BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 54, No. 2, 2006

Fragmentation-Coagulation Models of Phytoplankton

by

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Summary. We present two new models of the dynamics of phytoplankton aggregates. The first one is an individual-based model. Passing to infinity with the number of individuals, we obtain an Eulerian model. This model describes the evolution of the density of the spatial-mass distribution of aggregates. We show the existence and uniqueness of solutions of the evolution equation.

1. Introduction. In [4] the authors built a model of the phytoplankton dynamics, where the individual is an aggregate—a group of phytoplankton cells living together. Aggregates are structured by their size, which changes due to three processes: growth caused by cell division, fragmentation and coagulation. The size distribution of aggregates satisfies the equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial m}[g(m)u] + \Phi u + Cu, \tag{1}$$

where m is the size of an aggregate, $g(m)$ is the growth rate, and Φ and C are the operators of fragmentation and coagulation, respectively. The authors proved the existence and uniqueness theorem for equation (1) and checked the long-time behaviour of the distribution of size for some special cases.

In the present paper we construct an individual-based model which is additionally spatially structured and contains a process of random movement of aggregates. Our aim is to show that the limit passage in the model, when the number of individuals goes to infinity whereas the mass of a single cell tends to zero, leads to a transport equation of type (1) with a diffusion term. In many papers such a limit is a stochastic process with values in the space of measures, called a superprocess (see [9, 13, 16, 2, 15]). The measures which

2000 *Mathematics Subject Classification*: Primary 60K35; Secondary 47J35, 92D40.
Key words and phrases: phytoplankton dynamics, measure-valued processes, fragmentation-coagulation equation.

are values of this superprocess describe the distribution of particles in space. In our model we also obtain a limit but it is deterministic. In fact, we derive the evolution of the distribution density according to the equation

$$\frac{\partial u}{\partial t} = D(m)\Delta_x u + \frac{\partial}{\partial m}[\lambda(m)u] + \Phi^* u + \overline{C}u, \tag{2}$$

where Φ^* and $\overline{C}$ are operators responsible for fragmentation and coagulation (for their form, see (29) and (30) in Section 6). Finally, we prove the existence and uniqueness of solutions of our equation.

The approach resembling ours was applied to a model of coagulation with diffusion by Norris [24] and in a different setting of interacting particle systems by Kolokoltsov [20]. Measure-valued limits of interacting particle systems leading to so-called generalized Smoluchowski equations were also considered in [6, 14]. Similar equations, but used in a different context, appear e.g. in [1, 5, 7, 25], while in [11] one can find a survey of coagulation equations. Other results concerning this subject can be found in [3, 22] and the papers quoted therein. For the biological models that use similar methods we also refer to [10, 21, 23, 30]. We exploit methods that were developed by Dawson (cf. [9]) and other probabilists working on superprocesses (see also [16, 17, 13]).

The scheme of this paper is as follows. In the next section we introduce our model, which is mathematically formulated in Section 3. Section 4 concerns the rescaling of the individual model and the limit passage; the proof of the convergence theorem is given in Section 5. In Section 6 we derive the evolution equation that describes the behaviour of the limit process, and we prove the existence and uniqueness theorem.

2. Individual-based model of phytoplankton cells. We construct an individual-based model of phytoplankton. In our model an individual is an aggregate that consists of indistinguishable cells with equal masses joined by some organic glue. Cells in the aggregate may die or divide into two daughter cells, which causes the decrease or growth of the aggregate. An aggregate may shatter into two smaller aggregates or die (sink or be eaten). Thus the whole situation is described by the following processes:

- A single cell in the aggregate may die in a unit of time with probability $\lambda_m(m)$ depending on the mass (number of cells) m of the aggregate or may divide into two new cells with probability $\lambda_b(m)$.
- A whole aggregate moves according to the ε-random walk—i.e. it skips by a vector of length ε in one of $2d$ directions (parallel to one of the axes, d is the dimension of the space) with probability $(1/\varepsilon^2)D(m)$ (where D is a coefficient depending on the mass).
- The aggregate may die in a unit of time with probability $\lambda_d(m)$.

- The aggregate of mass m may split in a unit of time with probability $\lambda_f(m)$ into two parts with masses $\overline{m}$ and $m-\overline{m}$ with probability $p^{(1)}(m,\overline{m})$ (where $\sum_{\overline{m}=1}^{m} p^{(1)}(m,\overline{m})=1$). We assume that after fragmentation both new aggregates appear at the same location as their parent.
- Two aggregates may join up with probability $k^{(1)}$ depending on their masses and locations, and on the state of the whole population. More precisely, let the rate of coagulation of the ith aggregate be $c(m_i)$. Then the probability that it joins the jth aggregate is $c(m_j)/\sum_{k=1}^{N} c(m_k)$ and it is modified by a distance-dependent coefficient $v(x_i-x_j)$, thus $k^{(1)}$ takes the form

$$k^{(1)}(m_i,m_j,x_i-x_j,\nu)=\frac{c(m_i)c(m_j)}{\sum_k c(m_k)}v(x_i-x_j), \tag{3}$$

where the sum in the denominator extends over all living individuals.

Our model of the coagulation process is essentially different from standard physical models (e.g. Smoluchowski [29]) where the probability of coagulation is proportional to the square of the number of particles. We consider the more biologically justifiable case, when the ability of coagulation of a single aggregate is not unbounded, but approximately constant. Ability of coagulation depends on the concentration of some organic glue (TEP) [8, 26]. This means that the probability of joining is a function of production of TEP by an aggregate, which depends on the mass of the aggregate.

It should be noted that the probability of coagulation of two aggregates:

1) is proportional to the ability of both aggregates to coagulate,
2) depends on the distance of the aggregates,
3) is symmetrical, i.e. it does not depend on the order of the aggregates.

It seems difficult to find another model of coagulation which has all the above features and, at the same time, has good mathematical properties.

