpha+1/\varepsilon)\tau}h(X_\tau)\Big] = u(x).$$

5. Proof of Theorem 2.4. Let v_α denote the viscosity solution of (1) for $\alpha > 0$. Since

$$|v_\alpha(x) - v_{\alpha'}(x)| \le E\Big[\int_0^\tau |e^{-\alpha t} - e^{-\alpha' t}|\,dt\Big]\|g\| + E[|e^{-\alpha\tau} - e^{-\alpha'\tau}|]\|h\|,$$

we see by (12) that $\{v_\alpha\}$ is a Cauchy sequence, and v_α converges to $\overline{v}$ in $C(\overline{G})$ as $\alpha \to 0$. By the stability result for viscosity solutions [6, Lemma 6.2, p. 73], $\overline{v}$ is a viscosity solution of (8). Passing to the limit in (5) and (10), we deduce $\overline{v} = w$.

Now, in the same way as for [10, Thm. 3.9], we shall show the uniqueness for (8) under (11). Let w_i, $i = 1,2$, be two viscosity solutions of (8). Suppose that $\max_{\overline{G}}(w_1 - w_2) =: \vartheta > 0$. We choose $\varrho > 0$ such that

$$\varrho \max\{e^{\nu x_1} : x = (x_1, \ldots, x_N) \in \overline{G}\} \le \vartheta/2.$$

Define

$$\varPhi(x,y) = w_1(x) - w_2(y) - \frac{k}{2}|x-y|^2 + \varrho e^{\nu x_1}$$

for $k > 0$. Then there exists a maximum point $(x_k, y_k) \in \overline{G} \times \overline{G}$ of $\varPhi$ over $\overline{G} \times \overline{G}$. By compactness, extracting a subsequence, we may assume that

$(x_k, y_k) \to (\widehat{x}, \widehat{y}) \in \overline{G} \times \overline{G}$ as $k \to \infty$. Since $\Phi(x_k, y_k) \geq \max_{x \in \overline{G}}(w_1(x) - w_2(x) + \varrho e^{\nu x_1}) \geq \vartheta$, we have

$$|x_k - y_k|^2 \leq \frac{2}{k}\,(\max_{\overline{G}} w_1 - \min_{\overline{G}} w_2 + \vartheta/2),$$

which implies $\widehat{x} = \widehat{y}$. Moreover, $\Phi(x_k, y_k) \geq \Phi(x_k, x_k)$. Hence

$$\frac{k}{2}\,|x_k - y_k|^2 \leq w_2(x_k) - w_2(y_k) \ \to 0 \quad \text{ as } k \to \infty.$$

Thus

$$\Phi(\widehat{x}, \widehat{x}) = w_1(\widehat{x}) - w_2(\widehat{x}) + \varrho e^{\nu \widehat{x}_1} \geq \vartheta.$$

This implies that $\widehat{x} \in G$ and $(x_k, y_k) \in G \times G$ for sufficiently large k.

Ishii's lemma applied to $\Phi(x, y) = \widehat{w}(x) - w_2(y) - \frac{k}{2}|x-y|^2$, where $\widehat{w}(x) = w_1(x) + \varrho e^{\nu x_1}$, yields symmetric matrices X, Y such that

$$\begin{aligned}
&(k(x_k - y_k), X) \in \overline{J}^{2,+}\widehat{w}(x_k),\\
&(k(x_k - y_k), Y) \in \overline{J}^{2,-} w_2(y_k),\\
&\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3k \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.
\end{aligned}$$

Note that $Dx_1 = e_1, D^2 x_1 = I_1$ and

$$(\widehat{p}, \widehat{X}) := (k(x_k - y_k) - r(x_k)e_1, X - \nu r(x_k) I_1) \in \overline{J}^{2,+} w_1(x_k),$$

where $r(x) = \varrho \nu e^{\nu x_1}$. By the definition of viscosity solutions, we have

$$\begin{aligned}
&H_0(x_k, w_1(x_k), \widehat{p}, \widehat{X}) \leq 0,\\
&H_0(y_k, w_2(y_k), k(x_k - y_k), Y) \geq 0.
\end{aligned}$$

Thus

$$\begin{aligned}
0 &\leq H_0(y_k, w_2(y_k), k(x_k - y_k), Y) - H_0(x_k, w_1(x_k), \widehat{p}, \widehat{X})\\
&= -\frac{1}{2}\operatorname{tr}(\sigma(y_k)\sigma^\top(y_k) Y) - (b(y_k), k(x_k - y_k)) - g(y_k)\\
&\quad + \frac{1}{2}\operatorname{tr}(\sigma(x_k)\sigma^\top(x_k)\widehat{X}) + (b(x_k), \widehat{p}) + g(x_k)\\
&\leq \frac{3k}{2}\,\|\sigma(x_k) - \sigma(y_k)\|^2 - \frac{1}{2}\,\nu r(x_k)\operatorname{tr}(\sigma(x_k)\sigma^\top(x_k) I_1)\\
&\quad + k|b(x_k) - b(y_k)|\,|x_k - y_k| - r(x_k)(b(x_k), e_1) + g(x_k) - g(y_k).
\end{aligned}$$

Letting $k \to \infty$, we obtain $0 \leq -r(\widehat{x})\{\frac{1}{2}\nu \operatorname{tr}(\sigma(\widehat{x})\sigma^\top(\widehat{x}) I_1) + (b(\widehat{x}), e_1)\}$, which contradicts (11). The proof is complete.

6. Examples. We give some examples of ζ, λ of (6) to illustrate our theory.

Example 1. Let ϕ be a non-negative function in $C^2(\mathbb{R}^N)$ such that $\mathcal{L}\phi \geq \mu\phi$ for some constant $\mu > 0$. Let G be a bounded domain of the form

$$G = \{x : \phi(x) < 1\}, \quad \partial G = \{x : \phi(x) = 1\}.$$

Define $\zeta = 1 - \phi$. Then

$$\begin{aligned} -\alpha\zeta + \mathcal{L}\zeta + \lambda &\leq -\alpha + (\alpha - \mu)\phi + \lambda \\ &\leq -\alpha + \max(\alpha - \mu, 0) + \lambda \leq 0 \quad \text{in } G \end{aligned}$$

for a suitable choice of $\lambda > 0$.

Example 2. We consider

$$G = \{(x, y) : x^2 + y^2 < 1\} \subset \mathbb{R}^2$$

and

$$\sigma(x, y) = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \qquad b(x, y) = (x, y).$$

Take $\phi(x, y) = x^2 + y^2$. Then we have

$$\mathcal{L}\phi = 3\phi,$$

which satisfies the conditions of Example 1.

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