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ALGEBRAIC GEOMETRY

On the Łojasiewicz Exponent near the Fibre of a Polynomial

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Summary. The equivalence of the definitions of the Łojasiewicz exponent introduced by Ha and by Chądzyński and Krasiński is proved. Moreover we show that if the above exponents are less than -1 then they are attained at a curve meromorphic at infinity.

1. Introduction. Let $F : \mathbb{C}^n \to \mathbb{C}^m$ be a polynomial mapping and let $S \subset \mathbb{C}^n$ be an unbounded set. Put

$$N(F|S) := \{\nu \in \mathbb{R} : \exists_{A,D>0}\ \forall_{z\in S}\ (|z| \geq D \Rightarrow |F(z)| \geq A|z|^{\nu})\},$$

where $|\cdot|$ is an arbitrary norm in $\mathbb{C}^n$. By the *Łojasiewicz exponent at infinity* of $F|S$ we mean $\mathcal{L}_\infty(F|S) := \sup N(F|S)$.

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial in variables $z_1, \ldots, z_n$, where $n \geq 2$, and $\nabla f = (\partial f/\partial z_1, \ldots, \partial f/\partial z_n) : \mathbb{C}^n \to \mathbb{C}^n$ be its gradient. Let $\lambda \in \mathbb{C}$. Ha [5] introduces the following notion of Łojasiewicz exponent:

$$(1)\qquad \widetilde{\mathcal{L}}_{\infty,\lambda}(f) := \lim_{\delta\to 0^+} \mathcal{L}_\infty(\nabla f|S_{\lambda,\delta}),$$

where $S_{\lambda,\delta} = \{z \in \mathbb{C}^n : |f(z) - \lambda| < \delta\}$. He shows that in case $n = 2$, λ is a bifurcation point at infinity of f if and only if $\widetilde{\mathcal{L}}_{\infty,\lambda}(f) < -1$.

Chądzyński and Krasiński [2] introduce another notion of Łojasiewicz exponent:

$$(2)\qquad \mathcal{L}_{\infty,\lambda}(f) := \inf_{\Phi} \frac{\deg \nabla f \circ \Phi}{\deg \Phi},$$

where Φ is a meromorphic mapping defined in a neighbourhood of ∞ in $\overline{\mathbb{C}}$, $\deg \Phi > 0$ and $\deg(f - \lambda)\circ \Phi < 0$. They prove the equivalence of definitions

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(1) and (2) in case $n = 2$. In this paper we show that definitions (1) and (2) are equivalent for any $n \geq 2$ and $\lambda \in \mathbb{C}$ (Theorem 2.1 in Section 2). The essence is the Curve Selection Lemma.

Moreover Chądzyński and Krasiński proved that in case $n = 2$ if $\deg f = \deg_y f$ and $\mathcal{L}_{\infty,\lambda}(f) < -1$ then the exponent (2) is attained at a curve Ψ meromorphic at infinity such that $\deg \Psi > 0$, $\deg(f - \lambda) \circ \Psi < 0$ and $f'_y \circ \Psi = 0$ (see [2, Theorem 4.10 and Corollary 3.5]). In this article we prove that in case $n > 2$ if $\mathcal{L}_{\infty,\lambda}(f) < -1$ then the exponent (2) is also attained at a curve meromorphic at infinity (Theorem 3.1). From Theorem 3.1 we easily deduce that $\mathcal{L}_{\infty,\lambda}(f) \in \mathbb{Q} \cup \{-\infty\}$ provided $\mathcal{L}_{\infty,\lambda}(f) < -1$ (Corollary 3.3). We do not know if the assertion of Theorem 3.1 remains true without the additional assumption that $\mathcal{L}_{\infty,\lambda}(f) < -1$.

2. Equivalence of two definitions. We begin with some definitions. A real curve $\Phi : (R, +\infty) \to \mathbb{R}^N$, $R \in \mathbb{R}$, is called *meromorphic* at $+\infty$ if Φ is the sum of a Laurent series of the form

$$\Phi(t) = a_p t^p + a_{p-1} t^{p-1} + \cdots, \qquad a_i \in \mathbb{R}^N,\ p \in \mathbb{Z}.$$

If $\Phi \neq 0$ and $a_p \neq 0$ then p is called the *degree* of Φ and denoted by $\deg \Phi$. If $\Phi = 0$ then we put additionally $\deg \Phi = -\infty$.