3. Stochastic process describing the model. The state of our model is described by the vector $(k;x_1,m_1,\ldots,x_k,m_k)$, where k is the number of aggregates and x_i, m_i, for $i=1,\ldots,k$, denote, respectively, the location and mass of the ith aggregate. Since k (and so the number of variables) changes during evolution, and the order of pairs x_i,m_i is not important, we need a special state space. We use the set of measures

$$\mathcal{N}=\Big\{\sum_{i=1}^{k}\delta_{x_i,m_i} : k\in\mathbb{N},\ (x_i,m_i)\in\mathbb{R}^d\times\mathbb{N}\Big\},$$

i.e. we denote the aggregate of size m at position x by the Dirac delta

measure $\delta_{x,m}$ at $(x,m) \in \mathbb{R}^d \times \mathbb{N}$. The set $\mathcal{N}$ is a subspace of the space $\mathcal{M}$ of all finite Borel measures on $\mathbb{R}^d \times \mathbb{R}^+$ with the topology of weak convergence. Constrained by the nature of $\mathcal{N}$ (which is not even a Banach space), we use the formalism of $D([0,\infty),\mathcal{N})$ martingale problems. By $D([0,\infty),\mathcal{N})$ we denote the space of all càdlàg functions on $\mathcal{N}$, i.e. right continuous functions with left hand limits. Let us recall

DEFINITION 1. Let $B(\mathcal{N})$ be the space of measurable and bounded functions on $\mathcal{N}$ and let $\mathcal{L}$ be a linear operator defined on a subspace of $B(\mathcal{N})$ with values in $B(\mathcal{N})$. We say that a stochastic process $X(t)$ *solves the* $D([0,\infty),\mathcal{N})$ *martingale problem* for $\mathcal{L}$ and the initial state ν_0 if this process has $D([0,\infty),\mathcal{N})$-trajectories, $\mathrm{Prob}(X(0) = \nu_0) = 1$ and for every f from the domain of $\mathcal{L}$,

$$f(X(t)) - f(X(0)) - \int_0^t \mathcal{L}f(X(s))\,ds$$

is a martingale with respect to

$$\widehat{\mathcal{F}}_t = \sigma\Big(X(s), \int_0^s h(X(r))\,dr : s \le t,\ h \in B(\mathcal{N})\Big),$$

where $B(\mathcal{N})$ denotes the set of bounded Borel functions on $\mathcal{N}$.

Throughout this paper we omit the $D([0,\infty),E)$ and by the martingale problem we mean the $D([0,\infty),E)$ martingale problem. We will also speak of the $(\mathcal{L},\delta_{\nu_0})$-martingale problem, where δ_{ν_0} is the Dirac delta at the initial point. We will refer to $\mathcal{L}$ as the *generator* of the stochastic process. For an extensive guidebook to stochastic processes and martingale problems we refer to [17].

We formulate an individual version of the model in the setting of pure jump processes. We define a generator $\mathcal{L}^{(1)}$ as a jump operator

$$\begin{aligned}
(4)\quad & \mathcal{L}^{(1)}f(\nu) \\
&= \sum_{i=1}^{N}\Big[\frac{D(m_i)}{\varepsilon^2}\sum_{k=1}^{d}[f(\nu - \delta_{x_i,m_i} + \delta_{x_i+\varepsilon_k,\overline{m}}) + f(\nu - \delta_{x_i,m_i} + \delta_{x_i-\varepsilon_k,\overline{m}})] \\
&\quad + m_i\lambda_b(m_i)f(\nu - \delta_{x_i,m_i} + \delta_{x_i,m_i+1}) + m_i\lambda_m(m_i)f(\nu - \delta_{x_i,m_i} + \delta_{x_i,m_i-1}) \\
&\quad + \lambda_d(m_i)f(\nu - \delta_{x_i,m_i}) \\
&\quad + \lambda_f(m_i)\sum_{\overline{m}=1}^{m_i} p(m_i,\overline{m})f(\nu - \delta_{x_i,m_i} + \delta_{x_i,\overline{m}} + \delta_{x_i,m_i-\overline{m}})\Big] \\
&\quad + \sum_{i,j=1}^{N}\frac{c(m_i)c(m_j)v(x_i - x_j)}{\sum_k c(m_k)}f(\nu - \delta_{x_i,m_i} - \delta_{x_j,m_j} + \delta_{(x_i+x_j)/2,m_i+m_j}) \\
&\quad - \overline{\lambda}(\nu)f(\nu),
\end{aligned}$$

where

$$\overline{\lambda}(\nu) = \sum_{i=1}^{N} \left[\frac{2d\, D(m_i)}{\varepsilon^2} + m_i \lambda_b(m_i) + m_i \lambda_m(m_i) + \lambda_d(m_i) + \lambda_f(m_i) \right] + \sum_{i,j=1}^{N} \frac{c(m_i)c(m_j)v(x_i - x_j)}{\sum_k c(m_k)}$$

and ε_i is a d-dimensional vector with ε at the ith place and zeros elsewhere. In this section we assume that $\varepsilon = 1$, but in the next section we use a modified form of the operator $\mathcal{L}^{(1)}$ with $\varepsilon = 1/N$. We assume that the functions $D(m)$, $m\lambda_m(m)$, $m\lambda_b(m)$, $\lambda_f(m)$, $\lambda_d(m)$, $c(m)$ and $v(x - \overline{x})$ are bounded and continuous; moreover $c(m) > 0$ for all $m \in [0, \infty)$. Since the probability of extinction of the process is nonzero, we must also assume that for $\nu = 0$ we have $\mathcal{L}^{(1)} f(\nu) = 0$ (this means that after extinction the process remains in the state $\nu(t) = 0$).

