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DIFFERENTIAL GEOMETRY

Shape Operators and Structure Tensors of Real Hypersurfaces in Nonflat Quaternionic Space Forms

by

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Summary. We characterize curvature-adapted real hypersurfaces in nonflat quaternionic space forms in terms of their shape operators and structure tensors.

1. Introduction. In a nonflat quaternionic space form, which is either a quaternionic projective space or a quaternionic hyperbolic space, we have the following nice examples of homogeneous real hypersurfaces. In a quaternionic projective space $\mathbb{H}P^n(c)$ of quaternionic sectional curvature c, they are

(A) a tube of radius $r \in (0, \pi/\sqrt{c})$ around the canonically embedded totally geodesic $\mathbb{H}P^m(c)$ for some $m \in \{0, \ldots, n-2\}$,
(M) a tube of radius $r \in (0, \pi/2\sqrt{c})$ around the canonically embedded totally geodesic complex projective space $\mathbb{C}P^n(c)$,

and in a quaternionic hyperbolic space $\mathbb{H}H^n(c)$ of quaternionic sectional curvature c, they are

(A) a horosphere in $\mathbb{H}H^n(c)$ and a tube of some radius $r \in (0, \infty)$ around the canonically embedded totally geodesic $\mathbb{H}H^m(c)$ for some $m \in \{0, \ldots, n-1\}$,

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(M) a tube of some radius $r \in (0, \infty)$ around the canonically embedded totally geodesic complex hyperbolic space $\mathbb{C}H^n(c)$.

We call these examples a *hypersurface of type* (A) and *of type* (M) *in a nonflat quaternionic space form* $M^n(c;\mathbb{H})$ of quaternionic sectional curvature c $(\neq 0)$, respectively. In this note we study their shape operators and structure tensors induced from the quaternionic structure on $M^n(c;\mathbb{H})$.

2. Curvature-adapted real hypersurfaces. In order to study real hypersurfaces of type (A) and (M), Berndt [B] introduced the notion of curvature-adapted hypersurfaces in a Riemannian manifold $\widetilde{M}$. A hypersurface M of a Riemannian manifold $\widetilde{M}$ is called *curvature-adapted* if the normal Jacobi operator K and the shape operator A of M with respect to a unit normal vector field $\mathcal{N}$ are simultaneously diagonalizable (i.e. $K \circ A = A \circ K$). Here the *normal Jacobi operator* $K : TM \to TM$ of M with respect to $\mathcal{N}$ is defined by $K(\cdot) = \widetilde{R}(\cdot, \mathcal{N})\mathcal{N}$, where $\widetilde{R}$ is the curvature tensor of $\widetilde{M}$. For a real hypersurface M in a quaternionic Kähler manifold $\widetilde{M}$ with quaternionic Kähler structure $\mathcal{J}$, which is a rank 3 vector subbundle of the bundle of endomorphisms of the tangent bundle TM, we decompose TM into $\mathcal{D} \oplus \mathcal{D}^{\perp}$, where $\mathcal{D}$ is the maximal subbundle of TM which is invariant by $\mathcal{J}$. Here, a quaternionic Kähler structure $\mathcal{J}$ on a Riemannian manifold $\widetilde{M}$ of real dimension $4n$ is a rank 3 vector subbundle of the bundle of endomorphisms of $T\widetilde{M}$ with the following properties:

1) For each point $\widetilde{x} \in \widetilde{M}$ there is an open neighborhood $\widetilde{G}$ of $\widetilde{x}$ in $\widetilde{M}$ and sections J_1, J_2, J_3 of the restriction $\mathcal{J}|_{\widetilde{G}}$ over $\widetilde{G}$ such that
   (i) each J_i is an almost Hermitian structure on $\widetilde{G}$, that is, $J_i^2 = -\,\mathrm{id}$ and
   $$\langle J_i\widetilde{X}, \widetilde{Y}\rangle + \langle \widetilde{X}, J_i\widetilde{Y}\rangle = 0 \quad \text{for all vector fields } \widetilde{X} \text{ and } \widetilde{Y} \text{ on } \widetilde{G},$$
   where $\langle\ ,\ \rangle$ is the Riemannian metric of $\widetilde{M}$,
   (ii) $J_iJ_{i+1} = J_{i+2} = -J_{i+1}J_i$ (i mod 3) for $i = 1, 2, 3$.
2) $\widetilde{\nabla}_{\widetilde{X}}J$ is a section of $\mathcal{J}$ for each vector field $\widetilde{X}$ on $\widetilde{M}$ and section J of the bundle $\mathcal{J}$, where $\widetilde{\nabla}$ denotes the Riemannian connection of $\widetilde{M}$.

