BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 52, No. 3, 2004

OPERATOR THEORY

Bundle Convergence in a von Neumann Algebra and in a von Neumann Subalgebra

by

Barthélemy LE GAC and Ferenc MÓRICZ

Presented by Stanisław WORONOWICZ

Summary. Let H be a separable complex Hilbert space, $\mathcal{A}$ a von Neumann algebra in $\mathcal{L}(H)$, ϕ a faithful, normal state on $\mathcal{A}$, and $\mathcal{B}$ a commutative von Neumann subalgebra of $\mathcal{A}$. Given a sequence $(X_n : n \geq 1)$ of operators in $\mathcal{B}$, we examine the relations between bundle convergence in $\mathcal{B}$ and bundle convergence in $\mathcal{A}$.

1. Introduction. Bundle convergence in von Neumann algebras was introduced in 1996 by Hensz, Jajte and Paszkiewicz in their fundamental paper [2]. We refer to [2] for the definitions and basic properties of bundle convergence.

Let H be a separable complex Hilbert space, $\mathcal{L}(H)$ the algebra of all bounded linear operators acting on H, $\mathcal{A}$ a von Neumann algebra in $\mathcal{L}(H)$, ϕ a faithful, normal state on $\mathcal{A}$, and $\mathcal{B}$ a von Neumann subalgebra of $\mathcal{A}$. Clearly, the restriction of ϕ to $\mathcal{B}$ defines a faithful, normal state on $\mathcal{B}$. Thus, the following question seems to be quite natural.

QUESTION. Let $(X_n : n \geq 1)$ be a sequence of operators in $\mathcal{B}$ which is bundle convergent to O in $\mathcal{B}$, where O is the zero operator acting on H. Is then (X_n) bundle convergent in $\mathcal{A}$?

2000 *Mathematics Subject Classification*: Primary 46L53, 46L10.

Key words and phrases: von Neumann algebra and subalgebra, faithful and normal state, bundle convergence, almost uniform convergence, conditional expectation in a von Neumann algebra.

This research was completed while the second named author was a visiting professor at the University of Tennessee, Knoxville during the fall semester in 2003; and it was also partially supported by the Hungarian National Foundation for Scientific Research under Grants TS 044 782 and T 046 192.

We shall see in Section 2 that the answer to this question is negative in general. However, the answer is yes in the following two particular cases:

(i) If $\mathcal{A} := L_\infty(\Omega, \mathcal{F}, \mu)$, where $(\Omega, \mathcal{F}, \mu)$ is a classical probability space, and ϕ is defined by

$$\phi(A) := \int_\Omega A(\omega)\, d\mu(\omega), \qquad A \in \mathcal{A},$$

then the notion of bundle convergence in $\mathcal{A}$ coincides with that of almost sure convergence with respect to the probability measure μ. The positive answer to the above question follows from the well known fact that in this case, any von Neumann subalgebra is of the form $L_\infty(\Omega, \mathcal{G}, \mu)$, where $\mathcal{G}$ is a σ-subalgebra of $\mathcal{F}$.

(ii) If the sequence $(X_n : n \geq 1)$ is bounded in operator norm; this follows from the fact that bundle convergence in $\mathcal{A}$ (respectively, in $\mathcal{B}$) is equivalent to almost uniform convergence in $\mathcal{A}$ (respectively, in $\mathcal{B}$), by [2, Properties 3.7 and Theorem 4.1].

In this paper, we deal only with a commutative von Neumann subalgebra $\mathcal{B}$ of $\mathcal{A}$. In Section 2, we study a particular case of $\mathcal{A}$ which will be useful to construct counterexamples. In Section 3, we state some relations concerning bundle convergence of subsequences, and we consider the converse problem. Namely, assuming that a sequence (X_n) of operators in $\mathcal{B}$ is bundle convergent in $\mathcal{A}$, is it also bundle convergent in $\mathcal{B}$? It turns out that the answer depends on whether there exists a conditional expectation with respect to ϕ from $\mathcal{A}$ to $\mathcal{B}$. On closing, we raise two problems.

2. A particular case. Let H be a separable complex Hilbert space and fix an orthonormal basis $(e_j : j \geq 1)$ in H. We define a faithful, normal state ϕ on $\mathcal{A} := \mathcal{L}(H)$ in the following way:

$$\phi(A) := \sum_{j=1}^{\infty} 2^{-j} (Ae_j \,|\, e_j), \quad A \in A, \tag{2.1}$$

where $(\cdot|\cdot)$ is the inner product in H. In fact, ϕ is clearly a positive, linear functional on $\mathcal{L}(H)$, for the identity operator I we have $\phi(I) = 1$, and ϕ is faithful (since $2^{-j} > 0$ for all j). The normality of ϕ is a consequence of [3, Theorem, p. 121]. Let $\mathcal{D}$ be the von Neumann subalgebra of $\mathcal{L}(H)$ consisting of the operators in $\mathcal{L}(H)$ whose matrices are diagonal with respect to the orthonormal basis $(e_j : j \geq 1)$. Thus, every $X \in \mathcal{D}$ is of the form

$$X = \sum_{j=1}^{\infty} a_j P_{e_j}, \qquad \text{where } (a_j) \in \ell_\infty$$

and P_{e_j} is the (orthogonal) projection on the line $\mathbb{C}e_j$.

