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Continuous Selections in α-Convex Metric Spaces

by

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Summary. The existence of continuous selections is proved for a class of lower semicontinuous multifunctions whose values are closed convex subsets of a complete metric space equipped with an appropriate notion of convexity. The approach is based on the notion of pseudo-barycenter of an ordered n-tuple of points.

1. Introduction. It is well known that in the construction of continuous selections for multifunctions with values in a normed space the notion of barycenter plays a fundamental role.

An analogue of barycenter has been introduced by Michael [9] in nonlinear spaces, in order to extend his classical continuous selection theorem [8]. Further developments can be found in Michael [10] and Curtis [1]. Michael's construction is axiomatic and fairly general. However, as pointed out in [9], the axioms the barycenter has to satisfy are not always easily checked in some concrete situations.

An account of the above and related questions concerning continuous selections can be found in the comprehensive monographs of Repovš and Semenov [15], and Hu and Papageorgiou [7].

In the present paper we investigate continuous selection problems for multifunctions with values in a metric space. Our approach is in the same spirit of Michael [9], yet the barycenter calculus we develop seems more flexible to handle. It is based on an appropriate generalization of the notion of a segment joining two points, as in Dugundji [3], and Curtis [1]. More precisely, let Y be a metric space, and, for a continuous mapping $\alpha : Y \times Y \times [0,1] \to Y$, consider the following conditions (for a more general setting,

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see Section 2):

(i) $\alpha(y_0, y_0, t) = y_0$ for every $y_0 \in Y$ and $t \in [0,1]$;
(ii) $\alpha(y_1, y_2, 0) = y_1$, $\alpha(y_1, y_2, 1) = y_2$ for every $(y_1, y_2) \in Y \times Y$;
(iii) there is $r_\alpha > 0$ such that for every $(y_1, y_2), (\overline{y}_1, \overline{y}_2) \in Y \times Y$ with $d(y_1, \overline{y}_1), d(y_2, \overline{y}_2) < r_\alpha$, one has

$$h(\Lambda_\alpha(y_1, y_2), \Lambda_\alpha(\overline{y}_1, \overline{y}_2)) \le \max\{d(y_1, \overline{y}_1), d(y_2, \overline{y}_2)\}.$$

Here $\Lambda_\alpha(a,b) = \{\alpha(a,b,t) \mid t \in [0,1]\}$, $(a,b) \in Y \times Y$, and h is the Pompeiu–Hausdorff distance.

A metric space Y endowed with a continuous mapping α satisfying (i), (ii) (resp. (i), (ii), (iii)) is called a *convex* (resp. *Lipschitz α-convex*) *metric space*.

A subset A of a convex metric space is said to be *convex* if $\alpha(y_1, y_2, t) \in A$ for every $(y_1, y_2) \in A \times A$ and $t \in [0,1]$.

When Y is normed, (i)–(iii) are certainly satisfied if we set $\alpha(y_1, y_2, t) = (1-t)y_1 + ty_2$, thus Y is Lipschitz α-convex.

Let Y be a convex metric space. For an ordered n-tuple $(y_1, \ldots, y_n)$ of points $y_i \in Y$, with corresponding weights $(\lambda_1, \ldots, \lambda_n)$, where $\lambda_1 + \ldots + \lambda_n = 1$, $0 \le \lambda_i \le 1$, $i = 1, \ldots, n$, we use the notion of *pseudo-barycenter* $b_n(y_1, \ldots, y_n; \lambda_1, \ldots, \lambda_n)$ introduced in [2]. It keeps only a few properties of the barycenter, yet it turns out to be useful in continuous selection problems, where partition of unity techniques are employed.

Let M be a paracompact space, Y a complete metric space, 2^Y the family of all nonempty subsets of Y, and consider a multifunction $F : M \to 2^Y$. By a *selection* of F we mean any function $f : M \to Z$ satisfying $f(x) \in F(x)$ for every $x \in M$.

The following metric version of the classical continuous selection theorem due to Michael [8] will be established. If Y is Lipschitz α-convex, and F is lower semicontinuous in the sense of Michael [8], with closed convex values, then F admits a continuous selection. Actually the existence of continuous selections will be proved under the weaker assumption that Y is α-convex (see Definition 2.1).

We mention that in nonlinear spaces some other notions of convexity have been developed by Pasicki [13] and van de Vel [16], who have obtained, among many results, also a nonlinear version of Michael's continuous selection theorem.

It is worthwhile to point out that in our approach to convexity in metric spaces our major concern was to identify a minimum set of readily verifiable conditions, under which a flexible barycenter calculus could be developed. Conditions (i)–(iii) are perhaps not general enough, yet they are easily verifiable and also useful in some applications. In particular, condition (iii) makes it possible to show that our pseudo-barycenter is actually stable in the sense of Proposition 2.3, a crucial property in selection problems, which some authors introduce as an axiom.

The present paper consists of four sections. Section 2 contains notation and preliminaries, including a review of some properties of pseudo-barycenters established in [2]. A metric version of the Michael continuous selection theorem in an α-convex metric space is established in Section 3. Some examples of α-convex metric spaces are considered in Section 4.

2. Notation and preliminaries. Let Z be a metric space with distance d, and let 2^Z be the family of all nonempty subsets of Z. The open ball in Z with center $a \in Z$ and radius $r > 0$ is denoted by $B_Z(a, r)$. For $A \in 2^Z$ and $r > 0$, set

$$N_Z(A, r) = \{z \in Z \mid d(z, A) < r\} \quad \text{where} \quad d(z, A) = \inf_{a \in A} d(z, a).$$

The space of nonempty closed bounded subsets of Z is endowed with the *Pompeiu–Hausdorff metric*

$$h(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}.$$

For any nonempty set A we put $A^n = A \times \ldots \times A$, and denote by $(a_1, \ldots, a_n)$ an element of A^n, i.e. an ordered n-tuple of points $a_i \in A$, $i = 1, \ldots, n$.