When the ambient space $\widetilde{M}$ is a nonflat quaternionic space form, curvature-adapted real hypersurfaces are characterized in terms of $\mathcal{D}$ and the shape operators: The following three conditions on a real hypersurface M in $M^n(c;\mathbb{H})$ are equivalent:

(1) M is curvature-adapted.
(2) The subbundle $\mathcal{D}$ is invariant under the shape operator of M.
(3) The subbundle $\mathcal{D}^{\perp}$ is invariant under the shape operator of M.

It was shown by Berndt [B] that every curvature-adapted real hypersurface in $\mathbb{H}P^n(c)$ is locally congruent to a hypersurface of type (A) or (M) and that every curvature-adapted real hypersurface in $\mathbb{H}H^n(c)$ all of whose principal curvatures are constant is locally congruent to a hypersurface of type (A) or (M).

3. Structure tensors and the shape operator. Let M be a real hypersurface in a quaternionic Kähler manifold $\widetilde{M}$. For an endomorphism $J \in \mathcal{J}$ we define the *structure tensor* $\phi_J : TM \to TM$ associated with J by $\phi_J = \pi \circ J|_{TM}$, where $\pi : T\widetilde{M}|_M \to TM$ is the canonical projection. Let $\mathcal{S} = \{\phi_J \mid J \in \mathcal{J}\}$ be the set of all structure tensors. This is a rank 3 subbundle of the bundle of endomorphisms of TM. We set $\xi_J = -J\mathcal{N}$ for each $J \in \mathcal{J}$. It is clear that $\mathcal{D}_x^{\perp} = \{\xi_J(x) \mid J \in \mathcal{J}\}$ at each point $x \in M$ and that $\phi_J(\xi_J) = 0$, $\phi_J(\mathcal{D}^{\perp}) \subset \mathcal{D}^{\perp}$ and $\phi_J(v) = Jv$ for every $v \in \mathcal{D}$.

In a complex projective space, real hypersurfaces of type (A), which are tubes around canonically embedded totally geodesic complex projective spaces, are characterized as hypersurfaces with $A\phi = \phi A$. Here ϕ is the structure tensor induced by the complex structure of the ambient space.

We denote by $\mathfrak{F}(X)$ the set of real functions on a domain X. As in the case of complex space form, we consider an endomorphism $f\phi A + gA\phi$ of TM for $f, g \in \mathfrak{F}(TM)$ and $\phi \in \mathcal{S}$, which is given by $(f\phi A + gA\phi)(v) = f(v)\phi(Av) + g(v)A\phi(v)$ for $v \in TM$.

PROPOSITION 1. *Let M be a real hypersurface of a nonflat quaternionic space form $\widetilde{M}^n(c;\mathbb{H})$. Then the following conditions are equivalent*:

(1) *M is curvature-adapted.*
(2) *For every $\phi \in \mathcal{S}$ there exists $f \in \mathfrak{F}(TM)$ satisfying $(f\phi A + A\phi)(\mathcal{D}) \subset \mathcal{D}$.*
(2′) *$(f\phi A + gA\phi)(\mathcal{D}) \subset \mathcal{D}$ for every $\phi \in \mathcal{S}$ and $f, g \in \mathfrak{F}(TM)$.*
(3) *For every $\phi \in \mathcal{S}$ there exists $g \in \mathfrak{F}(TM)$ satisfying $(\phi A + gA\phi)(\mathcal{D}^{\perp}) \subset \mathcal{D}^{\perp}$.*
(3′) *$(f\phi A + gA\phi)(\mathcal{D}^{\perp}) \subset \mathcal{D}^{\perp}$ for every $\phi \in \mathcal{S}$ and $f, g \in \mathfrak{F}(TM)$.*

Proof. (1)⇒(2′)&(3′). This is trivial since $A(\mathcal{D}) \subset \mathcal{D}$ and $A(\mathcal{D}^{\perp}) \subset \mathcal{D}^{\perp}$. (2′)⇒(2) and (3′)⇒(3) are trivial.

(3)⇒(1). We decompose $A\xi_J$ as $A\xi_J = \widehat{\xi}_J + \xi_J^{\perp} \in \mathcal{D} \oplus \mathcal{D}^{\perp}$ for each $\xi_J \in \mathcal{D}^{\perp}$. We then have

$$\mathcal{D}^{\perp} \ni (\phi_J A + gA\phi_J)(\xi_J) = \phi_J A\xi_J = \phi_J(\widehat{\xi}_J) + \phi_J(\xi_J^{\perp}).$$

As $\phi_J(\widehat{\xi}) \in \mathcal{D}$ and $\phi_J(\xi_J^{\perp}) \in \mathcal{D}^{\perp}$, this implies $\phi_J(\widehat{\xi}) = 0$, so that $\widehat{\xi} = 0$. Thus we see that $A(\mathcal{D}^{\perp}) \subset \mathcal{D}^{\perp}$ and M is curvature-adapted.