As in the real case, a complex curve $\Psi : \{t \in \mathbb{C} : |t| > R\} \to \mathbb{C}^N$ is called *meromorphic at infinity* if Ψ is the sum of a Laurent series of the form

$$\Psi(t) = a_p t^p + a_{p-1} t^{p-1} + \cdots, \qquad a_i \in \mathbb{C}^N,\ p \in \mathbb{Z}.$$

If $\Psi \neq 0$ and $a_p \neq 0$ then p is called the degree of Ψ and denoted by $\deg \Psi$. If $\Psi = 0$ then we put additionally $\deg \Psi = -\infty$.

The first main result of the paper is the following

THEOREM 2.1. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial, $n \geq 2$ and $\lambda \in \mathbb{C}$. Then*

$$\widetilde{\mathcal{L}}_{\infty,\lambda}(f) = \mathcal{L}_{\infty,\lambda}(f). \tag{3}$$

Proof. Since $\widetilde{\mathcal{L}}_{\infty,\lambda}(f)$ does not depend on the choice of the norm in $\mathbb{C}^n$, we will use the Euclidean norm $\|\cdot\|$.

The inequality $\widetilde{\mathcal{L}}_{\infty,\lambda}(f) \leq \mathcal{L}_{\infty,\lambda}(f)$ follows directly from definitions (1) and (2). Indeed, it suffices to show that for every $\delta > 0$ we have

$$\mathcal{L}_{\infty,\lambda}(f) \geq \mathcal{L}_{\infty}(\nabla f | S_{\lambda,\delta}). \tag{4}$$

To prove (4) it suffices to show that for every $\nu \in N(\nabla f | S_{\lambda,\delta})$,

$$\mathcal{L}_{\infty,\lambda}(f) \geq \nu. \tag{5}$$

Let $\nu \in N(\nabla f | S_{\lambda,\delta})$. Then there exist $A, D > 0$ such that for $z \in S_{\lambda,\delta}$ we have

$$\|z\| \geq D \ \Rightarrow\ \|\nabla f(z)\| \geq A \|z\|^{\nu}. \tag{6}$$

Take any complex curve Φ meromorphic at infinity and such that $\deg\Phi > 0$ and $\deg(f-\lambda)\circ\Phi < 0$. We must show that

$$\frac{\deg\nabla f\circ\Phi}{\deg\Phi} \geq \nu. \tag{7}$$

Since $\deg\Phi > 0$ and $\deg(f-\lambda)\circ\Phi < 0$, there exists $R > 0$ such that for every $t \in \mathbb{C}$ with $|t| > R$ we have $\Phi(t) \in S_{\lambda,\delta}$ and $|\Phi(t)| > D$. Then (6) implies that for $|t| > R$ we have

$$\|\nabla f\circ\Phi(t)\| \geq A\|\Phi(t)\|^{\nu}.$$

Thus, $\deg\nabla f\circ\Phi \geq \nu\deg\Phi$, and since $\deg\Phi > 0$, we get (7). Because of arbitrariness of Φ, ν and δ we get (5), (4) and the "$\leq$" inequality of (3).

Now it suffices to prove

$$\widetilde{\mathcal{L}}_{\infty,\lambda}(f) \geq \mathcal{L}_{\infty,\lambda}(f). \tag{8}$$

Assume to the contrary that (8) does not hold. Hence there exists a rational number α such that

$$\widetilde{\mathcal{L}}_{\infty,\lambda}(f) < \alpha < \mathcal{L}_{\infty,\lambda}(f). \tag{9}$$

Since the mapping $(0,+\infty) \ni \delta \mapsto \mathcal{L}_{\infty}(\nabla f|S_{\lambda,\delta}) \in \mathbb{R}$ is nonincreasing, for every $\delta > 0$ we have

$$\mathcal{L}_{\infty}(\nabla f|S_{\lambda,\delta}) < \alpha.$$

Hence, $\alpha \notin N(\nabla f|S_{\lambda,\delta})$ for every $\delta > 0$. Thus, for every $\delta > 0$ there is $z^0 \in \mathbb{C}^n$ such that

$$|f(z^0)-\lambda| < \delta \wedge \|z^0\| > 1/\delta \wedge \|z^0\|^{\alpha} > \|\nabla f(z^0)\|. \tag{10}$$