For any map $\alpha : Y \times Y \times [0, 1] \to Y$ and an ordered pair $(y_1, y_2) \in Y \times Y$, we define the (y_1, y_2)-*locus* induced by α to be the set

$$\Lambda_\alpha(y_1, y_2) = \{y \in Y \mid y = \alpha(y_1, y_2, t) \text{ for some } t \in [0, 1]\}.$$

Definition 2.1. Let Y be a metric space. For a continuous mapping $\alpha : Y \times Y \times [0, 1] \to Y$, consider the following conditions:

(i) $\alpha(y_0, y_0, t) = y_0$ for every $y_0 \in Y$ and $t \in [0, 1]$;

(ii) $\alpha(y_1, y_2, 0) = y_1$, $\alpha(y_1, y_2, 1) = y_2$ for every $(y_1, y_2) \in Y \times Y$;

(iii) there is $r_\alpha > 0$ such that for every $0 < \varepsilon < r_\alpha$, there exists $0 < \eta \le \varepsilon$ such that $(y_1, y_2), (\overline{y}_1, \overline{y}_2) \in Y \times Y$ with $d(y_1, \overline{y}_1) < \varepsilon$ and $d(y_2, \overline{y}_2) < \eta$, one has

$$h(\Lambda_\alpha(y_1, y_2), \Lambda_\alpha(\overline{y}_1, \overline{y}_2)) < \varepsilon.$$

The space Y equipped with a continuous mapping $\alpha : Y \times Y \times [0, 1] \to Y$ which satisfies (i), (ii), (iii) (resp. (i), (ii)) is called an α-*convex* (resp. *convex*) *metric space*.

Instead of (iii) one can consider the following conditions:

(iii)′ there is $r_\alpha > 0$ such that for all $(y_1, y_2), (\overline{y}_1, \overline{y}_2) \in Y \times Y$ with $d(y_i, \overline{y}_i) < r_\alpha$, $i = 1, 2$, one has

$$h(\Lambda_\alpha(y_1, y_2), \Lambda_\alpha(\overline{y}_1, \overline{y}_2)) \le \max\{d(y_1, \overline{y}_1), d(y_2, \overline{y}_2)\};$$

(iii)$''$ there is $r_\alpha > 0$ such that for every $0 < \varepsilon < r_\alpha$ there exists $0 < \eta \leq \varepsilon$ such that for all $(y_1, y_2), (\overline{y}_1, \overline{y}_2) \in Y \times Y$ with $d(y_1, \overline{y}_1) < \varepsilon$ and $d(y_2, \overline{y}_2) < \eta$, one has

$$d(\alpha(y_1, y_2, t), \alpha(\overline{y}_1, \overline{y}_2, t)) < \varepsilon \quad \text{for every } t \in [0,1].$$

DEFINITION 2.2. Let $\alpha : Y \times Y \times [0,1] \to Y$ be a continuous mapping which, in addition to conditions (i) and (ii) of Definition 2.1, satisfies also (iii)$'$ (resp. (iii)$''$). Then Y equipped with the mapping α is called a *Lipschitz* (resp. *geodesically*) *α-convex metric space*.

In the above definitions, α is also called the *convexity mapping* of Y.

REMARK 2.1. The notion of geodesically α-convex space is similar to the notion of geodesic structure, introduced by Michael in [9], where α is continuous in t and satisfies some additional conditions which include (i), (ii) and (iii)$''$. It is worthwhile to observe that, from (iii)$''$ and the continuity of α in t, it follows that α is actually continuous in (y_1, y_2, t), as required in Definition 2.2.

REMARK 2.2. The following implications are immediate: Y is Lipschitz α-convex $\Rightarrow$ Y is α-convex $\Rightarrow$ Y is convex. Moreover, Y is geodesically α-convex $\Rightarrow$ Y is α-convex. The opposite implications are not true in general (see Examples 4.1, 4.2 below and [2, Example 3.1]). For examples of Lipschitz α-convex metric spaces see Example 4.3 and [2, Example 3.2.]

DEFINITION 2.3. A nonempty set A contained in an α-convex (or convex) metric space Y is called *convex* if $(y_1, y_2) \in A \times A$ and $t \in [0,1]$ imply $\alpha(y_1, y_2, t) \in A$.

REMARK 2.3. Let Y be an α-convex (or convex) metric space. Then the intersection of a family of convex subsets of Y is convex (the empty set is assumed to be convex). Moreover, Y and the singleton subsets of Y are convex. The closure of a convex subset of Y is also convex.

We now recall the notion of pseudo-barycenter introduced in [2] and review some of its properties.

Throughout, Y stands for an α-convex metric space. For $n \geq 1$, set

$$Y^n = \{(y_1, \dots, y_n) \mid y_i \in Y,\ i = 1, \dots, n\}$$
$$\Sigma^n = \{(\lambda_1, \dots, \lambda_n) \mid 0 \leq \lambda_i \leq 1,\ i = 1, \dots, n,\ \lambda_1 + \dots + \lambda_n = 1\}.$$

If $(y_1, \dots, y_n) \in Y^n$ and $(\lambda_1, \dots, \lambda_n) \in \Sigma^n$ are given, we say that λ_i is the *weight* assigned to y_i, $i = 1, \dots, n$, or, for brevity, that $(\lambda_1, \dots, \lambda_n)$ is the weight assigned to $(y_1, \dots, y_n)$.

DEFINITION 2.4. Let Y be an α-convex metric space. For $(y_1, \dots, y_n) \in Y^n$ and $(\lambda_1, \dots, \lambda_n) \in \Sigma^n$, $n \geq 1$, the *pseudo-barycenter* $b_n(y_1, \dots, y_n;$

$\lambda_1, \dots, \lambda_n)$ is defined as follows:

$$b_1(y_1; \lambda_1) = y_1 \quad \text{if } y_1 \in Y^1 \text{ and } \lambda_1 \in \Sigma^1, \text{ i.e. } \lambda_1 = 1,$$

and, for $n \geq 2$,

$$b_n(y_1, \dots, y_n; \lambda_1, \dots, \lambda_n)$$
$$= \begin{cases} y_n & \text{if } \lambda_n = 1, \\ \alpha\left(b_{n-1}\left(y_1, \dots, y_{n-1}; \dfrac{\lambda_1}{1-\lambda_n}, \dots, \dfrac{\lambda_{n-1}}{1-\lambda_n}\right), y_n, \lambda_n\right) & \text{if } \lambda_n < 1. \end{cases}$$

Observe that for $n = 2$ one has

$$b_2(y_1, y_2; \lambda_1, \lambda_2) = \alpha(y_1, y_2, \lambda_2) \quad \text{if } (y_1, y_2) \in Y^2 \text{ and } (\lambda_1, \lambda_2) \in \Sigma^2.$$

The following properties of the pseudo-barycenter have been proved in [2].