(2)⇒(1). For each $x \in M$ we take a local basis $J_1, J_2, J_3 \in \mathcal{J}|_G$ on a neighborhood of x with $J_i^2 = -1$ and $J_i \circ J_{i+1} = J_{i+2} = -J_{i+1} \circ J_i$ (i mod 3).

Putting $\phi_i = \phi_{J_i}$ and $\xi_i = \xi_{J_i}$, we express Av for each $v \in \mathcal{D}$ as

$$Av = \widehat{v} + \eta_1(v)\xi_1 + \eta_2(v)\xi_2 + \eta_3(v)\xi_3 \quad \text{with } \widehat{v} \in \mathcal{D}.$$

Then by assumption we have

$$\begin{aligned} \mathcal{D} \ni\, & (f\phi_1 A + A\phi_1)(v) \\ &= f(v)\{\phi_1(\widehat{v}) + \eta_2(v)\xi_3 - \eta_3(v)\xi_2\} \\ &\quad + \{\widehat{\phi_1(v)} + \eta_1(\phi_1(v))\xi_1 + \eta_2(\phi_1(v))\xi_2 + \eta_3(\phi_1(v))\xi_3\}. \end{aligned}$$

Hence $\eta_1(\phi_1(v)) = 0$, and similarly $\eta_2(\phi_2(v)) = \eta_3(\phi_3(v)) = 0$. Thus we can see that $\eta_i(v) = \eta_i(\phi_i(-\phi_i(v))) = 0$ for each $i = 1, 2, 3$, so that $A(\mathcal{D}) \subset \mathcal{D}$. Therefore M is curvature-adapted in $M^n(c; \mathbb{H})$. ∎

As a consequence of Proposition 1 we establish the following characterization of hypersurfaces of type (A) in $\mathbb{H}P^n(c)$.

THEOREM 1. *The following conditions on a real hypersurface M of $\mathbb{H}P^n(c)$ are equivalent*:

(1) *M is of type* (A).
(2) *$\phi A = A\phi$ for each $\phi \in \mathcal{S}$.*
(3) *For each $\phi \in \mathcal{S}$ there exists $g \in \mathfrak{F}(TM)$ with $\phi A + gA\phi = 0$ on $\mathcal{D}^\perp$.*
(4) *For each $\phi \in \mathcal{S}$ there exists $f \in \mathfrak{F}(TM)$ with $f\phi A + A\phi = 0$ on $\mathcal{D}$.*

Proof. Proposition 1 guarantees that M is curvature-adapted in $\widetilde{M}^n(\mathbb{H}; c)$, hence M is either of type (A) or of type (M) under each condition. We denote by λ_j an eigenvalue of $A|_{\mathcal{D}}$ and by μ_j that of $A|_{\mathcal{D}^\perp}$, and by $m(\nu)$ and V_ν the multiplicity and the eigenspace corresponding to the eigenvalue ν, respectively. The following is due to Berndt [B]:

When M is a hypersurface of type (A), its tangent bundle decomposes as $TM = V_{\lambda_1} \oplus V_{\lambda_2} \oplus V_{\mu_1}$ with

$$\lambda_1 = \frac{\sqrt{c}}{2}\cot\frac{\sqrt{c}\,r}{2}, \qquad \lambda_2 = -\frac{\sqrt{c}}{2}\tan\frac{\sqrt{c}\,r}{2}, \qquad \mu_1 = \sqrt{c}\cot(\sqrt{c}\,r),$$

and each of the eigenspaces $V_{\lambda_1}, V_{\lambda_2}$ and V_{μ_1} is invariant under every $\phi \in \mathcal{S}$. (For the case when M is a geodesic sphere, $V_{\lambda_2} = \{0\}$.) Therefore it is clear that $\phi A = A\phi$ for each $\phi \in \mathcal{S}$ in this case. When M is of type (M), its tangent bundle decomposes as $TM = V_{\lambda_1} \oplus V_{\lambda_2} \oplus V_{\mu_1} \oplus V_{\mu_2}$, where

$$\begin{aligned} \lambda_1 &= \frac{\sqrt{c}}{2}\cot\frac{\sqrt{c}\,r}{2}, \qquad \lambda_2 = -\frac{\sqrt{c}}{2}\tan\frac{\sqrt{c}\,r}{2}, \\ \mu_1 &= \sqrt{c}\cot(\sqrt{c}\,r), \qquad \mu_2 = -\sqrt{c}\tan(\sqrt{c}\,r), \end{aligned}$$

