On Functions with the Cauchy Difference Bounded by a Functional

by

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Summary. K. Baron and Z. Kominek [2] have studied the functional inequality

\[ f(x + y) - f(x) - f(y) \geq \phi(x, y), \quad x, y \in X, \]

under the assumptions that \( X \) is a real linear space, \( \phi \) is homogeneous with respect to the second variable and \( f \) satisfies certain regularity conditions. In particular, they have shown that \( \phi \) is bilinear and symmetric and \( f \) has a representation of the form

\[ f(x) = \frac{1}{2} \phi(x, x) + L(x) \]

for \( x \in X \), where \( L \) is a linear function.

The purpose of the present paper is to consider this functional inequality under different assumptions upon \( X, f \) and \( \phi \). In particular we will give conditions which force biadditivity and symmetry of \( \phi \) and the representation

\[ f(x) = \frac{1}{2} \phi(x, x) - A(x) \]

for \( x \in X \), where \( A \) is a subadditive function.

Let \((X, +)\) be an abelian group. We consider the functional inequality

\[ f(x + y) - f(x) - f(y) \geq \phi(x, y), \quad x, y \in X, \]

where \( \phi: X \times X \to \mathbb{R} \) and \( f: X \to \mathbb{R} \) are unknown mappings.

It is easy to check that if \( \phi: X \times X \to \mathbb{R} \) is biadditive and symmetric, \( A: X \to \mathbb{R} \) is subadditive and \( f: X \to \mathbb{R} \) is defined by the formula

\[ f(x) := \frac{1}{2} \phi(x, x) - A(x) \]

for \( x \in X \), then (1) holds. We are going to provide conditions under which the converse implication is valid.

**Proposition.** If \( f: X \to \mathbb{R} \) and \( \phi: X \times X \to \mathbb{R} \) satisfy (1) and

\[ \phi(x, -x) \geq -\phi(x, x), \quad x \in X, \]

then: (a) \( f(0) \leq 0 \); (b) \( f(x) + f(-x) \leq \phi(x, x) \) for \( x \in X \); (c) \( f(2x) \geq 3f(x) + f(-x) \) for \( x \in X \).

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Proof. The assumption (2) implies that \( \phi(0,0) \geq 0 \); thus applying (1) with \( x = 0 \) and \( y = 0 \) we get \( f(0) \leq 0 \). Using this and substituting \( y := -x \) in (1) we derive (b). Substituting \( y := x \) in (1) and using (b) proves (c). This completes the proof.

In what follows we make use of a result of Karol Baron (see S. Rolewicz [4, Lemma 5.7]). A careful inspection of the original proof allows us to weaken certain assumptions of this lemma. The original result reads as follows.

**Lemma (K. Baron).** Assume that \( f: X \to \mathbb{R} \) and \( \phi: X \times X \to \mathbb{R} \) satisfy (1). If \( f \) is even, \( f(2x) = 4f(x) \) and \( \phi(x,\cdot) \) is odd for every \( x \in X \), then there exists a biadditive and symmetric functional \( B: X \times X \to \mathbb{R} \) such that \( \phi = 2B \) and \( f(x) = B(x,x) \) for every \( x \in X \).

We have the following modification of this lemma.

**Lemma 1.** Assume that \( f: X \to \mathbb{R} \) and \( \phi: X \times X \to \mathbb{R} \) satisfy (1). If

\begin{align}
\label{eq:phi_inequality}
\phi(x,-y) &\geq -\phi(x,y), \quad x,y \in X, \\
\label{eq:f_inequality}
f(2x) &\leq 4f(x), \quad x \in X,
\end{align}

then

\[f(x) = \frac{1}{2} \phi(x,x), \quad x \in X.\]

Moreover, \( \phi \) is biadditive and symmetric.

**Proof.** Using the inequality (c) of the Proposition and (4) we see that \( f(x) \geq f(-x) \) for \( x \in X \), which proves that \( f \) is even. Setting \(-y\) instead of \( y \) in (1) we obtain

\[f(x-y) - f(x) - f(-y) \geq \phi(x,-y) \geq -\phi(x,y), \quad x,y \in X.
\]

Adding this to (1) and using the evenness of \( f \) leads to

\[f(x+y) + f(x-y) \geq 2f(x) + 2f(y), \quad x,y \in X.
\]

Fix \( u,v \in X \). Applying the above inequality with \( x = u + v \) and \( y = u - v \) we infer that

\[4f(u) + 4f(v) \geq f(2u) + f(2v) \geq 2f(u+v) + 2f(u-v), \quad u,v \in X.
\]