PROPOSITION 2.1. *For fixed $(y_1, \dots, y_n) \in Y^n$, $n \geq 1$, the pseudo-barycenter $b_n(y_1, \dots, y_n;\ \lambda_1, \dots, \lambda_n)$ is a continuous function of the weight $(\lambda_1, \dots, \lambda_n) \in \Sigma^n$.*

PROPOSITION 2.2. *Let $(y_1, \dots, y_n) \in Y^n$ and $(\lambda_1, \dots, \lambda_n) \in \Sigma^n$, $n \geq 2$. Let $i_1, \dots, i_k$, $1 \leq k \leq n-1$, be a subset of $\{1, \dots, n\}$ with $i_1 < \dots < i_k$ such that*

$$\lambda_i > 0 \quad \text{if } i \in \{i_1, \dots, i_k\}, \qquad \lambda_i = 0 \quad \text{if } i \in \{1, \dots, n\} \setminus \{i_1, \dots, i_k\}.$$

Then

$$b_n(y_1, \dots, y_n; \lambda_1, \dots, \lambda_n) = b_k(y_{i_1}, \dots, y_{i_k}; \lambda_{i_1}, \dots, \lambda_{i_k}).$$

PROPOSITION 2.3. *Let r_α correspond to Y as in Definition 2.1. For each $0 < \varepsilon < r_\alpha$ there exists $0 < \eta \leq \varepsilon$ such that for all $(y_1, \dots, y_n), (z_1, \dots, z_n) \in Y^n$, $n \geq 2$ arbitrary, with $d(y_i, z_i) < \eta$, $i = 1, \dots, n$, one has*

$$h(\Lambda_\alpha(y_1, \dots, y_n), \Lambda_\alpha(z_1, \dots, z_n)) < \varepsilon,$$

where

$$\Lambda_\alpha(y_1, \dots, y_n) = b_n(y_1, \dots, y_n; \Sigma^n), \qquad \Lambda_n(z_1, \dots, z_n) = b_n(z_1, \dots, z_n; \Sigma^n).$$

Moreover, for every nonempty α-convex set $C \subset Y$, $(y_1, \dots, y_n) \in Y^n$ with $d(y_i, C) < \eta$, $i = 1, \dots, n$, and $(\lambda_1, \dots, \lambda_n) \in \Sigma^n$, $n \geq 2$ arbitrary, one has

$$d(b_n(y_1, \dots, y_n; \lambda_1, \dots, \lambda_n), C) < \varepsilon.$$

REMARK 2.4. In a convex metric space the above definition of pseudo–barycenter makes sense, and Propositions 2.1 and 2.2 remain valid. Condition (iii) of Definition 2.1 plays a crucial role in showing that the pseudo–barycenter is stable in the sense of Proposition 2.3.

PROPOSITION 2.4. (Dugundji [4, p. 83]). *Let M and Z be topological spaces. Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a covering of M, where the sets $A_\lambda \subset M$ are open,* *and let* $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ *be a family of continuous functions* $\varphi_\lambda : A_\lambda \to Z$ *such that for any* $\lambda, \mu \in \Lambda$ *with* $A_\lambda \cap A_\mu \neq \emptyset$,

$$\varphi_\lambda(x) = \varphi_\mu(x) \quad \textit{for every } x \in A_\lambda \cap A_\mu.$$

Then there is a unique continuous function $f : M \to Z$ *which is an extension of each* φ_λ, *that is*, *for each* $\lambda \in \Lambda$,

$$f(x) = \varphi_\lambda(x) \quad \textit{for every } x \in A_\lambda.$$

3. Selections in α-convex metric spaces. In this section we establish a Michael type continuous selection theorem for multifunctions with values in an α-convex metric space.

Definition 3.1. Let M and Z be topological spaces. A multifunction $F : M \to 2^Z$ is called *lower semicontinuous* if, for every open $A \subset Z$, the set $\{x \in M \mid F(x) \cap A \neq \emptyset\}$ is open in M.

The following proposition is known yet, for the sake of completeness, the proof is included.

Proposition 3.1. *Let* M *be a topological space*, *and* Z *a metric space. Let* $F : M \to 2^Z$ *be lower semicontinuous*, $g : X \to Z$ *continuous*, *and let* $\theta : M \to (0,\infty)$ *be a lower semicontinuous function satisfying* $\theta(x) > d(g(x), F(x))$ *for every* $x \in M$. *Then the multifunction* $\Phi : M \to 2^Z$ *given by*

$$\Phi(x) = F(x) \cap B_Z(g(x), \theta(x)) \quad \textit{for every } x \in M$$

is lower semicontinuous.

Proof. For each open $A \subset Z$ the set $U = \{x \in M \mid \Phi(x) \cap A \neq \emptyset\}$ is open in M. In fact, let $x_0 \in U$ (if $U = \emptyset$ there is nothing to prove), and take $y_0 \in \Phi(x_0) \cap A$. Since g is continuous, θ lower semicontinuous and A open, there exist $\sigma > 0$ and an open neighborhood V of x_0 such that

$$B_Z(y_0, \sigma) \subset B_Z(g(x), \theta(x)) \cap A \quad \text{for every } x \in V. \tag{3.1}$$

The set $\{x \in M \mid F(x) \cap B_Z(y_0, \sigma) \neq \emptyset\}$ is open, for F is lower semicontinuous, and it contains x_0, since $y_0 \in F(x_0)$. Thus there is an open neighborhood W of x_0, $W \subset V$, such that $x \in W$ implies $F(x) \cap B_Z(y_0, \sigma) \neq \emptyset$. Combining the latter with (3.1) gives $\Phi(x) \cap A \neq \emptyset$ for every $x \in W$. As $x_0 \in U$ is arbitrary the set U is open and thus Φ is lower semicontinuous.

Theorem 3.1. *Let* M *be a paracompact space*, *and* Y *an* α-*convex complete metric space. Then every lower semicontinuous multifunction* $F : M \to 2^Y$ *with closed convex values admits a continuous selection.*

Proof. Let α be the convexity mapping of Y, and let r_α be as in Definition 2.1. By Proposition 2.3 there exist strictly decreasing sequences $\{\eta_n\}$

and $\{\theta_n\}$ with

$$0 < \eta_n < \theta_n/2, \quad \theta_n < \varepsilon_n = r_\alpha/2^n, \quad n = 1, 2, \ldots,$$

for which the following two conditions (S_1), (S_2) are satisfied.