and $m(\mu_1) = 1, m(\mu_2) = 2$. For each point we can take a local basis $J_i,\ i = 1, 2, 3$, satisfying $J_i^2 = -1,\ J_i \circ J_{i+1} = J_{i+2} = -J_{i+1} \circ J_i$ (i mod 3) and

$$\begin{aligned}
&\phi_1(V_{\lambda_j}) = V_{\lambda_j}\ (j=1,2), \quad \phi_1(V_{\mu_1}) = \{0\}, \quad \phi_1(V_{\mu_2}) = V_{\mu_2},\\
(3.1)\qquad &\phi_i(V_{\lambda_1}) = V_{\lambda_2}, \quad \phi_i(V_{\lambda_2}) = V_{\lambda_1}, \quad \phi_i(V_{\mu_1}) \subset V_{\mu_2},\\
&\phi_i(V_{\mu_2}) = V_{\mu_1}\ (i=2,3),
\end{aligned}$$

where $\phi_i = \phi_{J_i}$.

What we have to show is that condition (3) or (4) implies M is of type (A).

(3)⇒(1). For each $J \in \mathcal{J}$, (3) leads us to $\phi_J(A\xi_J) = -g(\xi)A\phi_J(\xi_J) = 0$. Hence $A\xi_J$ is proportional to ξ_J, which shows ξ_J is principal. As $D_x^\perp = \{\xi_J(x) \mid J \in \mathcal{J}\}$ for each x, it should be an eigenspace of $A|_{\mathcal{D}_x}$. Considering principal curvatures of real hypersurfaces of type (A) and of type (M), we find that M is not of type (M). Every hypersurface of type (A) clearly satisfies (3) with $g \equiv -1$. Thus M is of type (A).

(4)⇒(1). By assumption, $A\phi v = -f(v)\phi Av$ for every $v \in \mathcal{D}$. When M is of type (M), we consider a vector $v = a_1v_1 + a_2v_2 \in \mathcal{D}$ with $a_1, a_2 \in \mathbb{R}$ and $v_1 \in V_{\lambda_1}, v_2 \in V_{\lambda_2}$, $v_j \neq 0$. Since $\lambda_1 \neq \lambda_2$, we see that $A\phi_i(v)$ is not proportional to ϕ_iAv for the structure tensor ϕ_i, $i = 2, 3$, associated with the local basis given above. When M is of type (A), it satisfies (4) with $f \equiv -1$. Thus M is of type (A). ■

Inspecting the proof of Proposition 1, we can improve the statement as follows:

PROPOSITION 2. *For a real hypersurface M in a nonflat $M^n(c;\mathbb{H})$, the following conditions are equivalent*:

(1) *M is curvature-adapted in $\widetilde{M}^n(c;\mathbb{H})$.*

(2″) *For each $x \in M$ there exists a basis $\{K_1, K_2, K_3\}$ of $\mathcal{J}_x$ and functions $f_1, f_2, f_3 \in \mathfrak{F}(T_xM)$ satisfying $(f_i\phi_{K_i}A + A\phi_{K_i})(\mathcal{D}_x) \subset \mathcal{D}_x$ for $i = 1, 2, 3$.*

(3″) *For each $x \in M$ there exists a basis $\{K_1, K_2, K_3\}$ of $\mathcal{J}_x$ and functions $g_1, g_2, g_3 \in \mathfrak{F}(T_xM)$ satisfying $(\phi_{K_i}A + g_iA\phi_{K_i})(\mathcal{D}_x^\perp) \subset \mathcal{D}_x^\perp$ for $i = 1, 2, 3$.*

In this context we can improve Theorem 1 in the following manner.

THEOREM 2. *For a real hypersurface M in a quaternionic projective space $\mathbb{H}P^n(c)$ the following conditions are equivalent*:

(1) *M is of type (A).*

(2′) *For each $x \in M$ there exists a basis $\{K_1, K_2, K_3\}$ of $\mathcal{J}_x$ satisfying $\phi_{K_i}A = A\phi_{K_i}$ for $i = 1, 2, 3$.*

(3′) *For each $x \in M$ there exists a basis $\{K_1, K_2, K_3\}$ of $\mathcal{J}_x$ and $g_1, g_2, g_3 \in \mathfrak{F}(T_xM)$ satisfying $\phi_{K_i}A + g_iA\phi_{K_i} = 0$ on $\mathcal{D}_x^\perp$ for $i = 1, 2, 3$.*

(4′) *For each $x \in M$ there exists a basis $\{K_1, K_2, K_3\}$ of $\mathcal{J}_x$ and $f_1, f_2, f_3 \in \mathfrak{F}(T_xM)$ satisfying $f_i\phi_{K_i}A + A\phi_{K_i} = 0$ on $\mathcal{D}_x$ for $i = 1, 2, 3$.*