Now, for every $\alpha := (\alpha_1, \alpha_2, \dots) \in \ell_2$, $\alpha \neq (0, 0, \dots)$, let us define a vector u depending on α as follows:

$$u := K \sum_{j=1}^{\infty} \alpha_j 2^{-j/2} e_j, \tag{2.2}$$

where the constant $K > 0$ is chosen so that $\|u\| = 1$. Denote by P_u the projection on the line $\mathbb{C}u$.

THEOREM 1. *The projection P_u belongs to each bundle in $\mathcal{L}(H)$.*

Proof. Let $\mathcal{P}$ be a bundle in $\mathcal{L}(H)$. By definition, $\mathcal{P}$ is determined by some sequence $(D_n : n \geq 1)$ of positive operators in $\mathcal{L}(H)$ such that

$$\sum_{n=1}^{\infty} \phi(D_n) < \infty. \tag{2.3}$$

We associate with each operator D_n its infinite matrix $(d_{n,j,k})$ in the orthonormal basis (e_j), where

$$d_{n,j,k} := (D_n e_k \,|\, e_j), \quad n, j, k = 1, 2, \dots. \tag{2.4}$$

Taking into account that by the positivity of D_n,

$$D_n = C_n^* C_n \quad \text{for some } C_n \in \mathcal{L}(H),$$

where C_n^* is the adjoint operator to C_n, and making use of the Cauchy–Schwarz inequality, we conclude that

$$|d_{n,j,k}|^2 \leq d_{n,j,j} d_{n,k,k}, \quad n, j, k = 1, 2, \dots. \tag{2.5}$$

By (2.1) and (2.4), we may write

$$\phi(D_n) = \sum_{j=1}^{\infty} 2^{-j} d_{n,j,j}, \quad n = 1, 2, \dots. \tag{2.6}$$

Let x be an arbitrary vector in H. Then

$$x = \sum_{j=1}^{\infty} x_j e_j \quad \text{for some } (x_j) \subset \ell_2.$$

Since $\|u\| = 1$, we have $P_u x = (x \,|\, u) u$ and thus

$$\begin{aligned} D_n P_u x &= (x \,|\, u) D_n u = K (x \,|\, u) \sum_{j=1}^{\infty} \alpha_j 2^{-j/2} D_n e_j \\ &= K (x \,|\, u) \sum_{j=1}^{\infty} \alpha_j 2^{-j/2} \sum_{k=1}^{\infty} (D_n e_j \,|\, e_k) e_k \\ &= K (x \,|\, u) \sum_{k=1}^{\infty} \Big(\sum_{j=1}^{\infty} \alpha_j 2^{-j/2} d_{n,k,j} \Big) e_k. \end{aligned} \tag{2.7}$$

Accordingly, we define

$$(2.8)\qquad y_n := \sum_{k=1}^{\infty} y_{n,k} e_k, \qquad y_{n,k} := \sum_{j=1}^{\infty} \alpha_j 2^{-j/2} d_{n,k,j}, \qquad n,k = 1,2,\ldots.$$

Thus, we can rewrite (2.7) in the form

$$D_n P_u x = K(x \,|\, u) y_n,$$

whence

$$P_u D_n P_u x = K(x \,|\, u) P_u y_n = K(x \,|\, u)(y_n \,|\, u) u;$$

in particular,

$$(2.9)\qquad \|P_u D_n P_u x\| = K|(x \,|\, u)| \cdot |(y_n \,|\, u)|, \qquad n = 1,2,\ldots.$$

Now, we estimate $|(y_n \,|\, u)|$. By (2.2) and (2.8), we have

$$\begin{aligned}(y_n \,|\, u) &= \sum_{k=1}^{\infty} y_{n,k}(e_k \,|\, u) = K \sum_{k=1}^{\infty} y_{n,k} \alpha_k 2^{-k/2} \\ &= K \sum_{k=1}^{\infty} \Big(\sum_{j=1}^{\infty} \alpha_j 2^{-j/2} d_{n,k,j} \Big) \alpha_k 2^{-k/2}.\end{aligned}$$

