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GENERAL TOPOLOGY

An Exactly Two-to-One Map from an Indecomposable Chainable Continuum

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Summary. It is shown that a certain indecomposable chainable continuum is the domain of an exactly two-to-one continuous map. This answers a question of Jo W. Heath.

In 1939 O. G. Harrold [4] proved that *the interval* $[0,1]$ *is not the domain of any* (*exactly*) two-to-one map ([1]). Further spaces do not support such maps: any connected graph with odd Euler characteristic [3], the real line [10], the cube $[0,1]^n$ for any finite n [1], the Knaster bucket handle continuum ([2]) [10], other Knaster type continua and some of solenoids [2]. In 1961 J. Mioduszewski [10] asked *if there exists a two-to-one map from the pseudo-arc*, the only (up to homeomorphism) hereditarily indecomposable ([3]) chainable continuum. This question is still open.

The first example of a chainable continuum that does support a two-to-one map was given in [5]. J. W. Heath [5–9] asked *if any indecomposable chainable continuum can be the domain of a two-to-one map.*

In this note we answer the question of Heath in the affirmative.

A continuum is *chainable* when for every $\varepsilon > 0$, the continuum is covered by open sets $U_1, \ldots, U_{n(\varepsilon)}$ of diameters $\leq \varepsilon$ such that U_i meets U_j iff

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([1]) All *maps* are meant to be continuous, and all *spaces* are metric. A function is called (*exactly*) *k-to-one* if each point-inverse of the function has exactly k elements.

([2]) A *continuum* is a connected compact space.

([3]) A continuum is *indecomposable* if it is not the union of two proper subcontinua. A continuum is *hereditarily indecomposable* if each subcontinuum is indecomposable.

$|i-j| \le 1$. We say that points x, y in a chainable continuum are *opposite end points* if for every $\varepsilon > 0$, we can assume that $x \in U_1$ and $y \in U_{n(\varepsilon)}$.

A chainable continuum need not necessarily have a pair of opposite end points. However, if it does, then it is irreducible between these points ([4]).

We need the following result from [5, Example 3], attributed to W. Lewis.

Example 1. *There exist*: *a* (*decomposable*) *chainable continuum* X *with opposite end points* $0, 1 \in X$ *and a map* f *from* X *such that* $\operatorname{card} f^{-1}f(x) = 2$ *iff* $x \in X \setminus \{0,1\}$. *Hence* $\operatorname{card} f^{-1}f(0) = \operatorname{card} f^{-1}f(1) = 1$. ∎

We proceed to the construction of our main example:

Example 2. *There exists an indecomposable chainable continuum that is the domain of an exactly two-to-one map.*

Construction. Consider the Cantor ternary set $C \subset \mathbb{R}$. Any element $\sum_{i=1}^{\infty} 2a_i/3^i \in C$, where $a_i \in \{0,1\}$, will be denoted by $a_1a_2\ldots a_i\ldots$. It will be convenient to construct the domain and the image of our two-to-one map simultaneously. So, take Lewis's continuum X of Example 1 and its image $f(X)$. We can assume that $f(0)=0$, $f(1)=1$, and $X \cap f(X) = \{0,1\}$. Then consider the following two products: $D = \{0,1\} \times C \subset [X \cup f(X)] \times C$. Their elements will be denoted by $a = a_0.a_1a_2a_3\ldots$, where $a_0 \in X \cup f(X)$ and $a_1a_2a_2\ldots \in C$. We also put $o = 0.000\ldots$.

We shall identify certain pairs of points in $D \setminus \{o\}$, which will resemble the well known construction of the Knaster bucket handle continuum. Consider the switching operation $\widehat{0} = 1$ and $\widehat{1} = 0$. For any element $a \in D \setminus \{o\}$ let us put $n(a) = \min\{i : a_i = 1\}$, and then let us identify

$$a \sim \underbrace{0.00\ldots0}_{n(a)\text{ zeros}}1\widehat{a_{n(a)+1}}\widehat{a_{n(a)+2}}\widehat{a_{n(a)+3}}\widehat{a_{n(a)+4}}\ldots.$$

Consistently, when $a_0 = 1$, we write $1.a_1a_2a_3\ldots \sim 1.\widehat{a_1}\widehat{a_2}\widehat{a_3}\ldots$ We have thus obtained an upper semicontinuous equivalence relation $\sim$ on $[X \cup f(X)] \times C$. Denote by p the natural projection that carries $a \in [X \cup f(X)] \times C$ to its equivalence class $[a]_\sim$. Let K, L, and E denote $p(X \times C)$, $p[f(X) \times C]$, and $p(D)$ respectively. See Figure 1.

The space K will be the domain of the map to be defined. So, let us look at some of its subspaces. Consider all copies $X \times \{c\}$ of X, where $c \in C$ are the ends of open intervals contained in $\mathbb{R} \setminus C$. The images $p(X \times \{c\})$ are no longer pairwise disjoint. However, we can arrange them in a sequence $\{F^n\}_{n=1}^{\infty}$ such that

([4]) A continuum C is *irreducible between points* $x, y \in C$ if no proper subcontinuum of C contains both x and y.