Proof. Proposition 2 guarantees that under each condition, M is curvature-adapted in $\mathbb{H}P^n(c)$, hence it is either of type (A) or of type (M). Reviewing the proof of Theorem 1, we only need to check that (4′) implies (1). When M is of type (M), we take a local basis J_i, $i = 1, 2, 3$, of $\mathcal{J}$ satisfying $J_i^2 = -1$, $J_i \circ J_{i+1} = J_{i+2} = -J_{i+1} \circ J_i$ ($i \bmod 3$) and (3.1). Setting $K_i = \sum_{j=1}^3 a_{ij} J_j$ we may suppose $a_{33} \neq 0$. We then have

$$A\phi_{K_3}(b_1\xi_{J_1} + b_2\xi_{J_2}) = -\mu_1 a_{33} b_2 \xi_1 + \mu_2 a_{33} b_1 \xi_2 + \mu_2(a_{31}b_2 - a_{32}b_1)\xi_3,$$
$$\phi_{K_3}A(b_1\xi_{J_1} + b_2\xi_{J_2}) = -\mu_2 a_{33} b_2 \xi_1 + \mu_1 a_{33} b_1 \xi_2 + (\mu_2 a_{31}b_2 - \mu_1 a_{32}b_1)\xi_3,$$

for some constants b_1, b_2. Hence $A\phi_{K_3}(b_1\xi_{J_1} + b_2\xi_{J_2})$ is not neccesarily proportional to $\phi_{K_3}A(b_1\xi_{J_1} + b_2\xi_{J_2})$, and M is not of type (M). When M is of type (A), it clearly satisfies (4′). ∎

In order to characterize homogeneous real hypersurfaces in a complex projective space $\mathbb{C}P^n$, Kimura [K] studied commutativity of two endomorphisms derived from the shape operators and structure tensors (see Proposition 3 below). Here, we also consider endomorphisms $P = P_{\phi,f} = \phi A + fA\phi$ and $Q = Q_{\phi,g,k} = \phi A + gA\phi + k\phi$ of TM for functions $f, g, k : M \to \mathbb{R}$. When we consider P, Q on a tangent space T_xM, we treat f, g, k as constants.

Lemma. *Let M be a real hypersurface in a quaternionic Kähler manifold $\widetilde{M}$. If $(P_{\phi_J,f}Q_{\phi_J,g,k} - Q_{\phi_J,g,k}P_{\phi_J,f})\xi_J(x) = 0$ at some $x \in M$ with some constants f, g, k with $f \neq g$, then $\xi_J(x)$ is a principal curvature vector of M in $\widetilde{M}$.*

Proof. Direct computation yields

$$\text{(3.2)}\quad \begin{aligned} &P_{\phi_J,f}Q_{\phi_J,g,k} - Q_{\phi_J,g,k}P_{\phi_J,f} \\ &\quad = (f-g)(-\phi_J A^2 \phi_J + A\phi_J^2 A) + k\{(1-f)\phi_J A \phi_J + fA\phi_J^2 - \phi_J^2 A\}, \end{aligned}$$

in particular,

$$\begin{aligned} (P_{\phi_J,f}Q_{\phi_J,g,k} - Q_{\phi_J,g,k}P_{\phi_J,f})\xi_J &= (f-g)A\phi_J^2 A\xi_J - k\phi_J^2 A\xi_J \\ &= (f-g)A\biggl(-A\xi_J + \frac{\langle A\xi_J, \xi_J\rangle}{\|\xi_J\|^2}\xi_J\biggr) - k\biggl(-A\xi_J + \frac{\langle A\xi_J, \xi_J\rangle}{\|\xi_J\|^2}\xi_J\biggr). \end{aligned}$$

Hence

$$\begin{aligned} 0 &= \langle (P_{\phi_J,f}Q_{\phi_J,g,k} - Q_{\phi_J,g,k}P_{\phi_J,f})\xi_J(x), \xi_J(x)\rangle \\ &= (f(x) - g(x))\biggl(-\|A\xi_J\|^2 + \frac{\langle A\xi_J(x), \xi_J(x)\rangle^2}{\|\xi_J\|^2}\biggr), \end{aligned}$$

which shows that $\|A\xi_J(x)\|^2 = \langle A\xi_J(x), \xi_J(x)\rangle^2 / \|\xi_J\|^2$. Thus we conclude that $\xi_J(x)$ is principal. ∎

Remark. On every hypersurface of type (A) in a nonflat $M^n(c; \mathbb{H})$ the commutation relation $PQ = QP$ holds for arbitrary $\phi \in \mathcal{S}$ and functions f, g, k because $\phi A = A\phi$.