PROPOSITION 1. *For any initial state $\nu_0 \in \mathcal{N}$ there exists a unique solution $\{\nu^{(1)}(t)\}_{t \geq 0}$ of the martingale problem for $(\mathcal{L}^{(1)}, \delta_{\nu_0})$.*

Proof. The operator $\mathcal{L}^{(1)}$ given by (4) is a jump operator with unbounded jump rate (for the theory of jump processes see [18] or [17]). To obtain the existence of the process generated by $\mathcal{L}^{(1)}$ we construct an approximating sequence of stochastic processes that are solutions of stopped martingale problems with operators with bounded jump rates. For any $n \in \mathbb{N}$ define $\mathcal{N}^{\leq n} = \{\nu \in \mathcal{N} : \langle 1, \nu \rangle \leq n\}$. Notice that the jump rate satisfies

$$\overline{\lambda}(\nu) \leq Cn \tag{5}$$

on $\mathcal{N}^{\leq n}$ with some constant C. That is why the solution of the stopped martingale problem for $(\mathcal{L}^{(1)}, \delta_{\nu_0}, \mathcal{N}^{\leq n})$ coincides with the solution of the stopped martingale problem with the operator bounded by Cn. Moreover, the stopping time

$$\tau_n = \inf\{t \geq 0 : \nu(t) \notin \mathcal{N}^{\leq n} \text{ or } \nu(t-) \notin \mathcal{N}^{\leq n}\}$$

is such that

$$\tau_n \geq \sum_{k=1}^{n} \frac{\Delta_k}{Cn} \xrightarrow{n \to \infty} \infty,$$

where $\{\Delta_n\}_{n \in \mathbb{N}}$ is a sequence of i.i.d. random variables, exponentially distributed with intensity one. We use Proposition 3.2 in Chapter 4 of [17] to end the proof. ∎

REMARK 1. Although the description of the process is now formulated in the language of jump processes, in the subsequent sections we will use a different setting. That is why we will write the operator (4) in a different form. Compare it with the approach used in papers on superprocesses (cf. [16],

[17]). Let $C^2_{\rm b}$ be the space of all bounded functions with bounded derivatives up to second order and $C^2_{\rm b,pos} = \{g : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R} : g \in C^2_{\rm b} \text{ and } \inf g > 0\}$. For a given $g \in C^2_{\rm b,pos}$ we define a function $F_g \in C_{\rm b}(\mathcal{M})$ (bounded and continuous on $\mathcal{M}$) by the formula $F_g(\nu) = \exp[\langle \log g, \nu\rangle]$. The generating operator $\mathcal{L}^{(1)}$ on the functions $F_g(\nu)$ has the form

$$\mathcal{L}^{(1)}F_g(\nu) = \exp[\langle \log g, \nu\rangle]\left[\left\langle \frac{L^{(1)}g + B^{(1)}g + \Phi^{(1)}g}{g}, \nu \right\rangle + C(g,\nu)\right], \tag{6}$$

where

$$L^{(1)}g(x,m) = \frac{D(m_i)}{\varepsilon^2}\sum_{k=1}^{d}[g(x+\varepsilon_k,m) + g(x-\varepsilon_k,m) - 2g(x,m)]$$

is the operator responsible for the spatial movement;

$$\begin{aligned} B^{(1)}g(x,m) = m[&\lambda_m(m)g(x,m-1) + \lambda_b(m)g(x,m+1) \\ &- (\lambda_m(m) + \lambda_b(m))g(x,m)] \end{aligned}$$

is the operator of birth and death inside aggregates;

$$\begin{aligned} \Phi^{(1)}g(x,m) = \lambda_f(m)&\Big[\sum_{\overline{m}=1}^{m} g(x,m-\overline{m})g(x,\overline{m})p^{(1)}(m,\overline{m}) - g(x,m)\Big] \\ &+ \lambda_d(m)(1-g(x,m)) \end{aligned}$$

is responsible for the fragmentation and death of whole aggregates; and

$$\begin{aligned} C(g,\nu) = &\int\int\int\int \frac{c(m)c(\overline{m})}{\int\int c(\overline{\overline{m}})\,\nu(d\overline{\overline{x}}\,d\overline{\overline{m}})} v(x-\overline{x}) \\ &\times \left[\frac{g((x+\overline{x})/2, m+\overline{m})}{g(x,m)g(\overline{x},\overline{m})} - 1\right]\nu(dx\,dm)\,\nu(d\overline{x}\,d\overline{m}) \end{aligned}$$

is the coagulation term.

4. The limit passage. Now, we construct a sequence of rescaled processes $\{\nu^{(N)}(t)\}_{t\geq 0}$, $N \in \mathbb{N}$, based on $\{\nu^{(1)}(t)\}_{t\geq 0}$ that will approximate some continuous model. Assume that the number of particles at time 0 increases to infinity as $N \to \infty$ and assume that the mass of each cell is $1/N$. The Nth process $\nu^{(N)}$ has values in the space

$$\mathcal{N}_N = \left\{\frac{1}{N}\sum_{i=1}^{k}\delta_{x_i,n_i/N} : k \in \mathbb{N},\ \left(x_i, \frac{n_i}{N}\right) \in \mathbb{R}^d \times \frac{1}{N}\mathbb{N}\right\}.$$

From now on we set $m = n/N$. The rescaling means that the process $N\nu^{(N)}$ behaves like $\nu^{(1)}$ with appropriate coefficients. Namely:

- The birth or death of a cell means the change of mass by a factor of $1/N$.
- We set the step of the random walk to be $\varepsilon = 1/N$.
- The result of fragmentation of an aggregate of size $m = n/N$ may have any mass in $1/N, 2/N, \dots, (n-1)/N$; so we assume that the coefficients $p^{(N)}$ are such that $\sum_{\overline{n}=1}^{n} p^{(N)}(n/N, \overline{n}/N) = 1$; moreover, we assume that there exists a continuous function $q : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $m, \overline{m} \in \mathbb{R}^+$ with $\overline{m} \le m$ and all sequences (n_N), $(\overline{n}_N)$ of positive integers such that $n_N/N \to m$ and $\overline{n}_N/N \to \overline{m}$ as $N \to \infty$ we have $Np^{(N)}(n_N/N, \overline{n}_N/N) \to q(m, \overline{m})$ and this convergence is uniform,
- the coagulation term remains unchanged.