(S_1) For every nonempty convex $A \subset Y$, $(y_1, \ldots, y_p) \in Y^p$, $(\lambda_1, \ldots, \lambda_p) \in \Sigma^p$, with $p \geq 1$ arbitrary, $d(y_i, A) < \theta_n$, $i = 1, \ldots, p$, implies

$$d(b_p(y_1, \ldots, y_p; \lambda_1, \ldots, \lambda_p), A) < \varepsilon_n, \quad n \in \mathbb{N}.$$

(S_2) For every nonempty convex $A' \subset Y$, $(y'_1, \ldots, y'_p) \in Y^p$, $(\lambda'_1, \ldots, \lambda'_p) \in \Sigma^p$, with $p \geq 1$ arbitrary, $d(y'_i, A) < \eta_n$, $i = 1, \ldots, p$, implies

$$d(b_p(y'_1, \ldots, y'_p; \lambda'_1, \ldots, \lambda'_p), A) < \theta_n/2, \quad n \in \mathbb{N}.$$

STEP 1. We will construct a continuous $f_1 : M \to Y$ satisfying

$$d(f_1(x), F(x)) < \theta_1/2 \quad \text{for every } x \in M. \tag{3.2}$$

Following Michael [8] for $y \in Y_0$, where $Y_0 = \bigcup_{x \in M} F(x)$, set

$$U_y = \{x \in M \mid F(x) \cap B_Y(y, \eta_1) \neq \emptyset\}.$$

Then U_y is open, for F is lower semicontinuous, and thus $\mathcal{U} = \{U_y\}_{y \in Y_0}$ is an open covering of M. As M is paracompact, $\mathcal{U}$ has a neighborhood finite refinement $\mathcal{V} = \{V_\beta\}_{\beta \in B}$. For $V_\beta \in \mathcal{V}$, the set

$$\mathcal{F}(V_\beta) = \{U_y \in \mathcal{U} \mid V_\beta \subset U_y\}$$

is nonempty. In each $\mathcal{F}(V_\beta)$ fix one set, say $U_{y(V_\beta)}$, thus

$$V_\beta \subset U_{y(V_\beta)} \quad \text{for every } \beta \in B. \tag{3.3}$$

By Dugundji [4, p. 170], there is a partition $\{p_{V_\beta}\}_{\beta \in B}$ of unity subordinate to $\mathcal{V}$, i.e. a family of continuous functions $p_{V_\beta} : M \to [0, 1]$ such that:

(j) $\operatorname{supp} p_{V_\beta} \subset V_\beta$ for every $\beta \in B$;
(jj) $\{\operatorname{supp} p_{V_\beta}\}_{\beta \in B}$ is a neighborhood finite closed covering of M;
(jjj) $\sum_{\beta \in B} p_{V_\beta}(x) = 1$ for every $x \in M$.

By Zermelo's theorem [4, p. 31], $\mathcal{V}$ admits a partial ordering $\prec$ which makes $\mathcal{V}$ into a well ordered set.

Let $u \in M$ be arbitrary. Since $\mathcal{V}$ is neighborhood finite, there exists an open neighborhood W_u of u such that the family

$$\mathcal{V}_{W_u} = \{V_\beta \in \mathcal{V} \mid V_\beta \cap W_u \neq \emptyset\}$$

is nonempty and finite and thus, for some $k \in \mathbb{N}$,

$$\mathcal{V}_{W_u} = (V_{\beta_1}, \ldots, V_{\beta_k}) \quad \text{where} \quad V_{\beta_1} \prec \cdots \prec V_{\beta_k}. \tag{3.4}$$

Let

$$(y(V_{\beta_1}), \ldots, y(V_{\beta_k})), \quad (U_{y(V_{\beta_1})}, \ldots, U_{y(V_{\beta_k})})$$

correspond according to (3.3).

Now define $\varphi_{W_u} : W_u \to Y$ by

$$\varphi_{W_u}(x) = b_k(y(V_{\beta_1}), \dots, y(V_{\beta_k}); p_{V_{\beta_1}}(x), \dots, p_{V_{\beta_k}}(x)), \quad x \in W_u. \tag{3.5}$$

By Proposition 2.1, φ_{W_u} is well defined and continuous.

We have

$$d(\varphi_{W_u}(x), F(x)) < \theta_1/2, \quad x \in W_u. \tag{3.6}$$

In fact, let $x \in W_u$. The set $\mathcal{V}^x_{W_u}$ of all $V_\beta \in \mathcal{V}$ such that $p_{V_\beta}(x) > 0$ is a nonempty subset of $\mathcal{V}_{W_u}$, as $x \in \operatorname{supp} p_{V_\beta} \subset V_\beta$. Hence, for some $1 \le p \le k$ and $1 \le i_1 < \dots < i_p \le k$, one has

$$\mathcal{V}^x_{W_u} = (V_{\beta_{i_1}}, \dots, V_{\beta_{i_p}}) \quad \text{where} \quad V_{\beta_{i_1}} \prec \dots \prec V_{\beta_{i_p}},$$

and so Proposition 2.2 implies

$$\varphi_{W_u}(x) = b_p(y(V_{\beta_{i_1}}), \dots, y(V_{\beta_{i_p}}); p_{V_{\beta_{i_1}}}(x), \dots, p_{V_{\beta_{i_p}}}(x)). \tag{3.7}$$

On the other hand, for $s = 1, \dots, p$,

$$x \in \operatorname{supp} p_{V_{\beta_{i_s}}} \subset V_{\beta_{i_s}} \subset U_{y(V_{\beta_{i_s}})} = \{z \in M \mid F(z) \cap B_Y(y(V_{\beta_{i_s}}), \eta_1) \neq \emptyset\},$$

and consequently

$$d(y(V_{\beta_{i_s}}), F(x)) < \eta_1, \quad s = 1, \dots, p.$$

Then, by (S_2) (with $n = 1$),

$$d(b_p(y(V_{\beta_{i_1}}), \dots, y(V_{\beta_{i_p}});\ p_{V_{\beta_{i_1}}}(x), \dots, p_{V_{\beta_{i_p}}}(x)), F(x)) < \theta_1/2,$$

and, by (3.7), as $x \in W_u$ is arbitrary, (3.6) follows.