In view of the Lemma we obtain the following:

THEOREM 3. *For a real hypersurface M in $\mathbb{H}P^n(c)$ the following conditions are equivalent*:

(1) *M is of type* (A).
(2) $P_{\phi,f}Q_{\phi,g,k} = Q_{\phi,g,k}P_{\phi,f}$ *for all $\phi \in \mathcal{S}$ and $f, g, k \in \mathfrak{F}(M)$.*
(2′) *For each $\phi \in \mathcal{S}$ there exist $f, g, k \in \mathfrak{F}(M)$ such that $f - g$ has no zeros and $P_{\phi,f}Q_{\phi,g,k} = Q_{\phi,g,k}P_{\phi,f}$.*
(3) $P_{\phi,f}Q_{\phi,g,k} = Q_{\phi,g,k}P_{\phi,f}$ *on $\mathcal{D}^\perp$ for all $\phi \in \mathcal{S}$ and $f, g, k \in \mathfrak{F}(M)$.*
(3′) *For each $\phi \in \mathcal{S}$ there exist $f, g, k \in \mathfrak{F}(M)$ such that $f - g$ has no zeros and $P_{\phi,f}Q_{\phi,g,k} = Q_{\phi,g,k}P_{\phi,f}$ on $\mathcal{D}^\perp$.*
(4) *For each $x \in M$ there exists a basis $\{K_1, K_2, K_3\}$ of $\mathcal{J}_x$ and constants f_i, g_i, k_i such that $f_i \neq g_i$ and*

$$P_{\phi_{K_i},f_i}Q_{\phi_{K_i},g_i,k_i} = Q_{\phi_{K_i},g_i,k_i}P_{\phi_{K_i},f_i} \quad \text{on } T_xM \text{ for } i = 1, 2, 3.$$

Proof. Under each condition it follows from the Lemma that $A\mathcal{D}^\perp \subset \mathcal{D}^\perp$, so that our real hypersurface M is curvature-adapted. When M is of type (A), these conditions trivially hold. Therefore we assume that M is of type (M). Since there is a non-principal vector in $\mathcal{D}^\perp$, condition (3′) does not hold. Suppose M satisfies (4). Then we may consider $\xi_{K_1} \in V_{\mu_1}$ and $\xi_{K_2}, \xi_{K_3} \in V_{\mu_2}$ by the Lemma. Since $\{K_1, K_2, K_3\}$ is a basis of $\mathcal{J}_x$, we see that $\phi_{K_2}(\xi_{K_3}) = a\xi_{K_1}$ with a nonzero constant a and $\phi_{K_2}(\xi_{K_1}) \in V_{\mu_2} \setminus \{0\}$. As $\phi_{K_2}(V_{\lambda_1}) = V_{\lambda_2}$, $\phi_{K_2}(V_{\lambda_2}) = V_{\lambda_1}$, for $v \in V_{\lambda_1}$ we find by (3.2) that

$$\begin{aligned}(P_{\phi_{K_2},f_2}Q_{\phi_{K_2},g_2,k_2} - Q_{\phi_{K_2},g_2,k_2}P_{\phi_{K_2},f_2})v \\ = (\lambda_2 - \lambda_1)\{(f_2 - g_2)(\lambda_1 + \lambda_2) + k_2(f_2 - 1)\}v,\\ (P_{\phi_{K_2},f_2}Q_{\phi_{K_2},g_2,k_2} - Q_{\phi_{K_2},g_2,k_2}P_{\phi_{K_2},f_2})\xi_{K_3} \\ = a(\mu_2 - \mu_1)\{(f_2 - g_2)(\mu_1 + \mu_2) + k_2(f_2 - 1)\}\phi_{K_2}(\xi_{K_1}).\end{aligned}$$

Since $\lambda_1 + \lambda_2 \neq \mu_1 + \mu_2$, this is a contradiction which proves our result. ■

In terms of $\mathcal{D}^\perp$, we have the following characterization of *all* curvature-adapted real hypersurfaces of $\mathbb{H}P^n(c)$:

THEOREM 4. *For a real hypersurface M in $\mathbb{H}P^n(c)$ the following conditions are equivalent*:

(1) *M is curvature-adapted.*
(2) *For each $x \in M$ there exists a basis $\{K_1, K_2, K_3\}$ of $\mathcal{J}_x$ and constants f_i, g_i, k_i such that $f_i \neq g_i$ and*

$$P_{\phi_{K_i},f_i}Q_{\phi_{K_i},g_i,k_i} = Q_{\phi_{K_i},g_i,k_i}P_{\phi_{K_i},f_i} \quad \text{on } \mathcal{D}_x^\perp \text{ for } i = 1, 2, 3.$$