By (2.5), we find that

$$\begin{aligned}(2.10)\qquad |(y_n \,|\, u)| &\le K \sum_{k=1}^{\infty} \Big(\sum_{j=1}^{\infty} |\alpha_j| 2^{-j/2} |d_{n,k,j}| \Big) |\alpha_k| 2^{-k/2} \\ &\le K \sum_{k=1}^{\infty} \Big(\sum_{j=1}^{\infty} |\alpha_j| 2^{-j/2} \sqrt{d_{n,j,j}} \Big) |\alpha_k| 2^{-k/2} \sqrt{d_{n,k,k}} \\ &= K \Big(\sum_{k=1}^{\infty} |\alpha_k| 2^{-k/2} \sqrt{d_{n,k,k}} \Big)^2.\end{aligned}$$

Applying the Cauchy inequality, by (2.6) and (2.10), we conclude that

$$(2.11)\qquad |(y_n \,|\, u)| \le K \|\alpha\|_2^2 \phi(D_n), \qquad n = 1,2,\ldots,$$

where $\|\alpha\|_2$ is the ℓ_2-norm of $\alpha = (\alpha_1, \alpha_2, \ldots)$. Combining (2.9) and (2.11) gives

$$\|P_u D_n P_u x\| \le K^2 \|\alpha\|_2^2 \|x\| \phi(D_n).$$

Since $x \in H$ is arbitrary, we have

$$\|P_u D_n P_u\|_\infty \le K^2 \|\alpha\|_2^2 \phi(D_n), \quad n = 1,2,\ldots.$$

By (2.3), it follows that $\|P_u D_n P_u\|_\infty \to 0$ as $n \to \infty$. An analogous argument shows that

$$\sup_{n \ge 1} \sum_{k=1}^{n} \|P_u D_k P_u\|_\infty < \infty.$$

Consequently, the projection P_u belongs to the bundle determined by (D_n), as claimed. ∎

Now, let $(X_n : n \geq 1)$ be a sequence of operators in $\mathcal{D}$. We shall examine the relations between bundle convergence in $\mathcal{D}$ and in $\mathcal{A} = \mathcal{L}(H)$. First, we need the following

LEMMA. *Let (X_n) be a sequence in $\mathcal{D}$. Then*

$$X_n \xrightarrow{\text{b},\mathcal{D}} O \quad \text{as } n \to \infty$$

if and only if

$$(X_n e_j \,|\, e_j) \to 0 \quad \text{as } n \to \infty \text{ for each } j = 1, 2, \ldots.$$

Proof. We may identify $\mathcal{D}$ with the L_∞-space of the probability space $(\mathbb{N}, \mathcal{F}, \mu)$, where $\mathbb{N}$ is the set of natural numbers, $\mathcal{F}$ is the family of all subsets of $\mathbb{N}$, and μ is given by

$$\mu(\{j\}) = 2^{-j}, \quad j = 1, 2, \ldots.$$

Thus, bundle convergence in $\mathcal{D}$ coincides with almost sure convergence with respect to μ (see, for example, [2, p. 29]). ∎

COROLLARY 1. *Let (X_n) be a sequence in $\mathcal{D}$. Then*

$$X_n \xrightarrow{\text{b},\mathcal{A}} O \quad \textit{implies} \quad X_n \xrightarrow{\text{b},\mathcal{D}} O \quad \text{as } n \to \infty.$$

Proof. Fix $j = j_0 \geq 1$. In (2.2), we choose $(\alpha_1, \alpha_2, \ldots)$ as follows:

$$\alpha_{j_0} = 2^{j_0/2}, \quad \alpha_j = 0 \quad \text{if } j \neq j_0.$$

Thus $u = e_{j_0}$. We deduce that

$$X_n P_u u = X_n e_{j_0} = (X_n e_{j_0} \,|\, e_{j_0}) e_{j_0}.$$

Hence we get

$$|(X_n e_{j_0} \,|\, e_{j_0})| \leq \|X_n P_{e_{j_0}}\|_\infty \to 0 \quad \text{as } n \to \infty,$$

by Theorem 1. Then $X_n \xrightarrow{\text{b},\mathcal{D}} 0$, as a consequence of the lemma. ∎

COROLLARY 2. *There exists a sequence (X_n) in $\mathcal{D}$ which is bundle convergent to O in $\mathcal{D}$, but fails to be bundle convergent in $\mathcal{L}(H)$.*

Proof. Let

$$X_n := n 2^{n/2} P_{e_n}, \quad n = 1, 2, \ldots.$$

Then $X_n \in \mathcal{D}$ and $X_n \xrightarrow{\text{b},\mathcal{D}} O$ as $n \to \infty$, since for every $j = 1, 2, \ldots$, we have $(X_n e_j \,|\, e_j) = 0$ as soon as $n > j$. Now, in (2.2) choose

$$u := K \sum_{j=1}^{\infty} j^{-1} 2^{-j/2} e_j; \tag{2.12}$$

it follows that

$$X_n P_u u = X_n u = K e_n, \quad n = 1, 2, \ldots . \tag{2.13}$$

By using (2.13) and the orthonomality of the system (e_j), we get

$$\begin{aligned} \|(X_{n+1} - X_n)P_u\|_\infty &\geq \|(X_{n+1} - X_n)P_u u\| \\ &= K\|e_{n+1} - e_n\| = K\sqrt{2}, \quad n = 1, 2, \ldots . \end{aligned} \tag{2.14}$$