Now define $f_1 : M \to Y$ by

$$f_1(x) = \varphi_{W_u}(x) \quad \text{for every } x \in W_u \text{ and } u \in M. \tag{3.8}$$

Each φ_{W_u}, $u \in M$, is continuous, hence, by Proposition 2.4, f_1 is well defined and continuous if we show that for any W_u, $W_{u'}$ with $W_u \cap W_{u'} \neq \emptyset$ we have

$$\varphi_{W_u}(x) = \varphi_{W_{u'}}(x) \quad \text{for every } x \in W_u \cap W_{u'}. \tag{3.9}$$

In fact, let $x \in W_u \cap W_{u'}$. Clearly,

$$\begin{aligned} \mathcal{V}^x_{W_u} &= \{V_{\beta_i} \in \mathcal{V}_{W_u} \mid p_{V_{\beta_i}}(x) > 0\} \\ &= \{V_{\beta_i} \in \mathcal{V}_{W_{u'}} \mid p_{V_{\beta_i}}(x) > 0\} = \mathcal{V}^x_{W_{u'}}. \end{aligned} \tag{3.10}$$

By (3.4) and (3.10), for some $1 \le p \le k$ and $1 \le i_1 < \dots < i_p \le k$, one has $\mathcal{V}^x_{W_u} = \mathcal{V}^x_{W_{u'}} = (V_{\beta_{i_1}}, \dots, V_{\beta_{i_p}})$ where $V_{\beta_{i_1}} \prec \dots \prec V_{\beta_{i_p}}$. Thus (3.9) follows, for

$$\varphi_{W_u}(x) = b_p(y(V_{\beta_{i_1}}), \dots, y(V_{\beta_{i_p}}); p_{V_{\beta_{i_1}}}, \dots, p_{V_{\beta_{i_p}}}(x)) = \varphi_{W_{u'}}(x),$$

and thus f_1 is well defined and continuous. Further, in view of (3.8) and (3.6), f_1 satisfies (3.2), and Step 1 is proved.

STEP 2. Assume that $n \geq 2$ continuous functions $f_i : M \to Y$ have been defined such that:

$$(3.11)_i \qquad d(f_i(x), F(x)) < \theta_i/2 \quad \text{for every } x \in M,\ i = 1, \dots, n,$$

$$(3.12)_i \qquad d(f_i(x), f_{i-1}(x)) < \varepsilon_{i-1} \quad \text{for every } x \in M,\ i = 2, \dots, n.$$

Then there exists a continuous $f_{n+1} : M \to Y$ so that $(3.11)_{n+1}$ and $(3.12)_{n+1}$ are satisfied.

By Step 1, there exists a continuous $f_1 : M \to Y$ for which $(3.11)_1$ holds. The proof of the existence of a continuous $f_2 : M \to Y$ satisfying $(3.11)_2$ and $(3.12)_2$ is similar to the induction argument of Step 2, and therefore it is omitted.

Now suppose that $n \geq 2$ continuous functions $f_i : M \to Y$ have been defined satisfying $(3.11)_i$, $i = 1, \dots, n$, and $(3.12)_i$, $i = 2, \dots, n$. To construct $f_{n+1} : M \to Y$, define $\Phi_n : M \to 2^Y$ by

$$\Phi_n(x) = F(x) \cap B_Y(f_n(x), \theta_n/2) \quad \text{for every } x \in M.$$

By Proposition 3.1, Φ_n is lower semicontinuous with nonempty (not necessarily convex) values.

For $y \in Y_0 = \bigcup_{x \in M} F(x)$, set

$$U_y = \{x \in M \mid \Phi_n(x) \cap B_Y(y, \eta_{n+1}) \neq \emptyset\}.$$

Since $\mathcal{U} = \{U_y\}_{y \in Y_0}$ is an open covering of M, a paracompact space, $\mathcal{U}$ admits a neighborhood finite refinement, say $\mathcal{V} = \{V_\beta\}_{\beta \in B}$. As in Step 1, associate with each $V_\beta \in \mathcal{V}$ a set $U_{y(V_\beta)} \in \mathcal{U}$ for which (3.3) holds. Let $\{p_{V_\beta}\}_{\beta \in B}$ be a partition of unity subordinate to $\mathcal{V}$, and equip $\mathcal{V}$ with a partial ordering $\prec$ which makes $\mathcal{V}$ into a well ordered set.

Let $u \in M$ be arbitrary. Since $\mathcal{V}$ is neighborhood finite, there exists an open neighborhood W_u of u such that the family $\mathcal{V}_{W_u} = \{V_\beta \in \mathcal{V} \mid V_\beta \cap W_u \neq \emptyset\}$ is nonempty and finite. Let $\mathcal{V}_{W_u}$ be given by (3.4), and let $(y(V_{\beta_1}), \dots, y(V_{\beta_k}))$, $(U_{y(V_{\beta_1})}, \dots, U_{y(V_{\beta_k})})$ satisfy (3.3). As in Step 1, one can show that the function $\varphi_{W_u} : W_u \to Y$ given by (3.5) is well defined and continuous.

Furthermore, we have:

$$(3.13) \qquad d(\varphi_{W_u}(x), F(x)) < \theta_{n+1}/2 \quad \text{for every } x \in W_u,$$

$$(3.14) \qquad d(\varphi_{W_u}(x), f_n(x)) < \varepsilon_n \quad \text{for every } x \in W_u.$$

Indeed to show (3.13), let $x \in W_u$. The set $\mathcal{V}^x_{W_u}$ of all $V_\beta \in \mathcal{V}$ such that $p_{V_\beta}(x) > 0$ is a nonempty subset of $\mathcal{V}_{W_u}$, whence, for some $1 \leq p \leq k$ and $1 \leq i_1 < \dots < i_p \leq k$, one has $\mathcal{V}^x_{W_u} = (V_{\beta_{i_1}}, \dots, V_{\beta_{i_p}})$ where $V_{\beta_{i_1}} \prec \dots \prec V_{\beta_{i_p}}$. Thus, by Proposition 2.2,

$$(3.15) \qquad \varphi_{W_u}(x) = b_p(y(V_{\beta_{i_1}}), \dots, y(V_{\beta_{i_p}}); p_{V_{\beta_{i_1}}}(x), \dots, p_{V_{\beta_{i_p}}}(x)).$$

Since for $s = 1, \ldots, p$,

$x \in \operatorname{supp} p_{V_{i_s}} \subset V_{\beta_{i_s}} \subset U_{y(V_{\beta_{i_s}})} = \{z \in M \mid \Phi_n(z) \cap B_Y(y(V_{\beta_{i_s}}), \eta_{n+1}) \neq \emptyset\}$,

it follows that

$$d(y(V_{\beta_{i_s}}), \Phi_n(x)) < \eta_{n+1}, \qquad s = 1, \ldots, p, \tag{3.16}$$

and, a fortiori,

$$d(y(V_{\beta_{i_s}}), F(x)) < \eta_{n+1}, \qquad s = 1, \ldots, p.$$

Then, by (S_2) (with $n+1$ in place of n), one has

$$d(b_p(y(V_{\beta_{i_1}}), \ldots, y(V_{\beta_{i_p}}); p_{V_{\beta_{i_1}}}(x), \ldots, p_{V_{\beta_{i_p}}}(x)), F(x)) < \theta_{n+1}/2$$

and, in view of (3.15), as $x \in W_u$ is arbitrary, (3.13) follows.