Let $B = \{z \in \mathbb{C}^n : \|z\| < 1\}$. The mapping $H : B \ni z \mapsto z/(1-\|z\|^2) \in \mathbb{C}^n$ is a homeomorphism and is rational. Hence, the set

$$X = \{(z,\delta) \in B\times(0,+\infty) : |f\circ H(z)-\lambda|^2 < \delta^2 \wedge \|H(z)\|^2 > 1/\delta^2 \\ \wedge \|H(z)\|^{2\alpha} > \|\nabla f\circ H(z)\|^2\}$$

is semialgebraic. (10) implies that there is a sequence of points $(w^k,\delta_k) \in X$ convergent to a point $(w^0,0)$ such that $w^0 \in \partial B$. Therefore by the Curve Selection Lemma (cf. [6, Lemma 3.1]) we easily see that there exists a real curve $\widetilde{\Psi} = (\widetilde{\Phi},\varphi_{2n+1}) : (R,+\infty) \to X$, meromorphic at infinity, such that $\lim_{t\to\infty}\widetilde{\Psi}(t) = (w^0,0)$. Hence, $\deg\varphi_{2n+1} < 0$. Putting $\Phi = H\circ\widetilde{\Phi}$, we obtain the real curve $\Psi = (\Phi,\varphi_{2n+1}) : (R,+\infty) \to \mathbb{C}^n\times\mathbb{R}$ meromorphic at infinity and such that $\lim_{t\to\infty}\varphi_{2n+1}(t) = 0$. By definition of X, if $t > R$, we have

$$|f\circ\Phi(t)-\lambda| < \varphi_{2n+1}(t) \wedge \|\Phi(t)\| > \frac{1}{\varphi_{2n+1}(t)} \\ \wedge \|\Phi(t)\|^{\alpha} > \|\nabla f\circ\Phi(t)\|. \tag{11}$$

Thus, $\deg\Phi > 0$ and $\deg(f-\lambda)\circ\Phi \leq \deg\varphi_{2n+1} < 0$. By the last inequality, $\deg\nabla f\circ\Phi \leq \alpha\deg\Phi$. Extending Φ to the complex domain we obtain a complex meromorphic curve at infinity. From the above we get

$$\frac{\deg \nabla f \circ \Phi}{\deg \Phi} \leq \alpha,$$

which contradicts the second inequality in (9). This ends the proof. ∎

3. Attaining the Łojasiewicz exponent. Let us turn to the next main result.

THEOREM 3.1. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial, $n \geq 2$, $\lambda \in \mathbb{C}$. If $\mathcal{L}_{\infty,\lambda}(f) < -1$, then there exists a complex curve Φ meromorphic at infinity such that* $\deg \Phi > 0$, $\deg(f-\lambda) \circ \Phi < 0$ *and*

$$\mathcal{L}_{\infty,\lambda}(f) = \frac{\deg \nabla f \circ \Phi}{\deg \Phi}.$$

Before we pass to the proof we quote two propositions.

PROPOSITION 3.2 (Łojasiewicz inequality, [4, Theorem 2.1]). *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial. Then there exist $C, \varepsilon > 0$ such that*

$$|f(z)| \leq \varepsilon \ \Rightarrow \ |z|\,|\nabla f(z)| \geq C|f(z)|.$$

Analogously to Proposition 1 in [3], by using the Tarski–Seidenberg Theorem (cf. [1, Remark 3.8]) we prove the following

PROPOSITION 3.3. *Let $F : \mathbb{C}^n \to \mathbb{C}^m$ be a polynomial mapping and let $S \subset \mathbb{C}^n$ be an unbounded closed semialgebraic set. Then there exists a real curve $\Phi : (R, +\infty) \to S$ meromorphic at infinity such that* $\deg \Phi > 0$ *and*

$$\mathcal{L}_\infty(F|S) = \frac{\deg F \circ \Phi}{\deg \Phi}.$$

Moreover if $\mathcal{L}_\infty(F|S) \neq -\infty$ then $\mathcal{L}_\infty(F|S) \in N(F|S)$.