Consequently, if (X_n) were bundle convergent in $\mathcal{L}(H)$ to some operator X, then $(X_{n+1} - X_n : n \geq 1)$ would be bundle convergent to O in $\mathcal{L}(H)$, due to the additivity of bundle convergence; in particular, we would have

$$\|(X_{n+1} - X_n)P_u\|_\infty \to 0 \quad \text{as } n \to \infty,$$

since P_u belongs to every bundle in $\mathcal{L}(H)$. But this contradicts (2.14), and the contradiction yields the conclusion of Corollary 2. ∎

Remark 1. The sequence

$$X_n := n 2^{n/2} P_{e_n}, \quad n = 1, 2, \ldots, \tag{2.15}$$

converges almost uniformly to O in $\mathcal{D}$; consequently, it converges almost uniformly to O in $\mathcal{L}(H)$, as well. In this way, we have obtained a simple example which illustrates the following known statement.

Corollary 3. *There exists a sequence $(X_n : n \geq 1)$ of operators in $\mathcal{L}(H)$ such that (X_n) converges almost uniformly, but fails to be bundle convergent in $\mathcal{L}(H)$.*

A more theoretic proof of Corollary 3 can be derived from [6, Proposition 4.6], where it is proved that almost uniform convergence (unlike bundle convergence) does not have the additivity property.

Corollary 4. *There exists a sequence $(Y_n : n \geq 1)$ of operators in $\mathcal{L}(H)$ such that (Y_n) is bundle convergent to O, but (Y_n^2) fails to be bundle convergent in $\mathcal{L}(H)$.*

Proof. Let (X_n) be given by (2.15) and

$$Y_n := X_n^{1/2} = n^{1/2} 2^{n/4} P_{e_n}, \quad n = 1, 2, \ldots .$$

By (2.1), we have

$$\phi(Y_n^2) = \phi(X_n) = n 2^{n/2} \phi(P_{e_n}) = n 2^{-n/2}.$$

Since

$$\sum_{n=1}^{\infty} \phi(Y_n^2) = \sum_{n=1}^{\infty} n 2^{-n/2} < \infty,$$

by [2, Proposition 3.1] we conclude that (Y_n) is bundle convergent to O as $n \to \infty$. But we have seen in the proof of Corollary 2 that the sequence $(Y_n^2 = X_n : n \geq 1)$ fails to be bundle convergent in $\mathcal{L}(H)$. ∎

3. Bundle convergence of subsequences. The sequence $(X_n : n \geq 1)$ we used in the proof of Corollary 2 does not admit a subsequence $(X_{n_k} : k \geq 1)$ bundle convergent in $\mathcal{L}(H)$, since, with u given by (2.12),

$$\|(X_{n_{k+1}} - X_{n_k})P_u\|_\infty \geq K\|e_{n_{k+1}} - e_{n_k}\| = K\sqrt{2}, \quad k = 1, 2, \ldots.$$

So the following result is of some interest.

THEOREM 2. *Let H be a separable complex Hilbert space, $\mathcal{A}$ a von Neumann algebra in $\mathcal{L}(H)$, ϕ a faithful, normal state on $\mathcal{A}$, and $\mathcal{B}$ a commutative von Neumann subalgebra of $\mathcal{A}$. Let $(X_n : n \geq 1)$ be a sequence in $\mathcal{B}$ such that*

$$\sup_{n\geq 1} \phi(|X_n|^\alpha) < \infty \quad \textit{for some } \alpha > 2, \tag{3.1}$$

$$X_n \xrightarrow{\mathrm{b},\mathcal{B}} O \quad \textit{as } n \to \infty. \tag{3.2}$$

Then there exists a subsequence $(X_{n_k} : k \geq 1)$ of (X_n) such that

$$X_{n_k} \xrightarrow{\mathrm{b},\mathcal{A}} O \quad \textit{as } k \to \infty. \tag{3.3}$$

Proof. There exists a probability space $(\Omega, \mathcal{F}, \mu)$ and an isomorphism $X \mapsto T_X$ of $\mathcal{B}$ onto $L_\infty(\Omega, \mathcal{F}, \mu)$ such that

$$\phi(X) = \int_\Omega T_X(\omega)\, d\mu(\omega)$$

for every X in $\mathcal{B}$. Let $f_n := T_{X_n}$. If A is a measurable set in Ω, then by using Hölder's inequality with $1/p + 1/q = 1$, $p := \alpha/2$, we find

$$\begin{aligned}\phi(|X_n|^2) &= \int_\Omega |f_n|^2\, d\mu = \int_A |f_n|^2\, d\mu + \int_{A^{\mathrm{c}}} |f_n|^2\, d\mu \\ &\leq \sup_{\omega\in A} |f_n(\omega)|^2 + \Big(\int_\Omega |f_n|^\alpha d\mu\Big)^{2/\alpha} \cdot \mu(A^{\mathrm{c}})^{(\alpha-2)/\alpha}.\end{aligned} \tag{3.4}$$