To show (3.14), let $x \in W_u$. Since $\Phi_n(x) \subset B_Y(f_n(x), \theta_n/2)$, in view of (3.16) one has

$$d(y(V_{\beta_{i_s}}), f_n(x)) < \eta_{n+1} + \theta_n/2 < \theta_{n+1}/2 + \theta_n/2 < \theta_n, \qquad s = 1, \ldots, p,$$

for $\eta_{n+1} < \theta_{n+1}/2 < \theta_n/2$. Then, by (S_1),

$$d(b_p(y(V_{\beta_{i_1}}), \ldots, y(V_{\beta_{i_p}}); p_{V_{\beta_{i_1}}}(x), \ldots, p_{V_{\beta_{i_p}}}(x)), f_n(x)) < \varepsilon_n$$

and thus, by (3.15), as $x \in W_u$ is arbitrary, (3.14) follows.

Now define $f_{n+1} : M \to Y$ by

$$f_{n+1}(x) = \varphi_{W_u}(x) \quad \text{for every } x \in W_u \text{ and } u \in M.$$

As in Step 1, one can show that f_{n+1} is well defined and continuous. By (3.13) and (3.14), f_{n+1} satisfies $(3.11)_{n+1}$ and $(3.12)_{n+1}$, and Step 2 is proved.

By Step 2, there exists a sequence $\{f_n\}$ of continuous functions $f_n : M \to Y$ satisfying the following conditions:

$$d(f_n(x), F(x)) < \theta_n/2 \quad \text{for every } x \in M,\ n = 1, 2, \ldots,$$
$$d(f_{n+1}(x), f_n(x)) < \varepsilon_n \quad \text{for every } x \in M,\ n = 1, 2, \ldots.$$

Since $\{f_n\}$ is Cauchy and Y is complete, $\{f_n\}$ converges to a continuous function $f : M \to Y$. Clearly $f(x) \in F(x)$ for every $x \in M$, thus f is a continuous selection of F. This completes the proof.

COROLLARY 3.1. *Let D be a nonempty closed subset of a paracompact space M, and let Y be an α-convex complete metric space. Let $\varphi : D \to Y$ be a continuous function. Then φ admits a continuous extension on M, that is, there exists a continuous function $f : M \to Y$ such that $f(x) = \varphi(x)$ for every $x \in D$.*

Proof. Define $F : M \to 2^Y$ by

$$F(x) = \begin{cases} \{\varphi(x)\} & \text{if } x \in D, \\ Y & \text{if } x \in M \setminus D. \end{cases}$$

Then F is lower semicontinuous, and takes closed convex values, by Remark 2.3. By Theorem 3.1, F admits a continuous selection $f : M \to Y$, which is an extension of φ on M.

The following corollary is a variant of a result obtained by Pasicki [12].

COROLLARY 3.2. *Let Y be an α-convex complete metric space. Then each nonempty closed α-convex set $C \subset Y$ is an absolute retract.*

Proof. It suffices to show that for every metrizable X and closed $D \subset X$, $D \neq \emptyset$, each continuous function $\varphi : D \to C$ admits a continuous extension $f : X \to C$. This follows from Corollary 3.1, because C is an α-convex complete metric space and X is paracompact, by Stone's theorem [4, p. 156].

By the generalized Schauder theorem [5, p. 94] and Corollary 3.2 we have

COROLLARY 3.3. *Let Y be an α-convex compact metric space. Then each continuous function $f : Y \to Y$ has a fixed point.*

REMARK 3.1. In Michael's selection theorem, when F takes values in a Banach space, the lower semicontinuity assumption on F can be relaxed, as shown by Gutev [6] and Przesławski and Rybiński [14]. In our α-convex metric space setting it is not clear if a similar relaxation of lower semicontinuity is also possible.

4. Some examples of convex metric spaces. For $p, q \in \mathbb{R}^2$ we denote by $[p, q]$ the closed (unoriented) linear segment with end points p, q. We put $J = [0, 1]$.

EXAMPLE 4.1. Let $Y = \mathbb{R}^2$ be equipped with the metric d induced by the Euclidean norm of $\mathbb{R}^2$. For $n \in \mathbb{N}$, set $a_n = (n, 0)$, $b_n = (n, 1)$, $\overline{a}_n = (n + 1/n, 0)$, $\overline{b}_n = (n + 1/n, 1)$, $\overline{c}_n = (\overline{a}_n + \overline{b}_n)/2 + (1, 0)$. Define now

$$\beta(p,q,t) =$$
$$\begin{cases} p & \text{if } (p,q,t) \in A_0 = \{(p,q,t) \in Y \times Y \times J \mid t = 0\}, \\ q & \text{if } (p,q,t) \in A_1 = \{(p,q,t) \in Y \times Y \times J \mid t = 1\}, \\ (1-t)a_n + tb_n & \text{if } (p,q,t) \in S_n = \{(a_n, b_n, t) \in Y \times Y \times J \mid t \in J\},\ n \in \mathbb{N}, \\ \gamma(\overline{a}_n, \overline{b}_n, t) & \text{if } (p,q,t) \in T_n = \{(\overline{a}_n, \overline{b}_n, t) \in Y \times Y \times J \mid t \in J\},\ n \in \mathbb{N}, \end{cases}$$

where

$$\gamma(\overline{a}_n, \overline{b}_n, t) = \begin{cases} (1-2t)\overline{a}_n + 2t\overline{c}_n, & t \in [0, 1/2], \\ (2-2t)\overline{c}_n + (2t-1)\overline{b}_n, & t \in [1/2, 1]. \end{cases}$$

Since β is continuous on $A_0 \cup A_1 \cup \bigcup_{n \in \mathbb{N}} (S_n \cup T_n)$, a closed set, and takes values in Y, by Dugundji's theorem [4, p. 188], β admits a continuous extension, say α, defined on $Y \times Y \times J$ with values in Y.