Therefore \( f \) is a quadratic function, i.e.

\[f(x+y) + f(x-y) = 2f(x) + 2f(y), \quad x,y \in X.
\]

So, there exists a biadditive and symmetric functional \( B: X \times X \to \mathbb{R} \) such that \( f(x) = B(x,x) \) for \( x \in X \) (see e.g. J. Aczél & J. Dhombres [1, Chapter 11, Proposition 1]). It is easy to check that

\[B(x,y) = \frac{1}{2} [f(x+y) - f(x) - f(y)], \quad x,y \in X.
\]
Now, assumption (3) and the biadditivity of $B$ imply that $2B = \phi$. This completes the proof.

**Theorem 1.** Assume that $f: X \to \mathbb{R}$ and $\phi: X \times X \to \mathbb{R}$ satisfy (1), (3) and
\[ \limsup_{n \to \infty} \frac{1}{4^n} \phi(2^n x, 2^n x) < \infty, \quad x \in X, \]
(5)
\[ \liminf_{n \to \infty} \frac{1}{4^n} \phi(2^n x, 2^n y) \geq \phi(x, y), \quad x, y \in X. \]

If $f$ is even, then there exists a subadditive function $A: X \to \mathbb{R}$ such that
\[ f(x) = \frac{1}{2} \phi(x, x) - A(x), \quad x \in X. \]
Moreover, $\phi$ is biadditive and symmetric.

**Proof.** Fix an $x \in X$ and a positive integer $n$. By the evenness of $f$ and the Proposition we get
\[ \frac{1}{4^n-1} f(2^{n-1} x) \leq \frac{1}{4^n} f(2^n x) \leq \frac{1}{4^n} \cdot \frac{1}{2} \phi(2^n x, 2^n x). \]
The first part of the assumption (5) implies that the right-hand side of this inequality is bounded by a real constant which does not depend on $n$. Therefore the formula
\[ Q(x) := \lim_{n \to +\infty} \frac{1}{4^n} f(2^n x), \quad x \in X, \]
correctly defines a map $Q: X \to \mathbb{R}$. Moreover, $Q(2x) = 4Q(x)$ for $x \in X$ and the following inequality is satisfied:
\[ Q(x + y) - Q(x) - Q(y) = \lim_{n \to \infty} \left[ \frac{1}{4^n} f(2^n x + 2^n y) - \frac{1}{4^n} f(2^n x) - \frac{1}{4^n} f(2^n y) \right] \]
\[ \geq \liminf_{n \to \infty} \frac{1}{4^n} \phi(2^n x, 2^n y) \geq \phi(x, y), \quad x, y \in X. \]

Lemma 1 states that $\phi$ is biadditive and symmetric and $Q(x) = \frac{1}{2} \phi(x, x)$ for $x \in X$. In particular $Q(x + y) - Q(x) - Q(y) = \phi(y, x)$ for $x, y \in X$. From this and (1), it is easy to check that $A := Q - f$ is subadditive. This completes the proof.

**Corollary 1.** Assume that $f: X \to \mathbb{R}$ and $\phi: X \times X \to \mathbb{R}$ satisfy (1), (3), (5) and
\[ \phi(-x, -y) = \phi(x, y), \quad x, y \in X. \]
(6)
Then there exists a subadditive function $A: X \to \mathbb{R}$ such that
\[ f(x) = \frac{1}{2} \phi(x, x) - A(x), \quad x \in X. \]
Moreover, $\phi$ is biadditive and symmetric.
Proof. Define $h: X \to \mathbb{R}$ by $h(x) := \frac{1}{2}(f(x) + f(-x))$ for $x \in X$. Assumption (6) implies that

$$h(x + y) - h(x) - h(y) \geq \phi(x, y), \quad x, y \in X.$$ 

Using Theorem 1 with $f$ replaced by $h$ we get the biadditivity and symmetry of $\phi$. Now, one may easily check that the map $A: X \to \mathbb{R}$ given by $A(x) := \frac{1}{2}\phi(x, x) - f(x)$ for $x \in X$ is subadditive. This completes the proof.

Now, we are going to provide conditions which, in particular, allow us to omit the assumption (6) and to weaken (5). We start with a lemma.

**Lemma 2.** If $f: X \to \mathbb{R}$ and $\phi: X \times X \to \mathbb{R}$ satisfy (1), (2) and $f$ is odd then $f(2x) = 2f(x)$ and $\phi(x, x) = 0$ for $x \in X$. Moreover, if $\phi$ satisfies (3), then $f$ is additive and $\phi = 0$.