Now, since bundle convergence in $L_\infty(\Omega, \mathcal{F}, \mu)$ is in fact almost sure convergence with respect to μ, by using Egorov's theorem we may construct a measurable set A in Ω such that $\mu(A^{\mathrm{c}})$ is arbitrarily small and $f_n \to 0$ as $n \to \infty$ uniformly on A. Then, by using (3.1) and (3.4), we derive that

$$\phi(|X_n|^2) \to 0 \quad \text{as } n \to \infty.$$

By a classical argument, there exists a subsequence $(X_{n_k} : k \geq 1)$ of (X_n) for which

$$\sum_{k=1}^\infty \phi(|X_{n_k}|^2) < \infty.$$

Then, by [2, Property 3.1, p. 30], we get

$$X_{n_k} \xrightarrow{\mathrm{b},\mathcal{A}} O \quad \text{as } k \to \infty. \ \blacksquare$$

Remark 2. For each $\alpha, 1 \le \alpha < 2$, we can exhibit a sequence $(X_n : n \ge 1)$ in the von Neumann subalgebra $\mathcal{D}$ defined in Section 2 such that

$$\sup_{n\ge1} \phi(|X_n|^\alpha) < \infty, \qquad X_n \xrightarrow{\text{b},\mathcal{D}} O \quad \text{as } n \to \infty,$$

but (X_n) does not admit a subsequence satisfying (3.3). To this end, let

$$X_n := 2^{n/\alpha} P_{e_n}, \quad n = 1, 2, \dots .$$

Then

$$\phi(|X_n|^\alpha) = 2^n \phi(P_{e_n}) = 1 \quad \text{and} \quad X_n \xrightarrow{\text{b},\mathcal{D}} O \quad \text{as } n \to \infty$$

by the same argument as in the proof of Corollary 2. On the other hand,

$$X_n P_u u = 2^{n/\alpha} n^{-1} 2^{-n/2} e_n,$$

where u is given by (2.12). Hence

$$\|X_n P_u\|_\infty \ge \frac{1}{n} 2^{n(1/\alpha - 1/2)} \to \infty \qquad \text{as } n \to \infty.$$

Remark 3. The case $\alpha = 2$ is open.

Theorem 3. *Let H be a separable complex Hilbert space, $\mathcal{A}$ a von Neumann algebra in $\mathcal{L}(H)$, ϕ a faithful, normal state on $\mathcal{A}$, and $\mathcal{B}$ a commutative von Neumann subalgebra of $\mathcal{A}$. Let $(X_n : n \ge 1)$ be a sequence in $\mathcal{B}$ such that*

$$\sup_{n\ge1} \phi(|X_n|) < \infty, \tag{3.5}$$

$$X_n \xrightarrow{\text{b},\mathcal{A}} O \quad \text{as } n \to \infty. \tag{3.6}$$

Then there exists a subsequence (X_{n_k}) of (X_n) such that

$$X_{n_k} \xrightarrow{\text{b},\mathcal{B}} O \quad \text{as } k \to \infty. \tag{3.7}$$

Proof. By (3.6), there exists a bundle $\mathcal{P}$ in $\mathcal{A}$ such that, for each $P \in \mathcal{P}$,

$$\|X_n P\|_\infty \to 0 \quad \text{as } n \to \infty.$$

Let

$$A_n := |X_n|^{1/2}, \quad n = 1, 2, \dots .$$

We get for each $P \in \mathcal{P}$,

$$\begin{aligned} \|A_n P\|_\infty^2 &= \|P A_n^* A_n P\|_\infty = \|P |X_n| P\|_\infty \le \|P\|_\infty \|X_n P\|_\infty \\ &\le \|X_n P\|_\infty \to 0 \qquad \text{as } n \to \infty. \end{aligned}$$

Thus,

$$A_n \xrightarrow{\text{b},\mathcal{A}} O \quad \text{as } n \to \infty. \tag{3.8}$$