Notice that the function q satisfies $\int_0^m q(m, \overline{m})\, d\overline{m} = 1$ for $m > 0$ and the probabilistic kernel $P(m, A) := \int_A q(m, \overline{m})\, d\overline{m}$ will describe the distribution of the size of the aggregates after fragmentation if the aggregate before fragmentation has size m.

So the operator governing this Nth approximation has the form

$$\mathcal{L}^{(N)} F_g(N\nu) = \exp\left[\langle \log g, N\nu\rangle\right] \times \left[\left\langle \frac{L^{(N)}g + B^{(N)}g + \Phi^{(N)}g}{g}, N\nu \right\rangle + C(g, N\nu)\right], \tag{7}$$

with $L^{(N)}$ equal to $L^{(1)}$ (at $m = n/N$ instead of n) and with

$$B^{(N)}g\left(x, \frac{n}{N}\right) = n\left(\lambda_m\left(\frac{n}{N}\right)g\left(x, \frac{n-1}{N}\right) + \lambda_b\left(\frac{n}{N}\right)g\left(x, \frac{n+1}{N}\right) - \left(\lambda_m\left(\frac{n}{N}\right) + \lambda_b\left(\frac{n}{N}\right)\right)g\left(x, \frac{n}{N}\right)\right),$$

$$\Phi^{(N)}g\left(x, \frac{n}{N}\right) = \lambda_f\left(\frac{n}{N}\right)\left[\sum_{\overline{n}=1}^{n} g\left(x, \frac{n-\overline{n}}{N}\right)g\left(x, \frac{\overline{n}}{N}\right)p^{(N)}\left(\frac{n}{N}, \frac{\overline{n}}{N}\right) - g\left(x, \frac{n}{N}\right)\right] + \lambda_d\left(\frac{n}{N}\right)\left(1 - g\left(x, \frac{n}{N}\right)\right).$$

The sequence of rescaled processes converges weakly to some measure-valued stochastic process (governed also by a martingale problem), but it turns out that the limit process describes a deterministic behaviour.

THEOREM 1. *Let $\nu^{(N)}(0) \xrightarrow{w} \nu_0$. The sequence of processes $\nu^{(N)}$ converges weakly in distribution to the deterministic measure-valued process*

uniquely determined by the equation

$$\langle h,\nu(t)\rangle - \langle h,\nu_0\rangle = \int_0^t [\langle (L+B+\Phi)h,\nu(s)\rangle + C(h,\nu(s))]\,ds \tag{8}$$

for all $h \in C_{\mathrm{b}}^2$, *with*

$$Lh(x,m) = D(m)\Delta_x h(x,m), \tag{9}$$

$$Bh(x,m) = m(\lambda_b(m) - \lambda_m(m))\,\frac{\partial}{\partial m}h(x,m), \tag{10}$$

$$\Phi h(x,m) = \lambda_f(m)\Big[2\int_0^m h(x,\overline{m})q(m,\overline{m})\,d\overline{m} - h(x,m)\Big] - \lambda_d(m)h(x,m), \tag{11}$$

$$C(h,\nu) = \iiiint \frac{c(m)c(\overline{m})v(x-\overline{x})}{\iint c(\overline{\overline{m}})\,\nu(d\overline{\overline{x}}\,d\overline{\overline{m}})} \times (h(x+\overline{x}/2, m+\overline{m}) - h(x,m) + h(\overline{x},\overline{m})) \times \nu(dx\,dm)\,\nu(d\overline{x}\,d\overline{m}). \tag{12}$$

The limit process ν has values in the space $\mathcal{M}$.

5. Proof of Theorem 1. The scheme of the proof is as follows. Firstly we define a new operator $\mathcal{L}$ (see (16)). Next we prove that if the process $\{\nu(t)\}_{t\geq 0}$ solves the $(\mathcal{L},\nu_0)$-martingale problem then it is a deterministic evolution of measure given by (8); moreover it is unique (i.e. there exists at most one solution of the $(\mathcal{L},\nu_0)$-martingale problem). Then we check that the sequence $\nu^{(N)}$ converges to the solution of this problem.

In the proof we will use the following auxiliary theorem of Kurtz and Ethier:

Proposition 2 ([17, Corollary 8.16, Chapter 4]). *Let* (E,r) *be a Polish space,* $A \subset C_{\mathrm{b}}(E)\times C_{\mathrm{b}}(E)$ *be an operator (possibly multivalued), and* P_0 *be a probability measure on* E. *Suppose that the martingale problem for* (A,P_0) *has at most one solution. For* $N = 1,2,\ldots$, *suppose that* Y_N *is a progressive Markov process in a Polish space* E_N *corresponding to a measurable contraction semigroup with generator* A_N *and* $\eta_N : E_N \to E$ *is Borel measurable. Let* $X_N = \eta_N \circ Y_N$. *Assume that: the distribution of* $X_N(0)$ *converges weakly to* P_0 *as* $N\to\infty$, X_N *satisfies the compact containment condition, and the closure of the linear span of* $\mathcal{D}(A)$ *contains an algebra that separates points. If, moreover, for all* $(f,g)\in A$ *and* $T>0$ *there exist sequences of functions* $(f_N,g_N)\in A_N$ *and sets* $\Gamma_N \subset E_N$ *such that*:

(i) $\lim_{N\to\infty} \mathrm{Prob}(\{Y_N(t)\in\Gamma_N,\ 0\leq t\leq T\}) = 1$,

(ii) $\sup_N \|f_N\| < \infty$,

(iii) $$\lim_{N\to\infty}\sup_{y\in\Gamma_N}|f\circ\eta_N(y)-f_N(y)|=\lim_{N\to\infty}\sup_{y\in\Gamma_N}|g\circ\eta_N(y)-g_N(y)|=0,$$

then there exists a solution X of the (A,P_0) martingale problem and X_N converges weakly in distribution to X. Here $\|\cdot\|$ is the maximum norm in $C_{\mathrm{b}}(E)$.