(3) *There exist constants f, g, k ($f \neq g$) such that for each x we can choose a basis $\{K_1, K_2, K_3\}$ of $\mathcal{J}_x$ satisfying*

$$P_{\phi_{K_i},f}Q_{\phi_{K_i},g,k} = Q_{\phi_{K_i},g,k}P_{\phi_{K_i},f} \quad \text{on } \mathcal{D}_x^\perp \text{ for } i = 1, 2, 3.$$

Proof. By the Lemma, (2) implies $A\mathcal{D}^\perp \subset \mathcal{D}^\perp$, hence M is curvature-adapted. On the other hand, when M is of type (M), we take $f=-1$, $g=1$ and $k=-(\mu_1+\mu_2)$. For a local basis $J_i,\ i=1,2,3$, of $\mathcal{J}$ satisfying $J_i^2=-1$, $J_i\circ J_{i+1}=J_{i+2}=-J_{i+1}\circ J_i$ (i mod 3) and (3.1), we see that $P_{\phi_{J_i},f}=0$ on V_{μ_1} and $Q_{\phi_{J_i},g,k}=0$ on V_{μ_2}. Hence (3) holds. Thus we obtain the result. ■

We end this paper with some results corresponding to Theorems 3 and 4 on real hypersurfaces in a nonflat complex space form $M^n(c;\mathbb{C})$ of constant holomorphic sectional curvature c ($\neq 0$), which is either a complex projective space or a complex hyperbolic space. We say that a real hypersurface M in $M^n(c;\mathbb{C})$ is a *Hopf hypersurface* if the characteristic vector ξ of M is principal.

PROPOSITION 3. *For a real hypersurface M in a nonflat complex space form $M^n(c;\mathbb{C})$, two endomorphisms $P=\phi A-A\phi$ and $Q_k=\phi A+A\phi+k\phi$ commute for some constant k if and only if M is locally congruent to a Hopf hypersurface all of whose principal curvatures are constant.*

Proof. For $c>0$, the statement was proved by Kimura [K].

As we have

$$PQ_k-Q_kP=2\phi A^2\phi-2A\phi^2A+k(2\phi A\phi-A\phi^2-\phi^2A) \tag{3.3}$$

and $\phi^2v=-v+\langle v,\xi\rangle\xi$ for an arbitrary tangent vector v, we see that

$$\begin{aligned}\langle (PQ_k-Q_kP)\xi,\xi\rangle &= \langle 2A^2\xi-\langle A\xi,\xi\rangle A\xi+k(A\xi-\langle A\xi,\xi\rangle\xi),\xi\rangle\\ &= 2\|A\xi\|^2-2\langle A\xi,\xi\rangle^2.\end{aligned}$$

Thus if $PQ_k-Q_kP=0$ we find that ξ is principal, so that the corresponding principal curvature α is constant (see [NR]).

Let v be a principal vector orthogonal to ξ. If $Av=\lambda v$, then we have $2(2\lambda-\alpha)A\phi v=(2\alpha\lambda+c)\phi v$. We first consider the case $2\lambda\neq\alpha$. Then ϕv is also a principal vector of principal curvature $(2\alpha\lambda+c)/\{2(2\lambda-\alpha)\}$ (see [NR]). By (3.3) we have

$$\left(\lambda-\frac{2\alpha\lambda+c}{2(2\lambda-\alpha)}\right)\left(\lambda+\frac{2\alpha\lambda+c}{2(2\lambda-\alpha)}+k\right)=0,$$

hence either $4\lambda^2-4\alpha\lambda+c=0$ or $4\lambda^2+4k\lambda-2k\alpha+c=0$. Therefore in this case each principal curvature function is locally constant on M. Next we study the case that there is a point such that $\lambda=\alpha/2$ is a principal curvature. By continuity of principal curvature functions the above argument guarantees that $\alpha/2$ is a principal curvature on some neighborhood of this point. So our real hypersurface M is locally congruent to a Hopf hypersurface with constant principal curvatures.