By (3.5), we also have

$$\sup_{n\ge1} \phi(A_n^2) < \infty. \tag{3.9}$$

Now, by using [5, Proposition, p. 451], we derive that

$$\phi(A_n) \to 0 \quad \text{as } n \to \infty.$$

Let $B_n := A_n^{1/2} = |X_n|^{1/4}$; since $\phi(B_n^2) \to 0$ as $n \to \infty$, there exists a subsequence $(B_{n_k} : k \geq 1)$ of (B_n) such that

$$\sum_{k=1}^{\infty} \phi(B_{n_k}^2) < \infty.$$

It follows that

$$B_{n_k} = |X_{n_k}|^{1/4} \xrightarrow{\text{b},\mathcal{B}} O \quad \text{as } k \to \infty.$$

Since $\mathcal{B}$ is commutative, we may derive that $X_{n_k} \xrightarrow{\text{b},\mathcal{B}} O$ as $k \to \infty$. Here we took into account that $\mathcal{B}$ is isomorphic to some $L_\infty(\Omega, \mathcal{F}, \mu)$. ∎

Now, the following question arises naturally: In the conclusion (3.7) of Theorem 3, is it possible to replace the subsequence (X_{n_k}) by the whole sequence (X_n)? We shall see in Theorem 4 below that the answer is positive if there exists a conditional expectation $\mathcal{E}$ with respect to ϕ from $\mathcal{A}$ to $\mathcal{B}$.

Before stating Theorem 4, we note the interesting fact that it may happen that $(X_n : n \geq 1)$ is a sequence in $\mathcal{A}$ which is bundle convergent to O in $\mathcal{A}$, but $(\mathcal{E}(X_n) : n \geq 1)$ fails to be bundle convergent to O in both $\mathcal{B}$ and $\mathcal{A}$. To see this, let $\mathcal{A} := L_\infty([0,1], \mathcal{F}, \lambda)$, where $\mathcal{F}$ is the Borel field on $[0,1]$, λ the Lebesgue measure, $\mathcal{B} = \mathbb{C}I_{[0,1]}$, and

$$\phi(X) := \int_0^1 X(t)\, dt, \qquad X \in \mathcal{A}.$$

Now, the conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$ is given by

$$\mathcal{E}(X) = \phi(X) I_{[0,1]}, \quad X \in \mathcal{A}.$$

Since bundle convergence in $\mathcal{A}$ is in fact a.e. convergence with respect to Lebesgue measure, it is easy to exhibit a sequence $(X_n : n \geq 1)$ such that $X_n \to O$ a.e. as $n \to \infty$, but $\int_0^1 X_n(t)\, dt$ fails to converge in $\mathbb{C}$. (Compare [4, Problem 3, p. 101].)

THEOREM 4. *Let H be a separable complex Hilbert space, $\mathcal{A}$ a von Neumann algebra in $\mathcal{L}(H)$, ϕ a faithful and normal state on $\mathcal{A}$, and $\mathcal{B}$ a commutative von Neumann subalgebra of $\mathcal{A}$ such that there exists a conditional expectation $\mathcal{E}$ with respect to ϕ from $\mathcal{A}$ onto $\mathcal{B}$. Then for every sequence $(X_n : n \geq 1)$ of operators in $\mathcal{B}$,*

$$X_n \xrightarrow{\text{b},\mathcal{A}} O \quad \textit{implies} \quad X_n \xrightarrow{\text{b},\mathcal{B}} O \quad \textit{as } n \to \infty. \tag{3.10}$$

Proof. In fact, instead of bundle convergence, it is sufficient to assume only that the sequence (X_n) is almost uniformly convergent to O in $\mathcal{A}$. Then for every natural number k, there exists a projection P_k in $\mathcal{A}$ such that

$$\phi(P_k) > (k-1)/k \quad \text{and} \quad \|X_n P_k\|_\infty \to 0 \quad \text{as } n \to \infty.$$

By using the properties of the conditional expectation $\mathcal{E}$ (see [7, p. 211]), we have

$$\|X_n \mathcal{E}(P_k)\|_\infty = \|\mathcal{E}(X_n P_k)\|_\infty \le \|X_n P_k\|_\infty, \tag{3.11}$$

$$\mathcal{E}(P_k) \text{ is positive}, \quad \phi(\mathcal{E}(P_k)) = \phi(P_k), \tag{3.12}$$

$$\|\mathcal{E}(P_k)\|_\infty \le \|P_k\|_\infty = 1. \tag{3.13}$$

We recall (cf. [1, Théorème 1, p. 118] and the proof of our Theorem 2 above) that there exist a probability space $(\Omega, \mathcal{F}, \mu)$ and an isomorphism $X \mapsto T_X$ of $\mathcal{B}$ onto $L_\infty(\Omega, \mathcal{F}, \mu)$ such that

$$\phi(X) = \int_\Omega T_X(\omega)\, d\mu(\omega), \quad X \in \mathcal{B}.$$

Then

$$\varrho_k := T_{\mathcal{E}(P_k)}, \quad k = 1, 2, \ldots,$$

is a nonnegative function on $L_\infty(\Omega, \mathcal{F}, \mu)$, and it follows from (3.12) and (3.13) that

$$\int_\Omega \varrho_k(\omega)\, d\mu(\omega) > (k-1)/k, \quad \|\varrho_k\|_\infty \le 1. \tag{3.14}$$