REMARK 2. By the *compact containment condition* we mean that for every $\varepsilon>0$ and $T>0$ there is a compact set $\Gamma_{\varepsilon,T}$ such that

$$\inf_N \mathrm{Prob}\{X_N(t)\in\Gamma_{\varepsilon,T},\ 0\le t\le T\}\ge 1-\varepsilon.$$

Moreover we need some lemmas:

LEMMA 1. *If the process $\{\nu(t)\}_{t\ge0}$ solves the $(\mathcal{L}^{(N)},\delta_{\nu_0})$-martingale problem (where $\mathcal{L}^{(N)}$ is given by (7)) then*

$$\mathrm{Prob}\Big(\sup_{0\le t\le T}\langle 1,\nu(t)\rangle\ge a\Big)\le\frac{\langle 1,\nu_0\rangle}{a}\exp[T(\|\lambda_f-\lambda_d\|+\|cv\|)]. \tag{13}$$

The proof is based on that of Lemma 4.1 in [17, Chapter 9]. Although it requires some calculation, it is not very interesting, so we omit it here.

LEMMA 2. *The operator $L+B+\Phi$ generates a strongly continuous semigroup on $C_0(\mathbb{R}^d\times\mathbb{R}^+)$.*

Proof. The operator $L+B$ generates a strongly continuous semigroup on $C_0(\mathbb{R}^d\times\mathbb{R}^+)$ (cf. [28]) and Φ is a bounded operator on $C_{\mathrm{b}}(\mathbb{R}^d\times\mathbb{R}^+)$, so the Phillips perturbation theorem [12] gives the result. ∎

Let us write $C(h,\nu)$ as $\langle h,\widehat{C}\nu\rangle$, where

$$\widehat{C}\nu(A)=\iiiint\frac{c(m)c(\overline{m})v(x-\overline{x})}{\iint c(\overline{\overline{m}})\,\nu(d\overline{\overline{x}}\,d\overline{\overline{m}})}\times[\mathbf{1}_A((x+\overline{x})/2,m+\overline{m})-\mathbf{1}_A(x,m)-\mathbf{1}_A(\overline{x},\overline{m})]\,\nu(dx\,dm)\,\nu(d\overline{x}\,d\overline{m}).$$

One can prove that for every measure $\nu\in\mathcal{M}$ we have $\widehat{C}\nu\in\mathcal{M}$.

LEMMA 3. *Let $h\in C(\mathbb{R}^d\times\mathbb{R}^+)$ be such that $\|h\|\le1$ and let $\nu,\mu\in\mathcal{M}$. Then*

$$|\langle h,\widehat{C}\nu-\widehat{C}\mu\rangle|\le\|\mu-\nu\|_{\mathrm{TV}}. \tag{14}$$

Here $\|\nu\|_{\mathrm{TV}}$ denotes the total variation norm of the measure ν (cf. e.g. [17]).

Proof. Set $\alpha(\nu)=\iint c(m)\,\nu(dx\,dm)$ and

$$\widehat{h}=h\left(\frac{x+\overline{x}}{2},m+\overline{m}\right)-h(x,m)-h(\overline{x},\overline{m}).$$

Fix $\mu \in \mathcal{M} \setminus \{0\}$ and let $\varepsilon = \|c\|^{-1}\alpha(\mu)$. Then $\alpha(\nu) \le 2\alpha(\mu)$ for $\|\nu-\mu\|_{\mathrm{TV}} \le \varepsilon$. Moreover

$$\begin{aligned}
&|\langle h, \widehat{C}\nu - \widehat{C}\mu\rangle| \\
&= \left| \frac{(\alpha(\mu)-\alpha(\nu))}{\alpha(\nu)\alpha(\mu)} \iiiint c(m)c(\overline{m})v(x-\overline{x})\widehat{h}\,\nu(dx\,dm)\,\nu(d\overline{x}\,d\overline{m}) \right. \\
&\quad \left. + \frac{1}{\alpha(\mu)} \iiiint c(m)c(\overline{m})v(x-\overline{x})\widehat{h}\,(\nu+\mu)(dx\,dm)\,(\nu-\mu)(d\overline{x}\,d\overline{m}) \right| \\
&\le 3\|v\| \frac{|\alpha(\mu)-\alpha(\nu)|}{\alpha(\nu)\alpha(\mu)}\,\alpha(\nu)\alpha(\nu) \\
&\quad + \frac{3\|v\|\,\|c\|\alpha(\mu+\nu)}{\alpha(\mu)} \iint f_{h,\mu,\nu}(\overline{x},\overline{m})\,(\nu-\mu)(d\overline{x}\,d\overline{m}),
\end{aligned}$$

where

$$f_{h,\mu,\nu}(\overline{x},\overline{m}) = \iint \frac{c(m)c(\overline{m})v(x-\overline{x})}{3\|v\|\,\|c\|\alpha(\mu+\nu)}\,\widehat{h}(\nu+\mu)(dx\,dm).$$

Notice that $f_{h,\mu,\nu}$ is bounded by 1. Therefore $\int f_{h,\mu,\nu}\,d(\mu-\nu) \le \|\mu-\nu\|_{\mathrm{TV}}$. Thus, going on with the above calculations, for $\|\nu-\mu\|_{\mathrm{TV}} \le \varepsilon$ we have

$$\begin{aligned}
|\langle h, \widehat{C}\nu - \widehat{C}\mu\rangle| &\le 3\|v\|\frac{2\alpha(\mu)}{\alpha(\mu)}\|c\|\,\|\nu-\mu\|_{\mathrm{TV}} \\
&\quad + 3\|v\|\,\|c\|\frac{3\alpha(\mu)}{\alpha(\mu)}\|\nu-\mu\|_{\mathrm{TV}} \\
&\le 15\|v\|\,\|c\|\,\|\nu-\mu\|_{\mathrm{TV}}.
\end{aligned} \tag{15}$$