**Proof.** Fix an $x \in X$. Since $f$ is odd, we get

$$f(2x) - 2f(x) = -[f(-2x) - 2f(-x)] \leq -\phi(-x, -x) \leq \phi(-x, x) \leq f(-x + x) - f(-x) - f(x) = 0,$$

whence, again by the oddness of $f$, we obtain $f(2x) = 2f(x)$ for $x \in X$ and, in consequence, $\phi(x, x) = 0$ for $x \in X$.

Now, assume (3) and let $x, y \in X$. Using the assumption (3) and (1) twice we obtain

$$f(x - y) - f(x) - f(-y) \geq \phi(x, -y) \geq -\phi(x, y) \geq -f(x + y) + f(x) + f(y),$$

which means that

$$f(x + y) + f(x - y) \geq 2f(x).$$

Interchanging the roles of $x$ and $y$ we obtain

$$f(y + x) + f(y - x) \geq 2f(y).$$

Summing up these two inequalities we derive the superadditivity of $f$, which together with its oddness implies that $f$ is additive and $\phi \leq 0$. Using this and (3) we finally get $\phi = 0$. This completes the proof.

The following lemma provides sufficient conditions for the function $f$ to satisfy the assumption (4).

Recall that a group $X$ is called *uniquely 2-divisible* if the map $X \ni x \mapsto x + x \in X$ is bijective.

**Lemma 3.** Assume that $X$ is uniquely 2-divisible, $f: X \to \mathbb{R}$ and $\phi: X \times X \to \mathbb{R}$ satisfy (1), (2) and

$$\phi(2x, 2x) \leq 4\phi(x, x), \quad x \in X.$$ 

If $f$ is nonnegative and even, then $f(x) = \frac{1}{2}\phi(x, x)$ for $x \in X$. 

Proof. By the Proposition, for every \( x \in X \) and every positive integer \( n \) we have \( 4^n f(x/2^n) \geq 4^{n+1} f(x/2^{n+1}) \geq 0 \). So, the sequence \( (4^n f(x/2^n))_{n \in \mathbb{N}} \) is pointwise convergent. In particular, \( \lim_{n \to \infty} 2^n f(x/2^n) = 0 \) for every \( x \in X \).

Now, fix an \( x \in X \). Using (1) and (7), by induction, we get
\[
2^k f(\frac{x}{2^{k-1}}) - 2^{k+1} f(\frac{x}{2^k}) \geq 2^k \phi\left(\frac{x}{2^k}, \frac{x}{2^k}\right) \geq \frac{1}{2^k} \phi(x, x)
\]
for all \( k \in \mathbb{N} \). Summing up these inequalities for \( k \in \{1, \ldots, n\} \) we get
\[
2f(x) - 2^{n+1} f(\frac{x}{2^n}) \geq \sum_{k=1}^{n} \frac{1}{2^k} \phi(x, x), \quad n \in \mathbb{N}.
\]
Letting \( n \) tend to \( +\infty \) yields \( 2f(x) \geq \phi(x, x) \). Since the Proposition provides the opposite inequality, the proof is complete.

The following result yields an analogue of Corollary 1 in the paper [2] of K. Baron and Z. Kominek.

**Theorem 2.** Assume \( X \) to be uniquely 2-divisible. If \( f: X \to \mathbb{R} \) and \( \phi: X \times X \to \mathbb{R} \) satisfy (1), (3), (7) and
\[
f(x) + f(-x) \geq 0, \quad x \in X,
\]
then there exists an additive function \( a: X \to \mathbb{R} \) such that
\[
f(x) = \frac{1}{2} \phi(x, x) + a(x), \quad x \in X.
\]
Moreover, \( \phi \) is biadditive and symmetric.