Now, let

$$\Omega_k := \{\omega \in \Omega : \varrho_k(\omega) = 0\}, \quad k = 1, 2, \ldots.$$

By (3.14), we have $\mu(\Omega_k) \le 1/k$. It follows from (3.11) that

$$\|T_{X_n} \varrho_k\|_\infty \to 0 \quad \text{as } n \to \infty,\ k = 1, 2, \ldots.$$

This means that

$$T_{X_n} \to O \quad \text{as } n \to \infty \text{ a.e. on } \Omega_k^{\mathrm{c}},\ k = 1, 2, \ldots.$$

Consequently, we have

$$T_{X_n} \to O \quad \text{a.e. on } \bigcup_{k=1}^\infty \Omega_k^{\mathrm{c}},$$

whose complement is a set of μ-measure zero. This completes the proof of (3.10). ■

Remark 4. Corollary 1 in Section 2 is a particular case of Theorem 4. In fact, the mapping from $\mathcal{A} := \mathcal{L}(H)$ to $\mathcal{D}$ which assigns to each operator in $\mathcal{A}$, represented by an infinite matrix with respect to a fixed orthonormal basis $(e_j : j \ge 1)$ in H, the "diagonal part" of its representation, is actually a conditional expectation from $\mathcal{A}$ to $\mathcal{D}$.

REMARK 5. It may happen that there exists no conditional expectation of a von Neumann algebra $\mathcal{A}$ onto its commutative von Neumann subalgebra $\mathcal{B}$. For example, if $\mathcal{A}$ is the von Neumann algebra of all bounded linear operators on $H := L_2(-\infty,\infty)$ and $\mathcal{B} := L_\infty(-\infty,\infty)$ acting on $L_2(-\infty,\infty)$ by pointwise multiplication, then there exists no conditional expectation from $\mathcal{A}$ to $\mathcal{B}$ with respect to any faithful, normal state ϕ. This fact was kindly communicated to us by Professor M. Takesaki in a private letter.

The following theorem is a complement to Theorem 4.

THEOREM 5. *Let $H := L_2(0,1)$ equipped with the Borel sets and Lebesgue measure, $\mathcal{A} := \mathcal{L}(H)$, $\mathcal{B} := L_\infty(0,1)$, and $(e_k : k \ge 1)$ the complex trigonometric system (rearranged into an ordinary sequence). If ϕ is defined on $\mathcal{A}$ by* (2.1), *then there exists a sequence (X_n) in $\mathcal{B}$, bounded in L_∞-norm and such that*

$$X_n \xrightarrow{\text{b},\mathcal{A}} O \quad \textit{as } n \to \infty,$$

but (X_n) fails to be bundle convergent to O in $\mathcal{B}$.

For example, we may use the trigonometric system $\{t \mapsto e^{2\pi i n t} : n \in \mathbb{Z}\}$ as a fixed orthonormal basis in the following rearrangement:

$$\begin{aligned} &e_1(t) := 1, && e_2(t) := e^{2\pi i t}, && e_3(t) := e^{-2\pi i t}, \\ &e_4(t) := e^{2\pi i 2t}, && e_5(t) := e^{-2\pi i 2t}, && \ldots . \end{aligned}$$

Proof. Since $\mathcal{B}$ acts on H by pointwise multiplication, we have

$$(X_n A f)(t) = X_n(t)(Af)(t) \quad \text{a.e.}, \ n \ge 1, \ X_n \in \mathcal{A}, \ f \in H.$$

It follows that

$$\|X_n A f\|_2^2 = \int_0^1 |X_n(t)|^2 |(Af)(t)|^2 \, dt. \tag{3.15}$$

By the reasoning following (2.1), for every $\varepsilon > 0$ there exists a natural number $n_0 = n_0(\varepsilon)$ such that

$$\phi(P_\varepsilon) > 1 - \varepsilon, \quad \text{where} \quad P_\varepsilon := \sum_{j=1}^{n_0} P_{e_j}.$$

Since

$$P_\varepsilon f = \sum_{j=1}^{n_0} (f \mid e_j) e_j, \quad f \in H,$$

we have

$$(P_\varepsilon f)(t) = \sum_{j=1}^{n_0} (f \mid e_j) e_j(t) \quad \text{a.e.} \tag{3.16}$$

Combining (3.15) (with P_ε in place of A) and (3.16) yields

$$\|X_nP_\varepsilon f\|_2^2 = \int_0^1 |X_n(t)|^2 \Big|\sum_{j=1}^{n_0} (f \mid e_j)e_j(t)\Big|^2\,dt.$$