By construction α satisfies conditions (i) and (ii) of Definition 2.1, and hence Y is a convex metric space. However Y is not α-convex. In fact

$\Lambda_\alpha(a_n, b_n) = [a_n, b_n]$, $\Lambda_\alpha(\overline{a}_n, \overline{b}_n) = [\overline{a}_n, \overline{c}_n] \cup [\overline{c}_n, \overline{b}_n]$, and thus

$$h(\Lambda_\alpha(a_n, b_n), \Lambda_\alpha(\overline{a}_n, \overline{b}_n)) > 1 \quad \text{for every } n \in \mathbb{N}.$$

Since $d(a_n, \overline{a}_n) = d(b_n, \overline{b}_n) = 1/n$, it follows that condition (iii) of Definition 2.1 is not satisfied, and hence Y is not α-convex.

EXAMPLE 4.2. Consider the Riemannian manifold $S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$, $x = (x_1, x_2, x_3)$, and denote by d the Riemannian metric of S^2. Let $Y = \{x \in S^2 \mid x_1^2 + x_2^2 < \sin^2 \pi/8\}$. For $(p, q) \in Y^2$ denote by $g_{p,q} : J \to Y$ the unique geodesic joining p and q. Define $\alpha : Y \times Y \times J \to Y$ by

$$\alpha(p, q, t) = g_{p,q}(t).$$

Clearly $\Lambda_\alpha(p, q)$ coincides with the arc of great circle contained in Y, with end points p and q. Moreover $d(p, q)$ is the length of $\Lambda_\alpha(p, q)$. By Nijenhuis's theorem [11] and Remark 2.1, α is well defined, continuous, and satisfies conditions (i), (ii), (iii)$''$ of Definition 2.2. Therefore, Y is geodesically α-convex and, a fortiori, also α-convex.

On the other hand, Y is not Lipschitz α-convex. For $n > \sqrt{2}/\sin \pi/8$ and $\varepsilon_n = 1/n$, put $a_n = (\varepsilon_n, \varepsilon_n, \sqrt{1 - 2\varepsilon_n^2})$, $b_n = (-\varepsilon_n, \varepsilon_n, \sqrt{1 - 2\varepsilon_n^2})$, $\overline{a}_n = (\varepsilon_n, -\varepsilon_n, \sqrt{1 - 2\varepsilon_n^2})$, $\overline{b}_n = (-\varepsilon_n, -\varepsilon_n, \sqrt{1 - 2\varepsilon_n^2})$. Observe that $c_n \in \Lambda_\alpha(a_n, b_n)$, $\overline{c}_n \in \Lambda_\alpha(\overline{a}_n, \overline{b}_n)$, where $c_n = (0, \varepsilon_n, \sqrt{1 - 2\varepsilon_n^2})$, $\overline{c}_n = (0, -\varepsilon_n, \sqrt{1 - 2\varepsilon_n^2})$. An easy calculation shows that, for every $v \in \Lambda_\alpha(\overline{a}_n, \overline{b}_n)$, we have $d(c_n, v) \geq d(c_n, \overline{c}_n)$, and so $d(c_n, \Lambda_\alpha(\overline{a}_n, \overline{b}_n)) \geq d(c_n, \overline{c}_n)$. On the other hand $d(c_n, \overline{c}_n) > d(a_n, \overline{a}_n) = d(b_n, \overline{b}_n)$, and consequently

$$\sup_{u \in \Lambda_\alpha(a_n, b_n)} d(u, \Lambda_\alpha(\overline{a}_n, \overline{b}_n)) > \max\{d(a_n, \overline{a}_n), d(b_n, \overline{b}_n)\}.$$

From this and the analogous inequality obtained by interchanging $\Lambda_\alpha(a_n, b_n)$ and $\Lambda_\alpha(\overline{a}_n, \overline{b}_n)$, we have

$$h(\Lambda_\alpha(a_n, b_n), \Lambda_\alpha(\overline{a}_n, \overline{b}_n)) > \max\{d(a_n, \overline{a}_n), d(b_n, \overline{b}_n)\}.$$

Since $a_n, \overline{a}_n, b_n, \overline{b}_n \to (0, 0, 1)$ as $n \to \infty$, it follows that condition (iii)$'$ of Definition 2.2 is not satisfied, and hence Y is not Lipschitz α-convex.

We now present an example of a Lipschitz α-convex metric space.

EXAMPLE 4.3. Set $Y = \{x \in \mathbb{R}^2 \mid x = (x_1, x_2),\ x_1 \leq 0 \text{ or } x_2 \leq 0\}$ and equip Y with the metric d induced by the Euclidean norm $\|\cdot\|$ of $\mathbb{R}^2$.

Let $(p, q) \in Y^2$, and assume that $[p, q] \not\subset Y$. For some $T' = T'(p, q)$, $T'' = T''(p, q)$, where $0 \leq T' < T'' \leq 1$, the oriented segment of equation $x(t) = (1-t)p + tq$, $t \in [0, 1]$, meets ∂Y at points $a'(p, q) = x(T')$, $a''(p, q) = x(T'')$. Clearly $a'(p, q)$ (resp. $a''(p, q)$) lies on the strictly positive (resp. negative) coordinate half-axis, thus $a'(p, q) \neq a''(p, q)$. Set $T = T(p, q) = T' + (T'' - T')\|a'\|/(\|a'\| + \|a''\|)$, where $a' = a'(p, q)$, $a'' = a''(p, q)$. Clearly

$T' < T < T''$, for $a', a'' \neq 0$. Moreover a', a'', T', T'', T are continuous at each $(p,q) \in Y^2$, with $[p,q] \not\subset Y$.

Define now $\alpha : Y \times Y \times J \to Y$ as follows: $\alpha(p,q,t) = (1-t)p + tq$ if $[p,q] \subset Y$, and otherwise

$$\alpha(p,q,t) = \begin{cases} (1-t/T')p + (t/T')a', & t \in [0,T'], \\ ((T-t)/(T-T'))a', & t \in [T',T], \\ ((t-T)/(T''-T))a'', & t \in [T,T''], \\ ((1-t)/(1-T''))a'' + ((t-T'')/(1-T''))q, & t \in [T'',1]. \end{cases}$$

When $T' = 0$ (resp. $T'' = 1$), it is tacitly assumed that, on the right hand side, the expression corresponding to $t \in [0,T']$ (resp. $t \in [T'',1]$) does not appear.

Observe that the family of convex subsets of Y contains those of the form $C \cap Y$, where C is any convex subset of $\mathbb{R}^2$ containing the origin.