We now check that every Hopf hypersurface with constant principal curvatures satisfies $PQ_k=Q_kP$ for some constant k. Such real hypersurfaces are classified completely. In a complex hyperbolic space $\mathbb{C}H^n(c)$ they are called real hypersurfaces of type (A) and (B) (for details, see [NR]). For a real hypersurface of type (A), which is either a horosphere or a tube of radius r $(0 < r < \infty)$ around a totally geodesic $\mathbb{C}H^d(c)$ with $0 \le d \le n-1$, as we have $P = \phi A - A\phi = 0$, the claim is obvious. For a real hypersurface M of type (B), which is a tube of radius r around a totally geodesic real hyperbolic space $\mathbb{R}H^n(c/4)$ of constant sectional curvature $c/4$, the tangent bundle decomposes as $TM = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \mathbb{R}\xi$, where

$$\lambda_1 = \frac{\sqrt{|c|}}{2}\coth\frac{\sqrt{|c|}\,r}{2}, \qquad \lambda_2 = \frac{\sqrt{|c|}}{2}\tanh\frac{\sqrt{|c|}\,r}{2}, \qquad \alpha = \sqrt{|c|}\tanh(\sqrt{|c|}\,r),$$

and $\phi(V_{\lambda_1}) = V_{\lambda_2}$, $\phi(V_{\lambda_2}) = V_{\lambda_1}$. Therefore $Q_k = 0$ with $k = -(\lambda_1 + \lambda_2) = -\sqrt{|c|}\coth(\sqrt{|c|}\,r)$, hence $PQ_k = Q_kP$.

In a complex projective space $\mathbb{C}P^n(c)$ Hopf hypersurfaces with constant principal curvatures are real hypersurfaces of types (A)–(E) (see [NR]). For a real hypersurface of type (A), which is a tube of radius r $(<\pi/\sqrt{c})$ around a totally geodesic $\mathbb{C}P^d(c)$ with $1 \le d \le n-1$, the statement is obvious as $P = 0$. For a real hypersurface M of type (B), which is a tube of radius r $(< \pi/(2\sqrt{c}))$ around a totally geodesic real projective space $\mathbb{R}P^n(c/4)$ of constant sectional curvature $c/4$, the tangent bundle decomposes as $TM = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \mathbb{R}\xi$, where

$$\lambda_1 = -\frac{\sqrt{c}}{2}\cot\frac{\sqrt{c}\,r}{2}, \qquad \lambda_2 = \frac{\sqrt{c}}{2}\tan\frac{\sqrt{c}\,r}{2}, \qquad \alpha = \sqrt{c}\tan(\sqrt{c}\,r),$$

and $\phi(V_{\lambda_1}) = V_{\lambda_2}$, $\phi(V_{\lambda_2}) = V_{\lambda_1}$. Therefore $Q_k = 0$ with $k = -(\lambda_1 + \lambda_2) = \sqrt{c}\cot(\sqrt{|c|}\,r)$, hence $PQ_k = Q_kP$. For a real hypersurface M of type (C), (D) or (E), which is a tube of radius r $(< \pi/(2\sqrt{c}))$ around $\mathbb{C}P^1(c) \times \mathbb{C}P^{(n-1)/2}(c)$, complex Grassmannian $\mathbb{C}G_{2,5}$ or $SO(10)/U(5)$, respectively, the tangent bundle decomposes as $TM = V_{\lambda_1} \oplus V_{\lambda_2} \oplus V_{\lambda_3} \oplus V_{\lambda_4} \oplus \mathbb{R}\xi$, where

$$\lambda_1 = \frac{\sqrt{c}}{2}\cot\frac{\sqrt{c}\,r}{2}, \qquad \lambda_2 = -\frac{\sqrt{c}}{2}\tan\frac{\sqrt{c}\,r}{2}, \qquad \lambda_3 = \frac{\sqrt{c}(1+\tan(\sqrt{c}\,r/2))}{2(1-\tan(\sqrt{c}\,r/2))},$$

$$\lambda_4 = -\frac{\sqrt{c}(1-\tan(\sqrt{c}\,r/2))}{2(1+\tan(\sqrt{c}\,r/2))}, \qquad \alpha = \sqrt{c}\cot(\sqrt{c}\,r),$$

$\phi(V_{\lambda_i}) = V_{\lambda_i}, i = 1, 2$, and $\phi(V_{\lambda_3}) = V_{\lambda_4}$, $\phi(V_{\lambda_4}) = V_{\lambda_3}$. We consider Q_k for $k = -(\lambda_3 + \lambda_4) = \sqrt{c}\tan\sqrt{c}\,r$. Since

$$P(V_{\lambda_i}) = 0, \qquad Q_k(V_{\lambda_i}) \subset V_{\lambda_i} \ (i = 1, 2),$$
$$Q_k(V_{\lambda_j}) = 0 \ (j = 3, 4), \qquad P(V_{\lambda_3}) \subset V_{\lambda_4}, \qquad P(V_{\lambda_4}) \subset V_{\lambda_3},$$

we find $PQ_k = Q_kP = 0$ and obtain our result. ∎

Remark. In Proposition 3 we cannot relax the condition on k. Even in a complex projective space there exist Hopf hypersurfaces satisfying $PQ_k =$

Q_kP for some function k and having some principal curvatures *not* constant (see [K] for details).

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