Proof of Theorem 3.1. In the proof we will use the Euclidean norm. We can assume that $\lambda = 0$. From Theorem 3.2 we conclude that there exist $\varepsilon > 0$ and $C > 0$ such that for every $z \in \mathbb{C}^n$ we have the implication

$$|f(z)| \leq \varepsilon \ \Rightarrow \ \|z\| \cdot \|\nabla f(z)\| \geq C \cdot |f(z)|. \tag{1}$$

Let $Y = \{w \in \mathbb{C}^n : C\|w\|^{-1-r}|f(w)| \leq 1\}$, where r is a rational number such that

$$\mathcal{L}_{\infty,0}(f) < r < -1. \tag{2}$$

Obviously Y is a closed semialgebraic set.

Let $\mathcal{M}_\infty$ be the set of all complex meromorphic curves at infinity and define

$$\mathcal{A} = \{\Psi \in \mathcal{M}_\infty : \deg \Psi > 0 \wedge \deg f \circ \Psi < 0 \wedge \deg \nabla f \circ \Psi / \deg \Psi < r\}.$$

The definition of $\mathcal{L}_{\infty,0}(f)$ and (2) imply that $\mathcal{A} \neq \emptyset$, and moreover

$$\mathcal{L}_{\infty,0}(f) = \inf_{\Psi\in\mathcal{A}} \frac{\deg \nabla f \circ \Psi}{\deg \Psi}. \tag{3}$$

Observe that for every $\Psi \in \mathcal{A}$,

$$\exists_{R>0}\ \forall_{t\in\mathbb{C}}\ (|t| > R \Rightarrow \Psi(t) \in Y). \tag{4}$$

Indeed, take any $\Psi \in \mathcal{A}$. Then by the definition of $\mathcal{A}$ there exists $R > 0$ such that for every $t \in \mathbb{C}$ if $|t| > R$ then

$$|f \circ \Psi(t)| \le \varepsilon \wedge \|\Psi(t)\|^{-r}\|\nabla f \circ \Psi(t)\| \le 1.$$

Hence (1) implies that for every $|t| > R$,

$$C\|\Psi(t)\|^{-1-r}|f \circ \Psi(t)| \le \|\Psi(t)\|^{-r}\|\nabla f \circ \Psi(t)\| \le 1.$$

From this and the definition of Y we have $\Psi(t) \in Y$ for $|t| > R$, and so (4) holds. Hence the set Y is nonempty and unbounded.

We will show that

$$\mathcal{L}_\infty(\nabla f|Y) \le \mathcal{L}_{\infty,0}(f). \tag{5}$$

By (3) it suffices to show

$$\mathcal{L}_\infty(\nabla f|Y) \le \inf_{\Psi\in\mathcal{A}} \frac{\deg \nabla f \circ \Psi}{\deg \Psi}. \tag{6}$$

If $\mathcal{L}_\infty(\nabla f|Y) = -\infty$, then (6) is obvious. Assume that $\mathcal{L}_\infty(\nabla f|Y) \neq -\infty$. Then by Proposition 3.3 there exist $A, D > 0$ such that for every $z \in Y$,

$$\|z\| \ge D \ \Rightarrow\ \|\nabla f(z)\| \ge A\|z\|^{\mathcal{L}_\infty(\nabla f|Y)}.$$

Therefore for every $\Psi \in \mathcal{A}$, by (4) we have $\deg \nabla f \circ \Psi \ge \deg \Psi \cdot \mathcal{L}_\infty(\nabla f|Y)$, which gives (6). So (5) holds.

By Proposition 3.3 there exists a real curve $\Phi : (R', +\infty) \to Y$ meromorphic at infinity such that $\deg \Phi > 0$ and

$$\mathcal{L}_\infty(\nabla f|Y) = \frac{\deg \nabla f \circ \Phi}{\deg \Phi}. \tag{7}$$

Extending Φ to the complex domain we get (8). Moreover $\deg f \circ \Phi < 0$ by definition of Y. Hence from (5) and (7) we get the assertion. ∎

Theorem 3.1 immediately yields

COROLLARY 3.4. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial, $n \ge 2$, $\lambda \in \mathbb{C}$. If $\mathcal{L}_{\infty,\lambda}(f) < -1$ then $\mathcal{L}_{\infty,\lambda}(f) \in \mathbb{Q} \cup \{-\infty\}$.*

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