Let us now take arbitrary measures $\mu, \nu \in \mathcal{M}\setminus\{0\}$. Let $\mu_t = (1-t)\mu + t\nu$ and $\overline{\varepsilon} = \|c\|^{-1}\inf_{0\le t\le 1}\alpha(\mu_t)$. Choose an n such that $\|\nu-\mu\|_{\mathrm{TV}}/n < \overline{\varepsilon}$. From inequality (15) it follows that

$$|\langle h, \widehat{C}\mu_{i/n} - \widehat{C}\mu_{(i-1)/n}\rangle| \le 15\|v\|\,\|c\|\,\|\mu_{i/n} - \mu_{(i-1)/n}\|_{\mathrm{TV}}$$

for $i = 1, \dots, n$. Therefore

$$\begin{aligned}
|\langle h, \widehat{C}\nu - \widehat{C}\mu\rangle| &\le \sum_{i=1}^{n} |\langle h, \widehat{C}\mu_{i/n} - \widehat{C}\mu_{(i-1)/n}\rangle| \\
&\le \sum_{i=1}^{n} 15\|v\|\,\|c\|\,\|\mu_{i/n} - \mu_{(i-1)/n}\|_{\mathrm{TV}} \\
&\le 15\|v\|\,\|c\|\,\|\nu-\mu\|_{\mathrm{TV}}. \ \blacksquare
\end{aligned}$$

Proof of Theorem 1. Define the operator

$$\mathcal{L}[\exp[-\langle h,\nu\rangle]] = \exp[-\langle h,\nu\rangle][\langle -Lh - Bh - \Phi h, \nu\rangle + C(h,\nu)] \tag{16}$$

with L, B, Φ and C given by (9)–(12) in Section 4.

Assume that $\{\nu(t)\}_{t\geq 0}$ solves the $(\mathcal{L},\nu_0)$-martingale problem. This means that

$$\mathrm{E}\Big[e^{-\langle h,\nu(t)\rangle} - e^{-\langle h,\nu(s)\rangle} - \int_s^t \mathcal{L}[e^{-\langle h,\nu(r)\rangle}]\,dr \;\Big|\; \mathcal{F}_s\Big] = 0 \tag{17}$$

for all $h \in C^2_{\mathrm{b}}$. Take $h = \theta\overline{h}$ and differentiate with respect to θ; setting now $\theta = 0$ we get

$$\mathrm{E}\Big[\langle\overline{h},\nu(s)\rangle - \langle\overline{h},\nu(t)\rangle + \int_s^t [\langle (L+B+\Phi)\overline{h},\nu(r)\rangle + C(h,\nu(r))]\,dr \;\Big|\; \mathcal{F}_s\Big] = 0.$$

That means that

$$\begin{aligned}\langle h,\nu(t)\rangle &= \langle h,\nu_0\rangle \\ &\quad + \int_0^t [\langle (L+B+\Phi)h,\nu(s)\rangle + C(h,\nu(s))]\,ds + M(t),\end{aligned} \tag{18}$$

where $M(t)$ is a P_{ν_0}-martingale. From the Itô formula (see e.g. [19]) we have

$$\begin{aligned}e^{-\langle h,\nu(t)\rangle} - e^{-\langle h,\nu_0\rangle} - \int_0^t e^{-\langle h,\nu(s)\rangle}[\langle Lh + Bh + \Phi h,\nu(r)\rangle + C(h,\nu(r))]\,dr \\ = \int_0^t e^{-\langle h,\nu(s)\rangle}\,dM(s) + \frac{1}{2}\int_0^t e^{-\langle h,\nu(s)\rangle}\,d\langle M\rangle(s),\end{aligned}$$

where $\langle M\rangle$ is the quadratic variational process of M. We know that the left hand side is a martingale with mean value 0 and the first integral on the right hand side has the same property. Therefore the integral

$$\int_0^t e^{-\langle h,\nu(s)\rangle}\,d\langle M\rangle(s)$$

is also a martingale with mean value 0. But it is the integral of a nonnegative, nontrivial function with respect to a quadratic variational process, which is increasing. Thus, since its mean value is 0, we know that $\langle M\rangle(s) = 0$. This means that also $M(s) = 0$. Therefore (18) implies that $\nu(t)$ satisfies (8) for all $h \in C^2_{\mathrm{b}}$.

Now we prove that this solution is unique. Assume that the (nonrandom) right continuous family $\{\nu(t)\}$ of measures satisfies (8) and $\nu(0) = \nu_0$. It follows from (8) that $\langle h,\nu(t)\rangle$ is differentiable as a function of time, therefore this equation can be rewritten as

$$\forall_{h\in C^2_{\mathrm{b}}} \quad \frac{d}{dt}\langle h,\nu(t)\rangle = \langle (L+B+\Phi)h,\nu(s)\rangle + \langle h,\widehat{C}\nu(s)\rangle. \tag{19}$$

Fix $h_0 \in C_0^2$ such that $\|h\| \leq 1$. By Lemma 2 the evolution equation

$$\begin{cases} \dfrac{dh}{dt} = (L + B + \Phi)h, \\ h(0) = h_0, \end{cases} \tag{20}$$

has a unique solution. Notice that this solution satisfies $\|h(t)\| \leq \|h_0\| e^{at} \leq e^{at}$ for some $a > 0$ that is independent of h_0. Since $h_0 \in \mathcal{D}(L+B+\Phi) \subset C_0^2$, we also have $h(t) \in C_0^2$ for all $t > 0$. Thus for any $\nu \in \mathcal{M}$ we can write

$$\frac{d}{dt}\langle h(t), \nu\rangle = \langle (L + B + \Phi)h(t), \nu\rangle. \tag{21}$$

Using (19) and (21) we can write

$$\frac{\partial}{\partial s}\langle h(t-s), \nu(s)\rangle = \langle h(t-s), \widehat{C}\nu(s)\rangle. \tag{22}$$

Integrating both sides of (22) with respect to s we get

$$\langle h_0, \nu(t)\rangle = \langle h(t), \nu_0\rangle - \int_0^t \langle h(t-s), \widehat{C}\nu(s)\rangle \, ds. \tag{23}$$