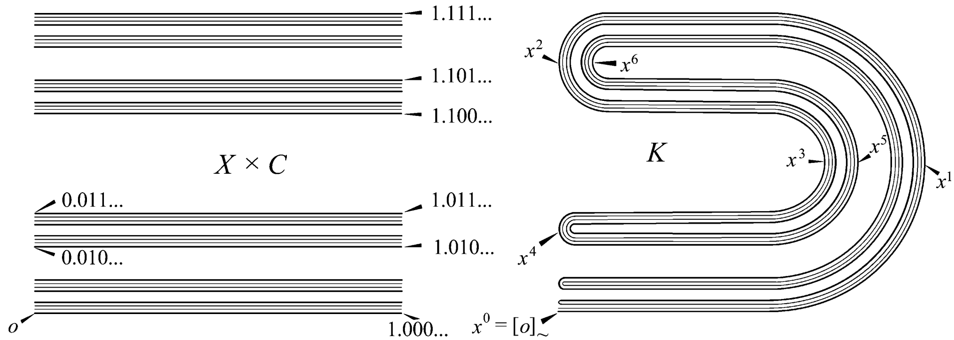

Fig. 1. The construction of the domain K of Example 2. In view of a remark at the end of the paper, the auxiliary continuum $L = g(K)$ has the same structure. Moreover, $[p|(f(X) \times C)] \circ (f \times \mathrm{id}_C) = g \circ [p|(X \times C)]$.

- $[o]_\sim \in F^1$,
- $\mathrm{card}(F^n \cap F^{n+1}) = 1$ for each n,
- $F^n \cap F^m = \emptyset$ whenever $|n - m| > 1$.

K is connected as it has a dense connected subspace, $Y = \bigcup_{n=1}^{\infty} F^n$. The chainability of K results from the fact that 0 and 1 are opposite end points of X. The indecomposability of K results from this claim: *If $M \subset K$ is a subcontinuum that contains $[o]_\sim$, then $M \subset Y$ or $M = K$.*

To prove the claim, let $x^0 = [o]_\sim$, and denote by x^n the only point in $F^n \cap F^{n+1}$. There are two possibilities: (1) There is an $x^n \notin M$. Then there is an open neighbourhood $V \ni x^n$ that does not meet M, and M is contained in some component of $K \setminus V$. Since $[o]_\sim \in M$, this component is contained in $(F^1 \cup \cdots \cup F^n) \setminus V$. Hence $M \subset Y$. (2) If each x^n is in M, then $F^n \subset M$ for each n. Indeed, chainable continua are hereditarily unicoherent ([5]) (see [11, 12.2 and 12.11]). Therefore, $F^n \cap M$ is a continuum. Moreover, F^n is homeomorphic to X; x^{n-1} and x^n are opposite end points of F^n. Since F^n is irreducible between x^{n-1} and x^n, we have $F^n \subset F^n \cap M \subset M$. Thus $Y \subset M$, and as Y is dense in K, we obtain $K = M$. The proof of the claim is complete.

The following formula defines a continuous function $g : K \xrightarrow{\text{onto}} L$:

$$g([a_0.a_1a_2a_3\ldots]_\sim) = [f(a_0).a_1a_2a_3\ldots]_\sim.$$

Observe that $g|E = \mathrm{id}_E$, and $L \setminus E$ is the two-to-one image of $K \setminus E$ under g. Since E is homeomorphic to the Cantor set, there exists a map h from L such that $h|E$ is two-to-one, $h|L \setminus E$ is one-to-one, and $h(E) \cap h(L \setminus E) = \emptyset$. Finally, $h \circ g$ is a two-to-one map from K onto $h(L)$. ■

([5]) A continuum is *hereditarily unicoherent* if any two subcontinua have connected intersection.

REMARKS. Our continuum K and the Knaster bucket handle continuum are constructed according to the same schema. Since different patterns (Lewis' continuum X or an arc, respectively) are used in these two cases, the resulting continua have different properties: *there exists no two-to-one map from the Knaster bucket handle continuum* ([10], see also [2]).

Lewis' construction [5] allows more than Example 1 says. His continuum X is built of certain continua C_i; each of them has opposite end points p_i and q_i, and the restrictions $f|C_i : C_i \to f(C_i)$ are open maps. Looking at the proof of Theorem 1.0 in [12] (the open image of a chainable continuum is chainable), we see that $f(p_i)$ and $f(q_i)$ are opposite end points of $f(C_i)$. Then we infer that $f(0)$ and $f(1)$ in Example 1 are opposite end points of the chainable image $f(X)$. Therefore, *we can prove that the continuum $L = g(K)$ in our construction of Example* 2 *is chainable and indecomposable.*

In general the image $h(L)$ need not be indecomposable, even if L is. The author does not know *whether the two-to-one image of an indecomposable chainable continuum can be indecomposable.*

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