By the Cauchy and then the Bessel inequalities, we find that

$$\|X_nP_\varepsilon f\|_2^2 \le \int_0^1 |X_n(t)|^2 \sum_{j=1}^{n_0} |(f \mid e_j)|^2 \sum_{j=1}^{n_0} |e_j(t)|^2\,dt$$
$$\le n_0\|f\|_2^2 \int_0^1 |X_n(t)|^2\,dt,$$

that is,

$$\|X_nP_\varepsilon\|_\infty \le \sqrt{n_0}\,\|X_n\|_2. \tag{3.17}$$

We recall (cf. (2.1)) that

$$\begin{aligned}\phi(X) &:= \sum_{j=1}^{\infty} 2^{-j}(Xe_j \mid e_j) = \sum_{j=1}^{\infty} 2^{-j}\int_0^1 X(t)e_j(t)\overline{e_j(t)}\,dt \\ &= \sum_{j=1}^{\infty} 2^{-j}\int_0^1 X(t)\,dt = \int_0^1 X(t)\,dt\end{aligned} \tag{3.18}$$

and that bundle convergence in $\mathcal{B}$ coincides with a.e. convergence on the interval $(0,1)$.

Now, it is a routine matter to find a sequence (X_n) of indicators on $(0,1)$ such that

$$\|X_n\|_2 = \|X_n\|_1 \to 0 \quad \text{as } n \to \infty$$

and (X_n) is not convergent to 0 a.e. on (0,1). On the other hand, by (3.17) we have $\|X_nP_\varepsilon\|_\infty \to 0$ as $n \to \infty$, that is,

$$X_n \to O \quad \text{almost uniformly as } n \to \infty.$$

Since (X_n) is bounded, it follows that (X_n) is bundle convergent to O in $\mathcal{A}$. ∎

Remark 6. By comparing Theorems 4 and 5, we see that there cannot exist any conditional expectation with respect to ϕ from $\mathcal{A}$ to $\mathcal{B}$, where ϕ, $\mathcal{A}$, and $\mathcal{B}$ are as in Theorem 5.

On closing, we raise two problems.

Problem 1. In the conclusion of Theorem 2, is it possible to replace the subsequence (X_{n_k}) by the whole sequence (X_n)?

Problem 2. In Theorem 4, is it possible to get rid of the condition that the subalgebra $\mathcal{B}$ is commutative and still have conclusion (3.10)?

Added in proof. The answer to the problem raised in Remark 3 in connection with Theorem 2 is in the negative. In fact, let H, $\mathcal{A}$ and $\mathcal{B}$ be as in Theorem 5. This time we define $X_n(t)$ to be the indicator of the interval $(0, 1/n)$ multiplied by $\sqrt{n}$, $n = 1, 2, \ldots$. Analogously to (3.18) in the proof of Theorem 5, we have

$$\phi(|X_n|^2) = \int_0^1 |X_n(t)|^2 \, dt = 1, \qquad n = 1, 2, \ldots.$$

So, condition (3.1) is satisfied. Since $X_n(t) \to 0$ a.e. as $n \to \infty$, (X_n) is bundle convergent to O in $\mathcal{B}$. On the other hand, no subsequence (X_{n_k}) of (X_n) can be bundle convergent to O in $\mathcal{A}$.

References

[1] J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien* (*Algèbres de von Neumann*), deuxième édition, Gauthier-Villars, Paris, 1969.

[2] E. Hensz, R. Jajte and A. Paszkiewicz, *The bundle convergence in von Neumann algebras and their L_2-spaces*, Studia Math. 120 (1996), 23–46.

[3] R. Jajte, *Strong Limit Theorems in Noncommutative Probability*, Lecture Notes in Math. 1110, Springer, Berlin, 1985.

[4] —, *Strong Limit Theorems in Noncommutative L_2-Spaces*, Lecture Notes in Math. 1477, Springer, Berlin, 1991.

[5] B. Le Gac and F. Móricz, *Beppo Levi and Lebesgue type theorems for bundle convergence in noncommutative L_2-spaces*, in: Oper. Theory Adv. Appl. 127, Birkhäuser, Basel, 2001, 447–464.

[6] A. Paszkiewicz, *Convergence in W^*-algebras*, J. Funct. Anal. 69 (1986), 143–154.

[7] M. Takesaki, *Theory of Operator Algebras II*, Encyclopaedia Math. Sci. 125, Springer, Berlin, 2003.

Barthélemy Le Gac
Université de Provence
Centre de Mathématiques et Informatique
39 rue Joliot-Curie
13453 Marseille Cedex 13, France
E-mail: barthelemy.legac@wanadoo.fr

Ferenc Móricz
Bolyai Institute
University of Szeged
Aradi vértanúk tere 1
6720 Szeged, Hungary
E-mail: moricz@math.u-szeged.hu

Received March 29, 2004;
received in final form September 20, 2004 (7389)