Now assume that $\{\nu(t)\}_{t\geq 0}$ and $\{\mu(t)\}_{t\geq 0}$ satisfy (8) with the same initial condition $\nu(0) = \mu(0) = \nu_0$. Then, using the above calculations and Lemma 3, we have

$$\begin{aligned} \langle h_0, \nu(t) - \mu(t)\rangle &= \int_0^t \langle h(t-s), \widehat{C}\mu(s) - \widehat{C}\nu(s)\rangle \, ds \\ &= e^{at}\int_0^t \langle h(t-s)e^{-at}, \widehat{C}\mu(s) - \widehat{C}\nu(s)\rangle \, ds \\ &\leq e^{at}\int_0^t \|\mu(s) - \nu(s)\|_{\mathrm{TV}} \, ds. \end{aligned}$$

Recall that this is valid for any $h_0 \in C_0^2$. Hence

$$\|\nu(t) - \mu(t)\|_{\mathrm{TV}} \leq e^{at}\int_0^t \|\mu(s) - \nu(s)\|_{\mathrm{TV}} \, ds, \tag{24}$$

and from the Gromwall inequality $\mu(t) = \nu(t)$ for all $t \geq 0$.

Our aim now is to prove that the sequence of the processes $\nu^{(N)}$ converges to a solution of the $(\mathcal{L}, \nu_0)$-martingale problem. In order to do it, we check the assumptions of Proposition 2. We have already checked that this martingale problem has at most one solution. To prove the compact containment of the sequence we will change our space a little: namely we replace $\mathbb{R}^d \times \mathbb{R}^+$ by its compactification $\widehat{E} = (\mathbb{R}^d \times \mathbb{R}^+) \cup \{\infty\}$ and so the processes X_N take values in the space $\widehat{\mathcal{M}} = \mathcal{M}(\widehat{E})$ of all finite Borel measures on the compactification

of $\mathbb{R}^d \times \mathbb{R}^+$. Observe how our situation fits into the frame of Proposition 2: In our case $E = \widehat{\mathcal{M}}$ and we can consider $E_N = \mathcal{N}_N$ as subsets of $\widehat{E}$ so that $X_N = \mu^{(N)}$ coincides with Y_N and η_N is just the identity. For the compact containment condition we use the fact that the set $\{\mu : \langle 1, \mu\rangle \le M\}$ is compact in $\widehat{E}$. So by Lemma 1,

$$\mathrm{Prob}(\nu_N(t) \in \{\mu : \langle 1, \mu\rangle \le M\} \text{ for } 0 \le t \le T) \ge 1 - \frac{\langle 1, \nu_0\rangle}{a}\, e^{T(\|\lambda_f - \lambda_d\| + \|cv\|)},$$

which proves the compact containment.

The family of functions $\{e^{-\langle h,\nu\rangle} : h \in C^2_{\mathrm{b}}\}$ is rich enough to separate points in $\widehat{\mathcal{M}}$. Fix $h \in C_{\mathrm{b,pos}}$. We now construct functions F_N such that F_N converges to $\exp[-\langle h, \nu\rangle]$ and $\mathcal{L}^{(N)}F_N$ converges to $\mathcal{L}\exp[-\langle h, \nu\rangle]$. Namely, let $F_N(\nu) = \exp[\langle N\log(1 - h/N), \nu\rangle]$ (for N sufficiently large $1 - h/N > 0$) and $\Gamma_N = \mathcal{N}_N$. Obviously $X_N(t) \in \Gamma_N$ for all $t \ge 0$ and F_N are uniformly bounded. Then

$$\begin{aligned}
&\sup_{\nu\in\mathcal{N}_N} |F_N(\nu) - \exp[-\langle h, \nu\rangle]| \\
&\qquad \le \sup_{\nu\in\mathcal{N}_N} \exp[-\langle 1, \nu\rangle \inf h]\,|\langle h + N\log(1 - h/N), \nu\rangle| \\
&\qquad \le \sup_{\nu\in\mathcal{N}_N} \exp[-\langle 1, \nu\rangle \inf h]\langle 1, \nu\rangle \|h - \log(1 - h/N)^{-N}\| \\
&\qquad \le \sup_{\nu\in\mathcal{N}_N} C\|h - \log(1 - h/N)^{-N}\| \xrightarrow{N\to\infty} 0,
\end{aligned}$$

where C is some constant. Similar calculations show that

$$\sup_{\nu\in\mathcal{N}_N} |\mathcal{L}^{(N)}F_N(\nu) - \mathcal{L}[\exp(-\langle h, \nu\rangle)]| \xrightarrow{N\to\infty} 0, \tag{25}$$

which completes the proof. ∎

6. Equation on densities. Consider the solution $\nu(t)$ of (8) and assume that it is absolutely continuous with respect to the Lebesgue measure, i.e. $\nu(t)(dx\,dm) = u(t, x, m)\,dx\,dm$.

REMARK 3. Since (8) implies uniqueness (cf. proof of Theorem 1) and Theorem 2 will give us an absolutely continuous solution for any initial density, it suffices to assume that the initial measure ν_0 in (8) is absolutely continuous.

Then by simple calculations one can check that (8) is the mild version of the equation

$$\frac{\partial u(t, x, m)}{\partial t} = L^*u(t, x, m) + B^*u(t, x, m) + \Phi^*u(t, x, m) + \overline{C}u(t, x, m), \tag{26}$$

where

$$L^*f(x,m) = D(m)\Delta_x f(x,m), \tag{27}$$

$$B^*f(x,m) = \frac{\partial}{\partial m}[m(\lambda_m(m) - \lambda_b(m))f(x,m)], \tag{28}$$

$$\begin{aligned}\Phi^*f(x,m) &= \lambda_f(m)\Big[2\int_m^\infty f(x,\overline{m})q(\overline{m},m)\,d\overline{m} - f(x,m)\Big] \\ &\quad - \lambda_d(m)f(x,m),\end{aligned} \tag{29}$$

