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The Fedoryuk Condition and the Łojasiewicz Exponent near a Fibre of a Polynomial

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Summary. We give a description of the set of points for which the Fedoryuk condition fails in terms of the Łojasiewicz exponent at infinity near a fibre of a polynomial.

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial and let $\operatorname{grad} f : \mathbb{C}^n \to \mathbb{C}^n$ be its gradient. We take $\lambda \in \mathbb{C}$ and consider the *Fedoryuk condition* (see [3]) at λ:

$$(1) \qquad \exists_{\delta,\eta,\,R>0}\ \forall_{z\in\mathbb{C}^n} \quad (|z| > R \wedge |f(z)-\lambda| < \delta \Rightarrow |\operatorname{grad} f(z)| > \eta),$$

where $|\cdot|$ is the policylindric norm in $\mathbb{C}^n$. We see that this condition restricts the asymptotic behaviour of $\operatorname{grad} f(z)$ as $|z| \to \infty$ and $f(z) \to \lambda$. One can prove that if condition (1) is fulfilled at λ and λ is not a critical value of f then f is topologically trivial over a small neighbourhood of λ in $\mathbb{C}$ (cf. [5], [7]).

We denote by $\widetilde{K}_\infty(f)$ the set of $\lambda \in \mathbb{C}$ at which the Fedoryuk condition fails. Recently Spodzieja [9] has shown another application of the Fedoryuk condition. He proved that the set $\widetilde{K}_\infty(f)$ is finite if and only if $\operatorname{grad} f$ and f are separated at infinity, which means that there exist $C, R > 0$ and $q \in \mathbb{R}$ such that if $|f(z)| \geq R$ then $|\operatorname{grad} f(z)| \geq C|f(z)|^q$.

Now we define the *Łojasiewicz exponent at infinity* of $\operatorname{grad} f$ near a fibre $f^{-1}(\lambda)$, where $\lambda \in \mathbb{C}$, by

$$(2) \qquad \mathcal{L}_{\infty,\lambda}(f) = \inf_{\varphi} \frac{\deg \operatorname{grad} f \circ \varphi}{\deg \varphi},$$

where $\varphi : \{t \in \mathbb{C} : |t| > r\} \to \mathbb{C}^n$, $r > 0$, is the sum of a Laurent series of

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the form

$$\varphi(t) = a_p t^p + \cdots + a_1 t + \sum_{k\geq 0} a_{-k} t^{-k}, \qquad a_i \in \mathbb{C}^n,\ p \in \mathbb{N}, \tag{3}$$

such that $\deg\varphi > 0$ and $\deg(f\circ\varphi-\lambda) < 0$. Here $\deg\varphi(t) = \sup\{i : a_i \neq 0\}$ if $\varphi \neq 0$, and $\deg 0 = -\infty$. We shall call mappings of the form (3) *meromorphic at infinity.*

The exponent $\mathcal{L}_{\infty,\lambda}(f)$ was defined by Chądzyński and Krasiński in [2].

Chądzyński and Krasiński [2] for $n = 2$ and Skalski [8] for arbitrary n proved that this definition is equivalent to the following definition introduced by Ha [4]:

$$\mathcal{L}_{\infty,\lambda}(f) = \lim_{\delta\to 0^+} \mathcal{L}_\infty(\operatorname{grad} f | f^{-1}(D_\delta)),$$

where $D_\delta = \{\xi \in \mathbb{C} : |\xi-\lambda| < \delta\}$ and $\mathcal{L}_\infty(\operatorname{grad} f|f^{-1}(D_\delta))$ is the Łojasiewicz exponent at infinity of the mapping $\operatorname{grad} f$ on the set $f^{-1}(D_\delta)$.

Our aim is to prove the following characterization of the set $\widetilde{K}_\infty(f)$ in terms of the Łojasiewicz exponent near fibre.

THEOREM 1. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial. Then $\widetilde{K}_\infty(f) = \{\lambda \in \mathbb{C} : \mathcal{L}_{\infty,\lambda}(f) < 0\}$.*

The key in the proof is the Curve Selection Lemma at infinity. We begin with a definition. A mapping $\varphi : (r, +\infty) \to \mathbb{R}^n$, $r > 0$, is called *meromorphic at* $+\infty$ if φ is the sum of a Laurent series of the form (3), where $a_i \in \mathbb{R}^n$. We define $\deg\varphi$ as above.

LEMMA 1 (Curve Selection Lemma at infinity; [6, Lemma 2], [1, Lemma 1]). *If $X \subset \mathbb{R}^n$ is an unbounded semialgebraic set*, *then there exists a curve $\varphi : (r, +\infty) \to X$ meromorphic at $+\infty$ such that* $\deg\varphi > 0$.

Proof of Theorem 1. The inclusion $\supset$ is clear.

Let $B = \{z \in \mathbb{C}^n : \|z\| < 1\}$, where $\|\cdot\|$ denotes the Euclidan norm in $\mathbb{C}^n$. The mapping

$$B \ni z \mapsto H(z) = \frac{z}{1 - \|z\|^2}$$

is a rational homeomorphism of B onto $\mathbb{C}^n$. Take any $\lambda \in \widetilde{K}_\infty(f)$. By the definition of H the set

$$X = \{(z, R) \in B \times \mathbb{R} : \|H(z)\| > R \wedge |f(H(z)) - \lambda| < 1/R \wedge \|\operatorname{grad} f(H(z))\| < 1/R\}$$

is unbounded and semialgebraic. Hence, by the Curve Selection Lemma at infinity there exists a mapping $\widetilde{\psi} = (\widetilde{\varphi}, \varphi_{2n+1}) : (r, +\infty) \to X \subset \mathbb{C}^n \times \mathbb{R}$ meromorphic at $+\infty$ such that $\deg\widetilde{\psi} > 0$. Since $\|\widetilde{\varphi}(t)\| < 1$ for $t \in (r, +\infty)$, we have $\deg\varphi_{2n+1} > 0$. Put $\varphi = H\circ\widetilde{\varphi}$. Since H is rational, φ is meromorphic at $+\infty$. Moreover for any $t \in (r, +\infty)$ we have

$$\|\varphi(t)\| > |\varphi_{2n+1}|,$$
$$|f(\varphi(t)) - \lambda| \cdot |\varphi_{2n+1}(t)| < 1,$$
$$\|\operatorname{grad} f(\varphi(t))\| \cdot |\varphi_{2n+1}(t)| < 1.$$

From the above inequalities we have

$$\deg \varphi \geq \deg \varphi_{2n+1} > 0,$$
$$\deg(f \circ \varphi - \lambda) + \deg \varphi_{2n+1} \leq 0,$$
$$\deg \operatorname{grad} f \circ \varphi + \deg \varphi_{2n+1} \leq 0.$$

On the other hand we can extend φ to a complex mapping meromorphic at infinity. Thus, by (2),

$$\mathcal{L}_{\infty,\lambda}(f) \leq \frac{\deg \operatorname{grad} f \circ \varphi}{\deg \varphi} < 0.$$

This ends the proof. ∎

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