It is routine to check that α is well defined, continuous, and satisfies conditions (i), (ii) of Definition 2.2. To show (iii)$'$, let $(p,q), (\overline{p},\overline{q}) \in Y^2$ and put, for brevity, $\Lambda = \Lambda_\alpha(p,q)$, $\overline{\Lambda} = \Lambda_\alpha(\overline{p},\overline{q})$. We claim that

$$\max_{x\in\Lambda} d(x,\overline{\Lambda}) \leq \max\{d(p,\overline{p}), d(q,\overline{q})\}. \tag{4.1}$$

To see this, observe that

$$d(x,\overline{\Lambda}) \leq d(x,[\overline{p},\overline{q}]) \quad \text{for every } x \in Y. \tag{4.2}$$

Now, take an $m \in \Lambda$ so that $d(m,\overline{\Lambda}) = \max_{x\in\Lambda} d(x,\overline{\Lambda})$.

Suppose $[p,q] \subset Y$, thus $\Lambda = [p,q]$. Since $d(\cdot,[\overline{p},\overline{q}])$ is convex and $m \in [p,q]$, we have $d(m,[\overline{p},\overline{q}]) \leq \max\{d(p,[\overline{p},\overline{q}]), d(q,[\overline{p},\overline{q}])\}$ and hence, by (4.2) (with $x = m$), (4.1) follows.

Suppose $[p,q] \not\subset Y$. Without loss of generality we assume $a' = (\alpha',0)$ and $a'' = (0,\alpha'')$ for some $\alpha',\alpha'' > 0$, and thus $\Lambda = [p,a'] \cup [0,a'] \cup [0,a''] \cup [a'',q]$. We consider three cases.

(a$_1$) $m \in [p,a'] \cup [a'',q]$. Since $m \in [p,q]$, (4.1) follows as before, in view of (2.2).

(a$_2$) $m = 0$. We claim that $0 \in \overline{\Lambda}$. Suppose not, and take $\overline{z} \in \overline{\Lambda}$ so that $d(0,\overline{\Lambda}) = \|\overline{z}\|$. Clearly $\overline{\Lambda} = [\overline{p},\overline{q}]$, for $0 \notin \overline{\Lambda}$, and hence $\overline{\Lambda}$ lies outside the open ball centered at 0 with radius $\|\overline{z}\|$. Set $\pi = \{x \in \mathbb{R}^2 \mid \langle x,\overline{z}\rangle < 0\}$. As $a',a'' \notin \pi$ we have $\overline{z}_1 \geq 0$, $\overline{z}_2 \geq 0$. Since $\overline{z} \in \overline{\Lambda} \subset Y$ it follows that $\overline{z} \in \partial Y$. Suppose that $\overline{z}$ lies on the half-axis $x_1 \geq 0$. Then $\overline{\Lambda} \subset \{x \in Y \mid x_1 \geq \|\overline{z}\|\}$, for $d(0,\overline{\Lambda}) = \|\overline{z}\|$, and hence $d(a'',\overline{\Lambda}) \geq \|a'' - \overline{z}\| > \|\overline{z}\| = d(0,\overline{\Lambda}) \geq \max_{x\in\Lambda} d(x,\overline{\Lambda})$, a contradiction. Therefore $0 \in \overline{\Lambda}$, proving the claim. As $m = 0 \in \Lambda$, (4.1) trivially holds.

(a$_3$) $m \in [0,a'] \cup [0,a'']$. Assume $m \in [0,a']$ (if $m \in [0,a'']$ the argument is similar). Suppose $[\overline{p},\overline{q}] \subset Y$, thus $\overline{\Lambda} = [\overline{p},\overline{q}]$. Since $d(\cdot,\overline{\Lambda})$ is convex and $m \in [0,a'] \subset \Lambda$, in view of the definition of m we have $d(m,\overline{\Lambda}) =$

$\max\{d(0,\overline{\Lambda}), d(a',\overline{\Lambda})\}$. If $d(m,\overline{\Lambda}) = d(0,\overline{\Lambda})$, then (a$_2$) implies $0 \in \overline{\Lambda}$ and hence (4.1) holds. Let $d(m,\overline{\Lambda}) = d(a',\overline{\Lambda})$. As $d(\cdot,\overline{\Lambda})$ is convex and $a' \in [p,q]$, we have

$$d(a',\overline{\Lambda}) \le \max\{d(p,[\overline{p},\overline{q}]), d(q,[\overline{p},\overline{q}])\} \le \max\{d(p,\overline{p}), d(q,\overline{q})\},$$

and hence (4.1) holds.

Suppose $[\overline{p},\overline{q}] \not\subset Y$. Then for some $\overline{a}' = (\overline{\alpha}',0)$ and $\overline{a}'' = (0,\overline{\alpha}'')$ with $\overline{\alpha}',\overline{\alpha}'' > 0$ we have $\overline{\Lambda} = [\overline{p},\overline{a}'] \cup [0,\overline{a}'] \cup [0,\overline{a}''] \cup [\overline{a}'',\overline{q}]$. By hypothesis $m \in [0,a']$. If $m \in [0,\overline{a}']$ we have $d(m,\overline{\Lambda}) = 0$ and (4.1) holds. If $m \notin [0,\overline{a}']$, we have $m \in [\overline{a}',a']$, and hence $m = (1-t)\overline{a}' + ta'$ for some $t \in [0,1]$. In view of (4.2) and the convexity of $d(\cdot,[\overline{p},\overline{q}])$, it follows that

$$d(m,\overline{\Lambda}) \le d(m,[\overline{p},\overline{q}]) \le (1-t)d(\overline{a}',[\overline{p},\overline{q}]) + td(a',[\overline{p},\overline{q}]) \le d(a',[\overline{p},\overline{q}]), \tag{4.3}$$

for $\overline{a}' \in [\overline{p},\overline{q}]$. Since $a' \in [p,q]$, the convexity of $d(\cdot,[\overline{p},\overline{q}])$ implies

$$d(a',[\overline{p},\overline{q}]) \le \max\{d(p,[\overline{p},\overline{q}]), d(q,[\overline{p},\overline{q}])\} \le \max\{d(p,\overline{p}), d(q,\overline{q})\}. \tag{4.4}$$

Combining (4.3) and (4.4) completes the proof of claim (4.1).

From (4.1) and the analogous inequality obtained by interchanging Λ and $\overline{\Lambda}$ it follows that $h(\Lambda,\overline{\Lambda}) \le \max\{d(p,\overline{q}), d(q,\overline{q})\}$, and hence also condition (iii)$'$ of Definition 2.2 is fulfilled. Consequently, Y is a Lipschitz α-convex metric space.

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