$$\begin{aligned}\overline{C}f(x,m) &= \int_{\mathbb{R}^d}\int_0^m 2^d \frac{c(m-\overline{m})c(\overline{m})v(2(x-\overline{x}))}{\int\int c(\overline{\overline{m}})f(\overline{\overline{x}},\overline{\overline{m}})\,d\overline{\overline{x}}\,d\overline{\overline{m}}} \\ &\quad \times f(2x-\overline{x}, m-\overline{m})f(\overline{x},\overline{m})\,d\overline{m}\,d\overline{x} \\ &\quad - \int_{\mathbb{R}^d}\int_0^\infty 2\frac{c(m)c(\overline{m})v(x-\overline{x})}{\int\int c(\overline{\overline{m}})f(\overline{\overline{x}},\overline{\overline{m}})\,d\overline{\overline{x}}\,d\overline{\overline{m}}} \\ &\quad \times f(x,m)f(\overline{x},\overline{m})\,d\overline{m}\,d\overline{x},\end{aligned} \tag{30}$$

where Δ_x is the Laplace operator with respect to the spatial variable x.

THEOREM 2. *Let $D(m) > 0$ and $c(m) > 0$ for $m \geq 0$. For any $u_0 \in L^1_+(\mathbb{R}^d \times \mathbb{R}^+)$ there exists a unique solution $u(t,x,m) \in L^1_+(\mathbb{R}^d \times \mathbb{R}^+)$ of (26) such that $u(0,x,m) = u_0(x,m)$.*

Proof. The operator $L^* + B^*$ generates a strongly continuous Markov semigroup of linear operators on $X = L^1(\mathbb{R}^d \times \mathbb{R}^+)$, which can be written in the form

$$P(t)\varphi(x,m) = \begin{cases} \displaystyle\int_{\mathbb{R}^d} \kappa^+(\pi_{-t}m, m, x, \overline{x})\varphi(\overline{x}, \pi_{-t}m)\,\frac{\partial}{\partial m}\pi_{-t}m\,d\overline{x} & \text{for } \lambda(m) > 0, \\ \displaystyle\int_{\mathbb{R}^d} \kappa^0(t; m; x, \overline{x})\varphi(\overline{x}, m)\,d\overline{x} & \text{for } \lambda(m) = 0, \\ \displaystyle\int_{\mathbb{R}^d} \kappa^-(\pi_{-t}m, m, x, \overline{x})\varphi(\overline{x}, \pi_{-t}m)\,\frac{\partial}{\partial m}\pi_{-t}m\,d\overline{x} & \text{for } \lambda(m) < 0. \end{cases} \tag{31}$$

Notice that, because λ is continuous, we can divide the half-line $\mathbb{R}^+$ into intervals where $\lambda < 0$ or $\lambda > 0$ and places—single points or intervals—where $\lambda = 0$. The term $\pi_t m$ is the solution of

$$\frac{d}{dt}\pi_t m = \lambda(\pi_t m)$$

with $\pi_0 m = m$ and $\lambda(m) = m[\lambda_m(m) - \lambda_b(m)]$. The functions $\kappa^{+/0/-}$ are

defined by

$$\kappa^+(\tau_0,\tau_1,x,\overline{x}) = \frac{1}{\sqrt{4\pi\int_{\tau_0}^{\tau_1}\frac{D(\tau)}{\lambda(\tau)}\,d\tau}^{\,d}}\,\exp\!\left(-\frac{|x-\overline{x}|^2}{4\int_{\tau_0}^{\tau_1}\frac{D(\tau)}{\lambda(\tau)}\,d\tau}\right),$$

$$\kappa^0(t;m;x,\overline{x}) = \frac{1}{\sqrt{4\pi D(m)\,t}^{\,d}}\,\exp\!\left(-\frac{|x-\overline{x}|^2}{4D(m)t}\right),$$

$$\kappa^-(\tau_0,\tau_1,x,\overline{x}) = \frac{1}{\sqrt{4\pi\int_{\tau_1}^{\tau_0}\frac{D(\tau)}{-\lambda(\tau)}\,d\tau}^{\,d}}\,\exp\!\left(-\frac{|x-\overline{x}|^2}{4\int_{\tau_1}^{\tau_0}\frac{D(\tau)}{-\lambda(\tau)}\,d\tau}\right).$$

The terms κ^+, κ^0, and κ^- have the following natural interpretation. Functions κ^+ and κ^- are fundamental solutions of the non-autonomous, respectively, forward and backward heat equation

$$\lambda(\tau)\,\frac{\partial}{\partial\tau}u(\tau,x) = D(\tau)\Delta u(\tau,x),$$

whereas κ^0 is the fundamental solution of the autonomous heat equation with constant diffusion $D(m)$.

Since λ_f, λ_d and p are bounded, Φ^* is a bounded linear operator on X. Thus, by the Phillips perturbation theorem, the operator $L^* + B^* + \Phi^*$ generates a strongly continuous semigroup of bounded positive operators on X.

One can prove that the operator $\overline{C}$ is Lipschitzian on $X_+ = L^1_+(\mathbb{R}^d\times\mathbb{R}^+)$. This proof is based on that of Theorem 1 in [4] and is similar to the proof of Lemma 3. The rest of the proof of the existence of the semigroup generated by (26) is a simple consequence of the method of variation of parameters (see e.g. [27]). ∎

Remark 4. We should underline that (26) is a fragmentation-coagulation equation, which, according to Theorem 2, has a unique solution that exists for all positive time. This feature distinguishes (26) from physical coagulation equations which do not have global solutions. This surprising property of (26) results from the special form of the coagulation term (30), which is homogeneous with respect to f.

Acknowledgements. This research was partially supported by the State Committee for Scientific Research (Poland) Grant No. 2 P03A 031 25 and by the 6 EU programme SPADE 2. R.R. also thanks The International Centre for Mathematical Sciences (ICMS) in Edinburgh for financial support of his participation in the ICMS Workshop on Coagulation-Fragmentation Processes, 2005.

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Received May 5, 2006;
received in final form July 11, 2006 (7523)