**Proof.** Define \( h, a: X \to \mathbb{R} \) by \( h(x) := \frac{1}{2}[f(x) + f(-x)] \) and \( a(x) := \frac{1}{2}[f(x) - f(-x)] \), \( x \in X \). Clearly \( h \) is even whereas \( a \) is odd. Next, define \( \phi_1: X \times X \to \mathbb{R} \) by \( \phi_1(x, y) := \frac{1}{2} [\phi(x, y) + \phi(-x, -y)] \) for \( x, y \in X \). It is easy to check that \( h \) and \( \phi_1 \) satisfy the assumptions of Lemma 3. So, \( h(x) = \frac{1}{2} \phi_1(x, x) \) for \( x \in X \). Now, observe that the assumptions of Lemma 1 are satisfied. Therefore \( \phi_1 \) is biadditive and symmetric and, in consequence,
\[
h(x + y) - h(x) - h(y) = \phi_1(x, y), \quad x, y \in X.
\]
Define \( \phi_2: X \times X \to \mathbb{R} \) by \( \phi_2 := \phi - \phi_1 \). Note that \( \phi_2(x, -y) \geq -\phi_2(x, y) \) for \( x, y \in X \) and
\[
a(x + y) - a(x) - a(y) \geq \phi_2(x, y), \quad x, y \in X.
\]
Now, Lemma 2 applied for \( f = a \) and \( \phi = \phi_2 \) states that \( a \) is additive and \( \phi_2 = 0 \), i.e. \( \phi = \phi_1 \). This completes the proof.

A similar reasoning allows us to derive the following corollary from Lemmas 2 and 3.

**Corollary 2.** Assume \( X \) to be uniquely 2-divisible. If \( f: X \to \mathbb{R} \) and \( \phi: X \times X \to \mathbb{R} \) satisfy (1), (2), (7) and (8), then there exists an odd function
a: \( X \rightarrow \mathbb{R} \) such that \( a(2x) = 2a(x) \) for \( x \in X \) and

\[
f(x) = \frac{1}{2} \phi(x, x) + a(x), \quad x \in X.
\]

Moreover, \( \phi(2x, 2x) = 4\phi(x, x) \geq 0 \) for \( x \in X \).

Proof. Define \( a, h, \phi_1 \) and \( \phi_2 \) as in the previous proof. Lemma 3 implies that \( h(x) = \frac{1}{2} \phi_1(x, x) \) and \( h(2x) = 4h(x) \) for \( x \in X \). We are going to show that \( \phi_2 \) satisfies (2). Since \( \phi_2 = \phi - \phi_1 \), it suffices to prove that \( \phi_1(x, -x) = -\phi_1(x, x) \) for \( x \in X \). But

\[
-2h(x) = h(-x + x) - h(-x) - h(x) \geq \phi_1(-x, x) \geq -\phi_1(-x, -x)
\]

\[
= -2h(x),
\]

which is what we wanted. Lemma 2 implies that \( a(2x) = 2a(x) \) and \( \phi_2(x, x) = 0 \), i.e. \( h(x) = \frac{1}{2} \phi_1(x, x) = \frac{1}{2} \phi(x, x) \) for \( x \in X \). This completes the proof.

We end this paper with some additional remarks.

Remark 1. If \( c \in (0, \infty) \), \( f : \mathbb{R} \rightarrow \mathbb{R} \) is constant and equal to \(-c\), \( \phi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) is constant and equal to \( c \), then (1), (3) and (7) are satisfied. So, the assumption (4) in Lemma 1 cannot be omitted, the assumption (5) in Theorem 1 and Corollary 1 cannot be replaced by (7), the nonnegativity of \( f \) in Lemma 3 cannot be replaced by its boundedness, and the assumption (8) in Theorem 2 cannot be omitted.

Remark 2. Let \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \) be a nonzero and even function which satisfies the equality

\[
\varphi(2t) = 2\varphi(t), \quad t \in \mathbb{R}
\]

(see e.g. M. Kuczma, B. Choczewski and R. Ger [3] for examples of such functions). Define \( f : \mathbb{R} \rightarrow \mathbb{R} \) and \( \phi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) by

\[
f(x) := (\varphi(x))^2, \quad x \in \mathbb{R},
\]

\[
\phi(x, y) := f(x + y) - f(x) - f(y), \quad x, y \in \mathbb{R}.
\]

Then inequality (1) is satisfied, and \( f \) is even, nonnegative and satisfies (4). Moreover, \( \phi \) satisfies (2), (5) and (6). So, in Lemma 1, Theorems 1 and 2 and Corollary 1, (3) cannot be replaced by (2).

Remark 3. Let \( (X; \cdot \| \cdot \|) \) be a normed linear space. Corollary 1 implies that the inequality

\[
f(x + y) - f(x) - f(y) \geq \|x\| \cdot \|y\|, \quad x, y \in X,
\]

has no solution. In fact, the function \( \phi(x, y) := \|x\| \cdot \|y\|, \quad x, y \in X \), satisfies (3), (5) and (6), but \( \phi \) fails to be biadditive.

In this inequality \( X \) may stand for an abelian group and the norm can be replaced by any real function, which is nonzero, nonnegative, even and 2-homogeneous.
References


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