BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 53, No. 1, 2005

SEVERAL COMPLEX VARIABLES AND ANALYTIC SPACES

On Compact Complex Manifolds with Finite Automorphism Group

by

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Summary. It is known that compact complex manifolds of general type and Kobayashi hyperbolic manifolds have finite automorphism groups. We give criteria for finiteness of the automorphism group of a compact complex manifold which allow us to produce large classes of compact complex manifolds with finite automorphism group but which are neither of general type nor Kobayashi hyperbolic.

Notations and conventions. Our notation and terminology are standard. We assume that complex spaces (in the sense of Serre) are connected and have a countable base of topology. By a *complex variety* we mean an irreducible complex space, and a *complex manifold* is an irreducible, nonsingular complex space. We write $\mathrm{Hol}(X,Y)$ for the totality of holomorphic maps of X into Y with the compact-open topology, and $\mathrm{Aut}(X)$ is the topological group of all holomorphic automorphisms of the complex space X.

We say that a compact complex variety X is *restricted* if X admits a holomorphic embedding into a complex projective space $\mathbb{P}_m$ for some m. By a *complex continuum* we mean a connected complex space of finite dimension.

We denote by $\mathrm{kod}(X)$ the Kodaira dimension of a compact complex manifold X, and by $\Pi_1(X)$ the fundamental group of X.

We say that a compact complex manifold X is a *Galois manifold* if the fundamental group $\Pi_1(X)$ is finite; if this group is moreover abelian, we say that X is an *abelian Galois manifold.*

If $a(X) := \#\,\mathrm{Aut}(X) < \infty$ we say that X is *modest*; if moreover X is abelian Galois then X is *very modest.* A variety X is called *primary* if

2000 *Mathematics Subject Classification*: Primary 32M99.
Key words and phrases: manifolds with finite automorphism group, non-Kähler manifolds.

it admits no primary decomposition $X = Z_1^{r_1} \times \cdots \times Z_q^{r_q}$, where $q \geq 2$, $r_1, \ldots, r_q$ are positive integers.

Results. We start by recalling Theorem 3 of [2], stating that if X and Y are compact complex spaces such that $a(X)$ and $a(Y)$ are finite, then $a(X)a(Y)$ divides $a(X \times Y)$, i.e. there exists a natural number $b(X \times Y)$ such that $a(X \times Y) = b(X \times Y)a(X)a(Y)$.

From this we can infer the following corollaries.

COROLLARY 1. *Let X and Y be compact modest complex continua. Then*

(1) *$X \times Y$ is modest and $a(X \times Y) = b(X \times Y)a(X)a(Y)$, where $b(X \times Y)$ is a positive integer.*
(2) *Assume that $W_1^{r_1} \times \cdots \times W_p^{r_p}$ and $Z_1^{s_1} \times \cdots \times Z_q^{s_q}$ are primary decompositions of X and Y respectively such that W_k is not biholomorphic to W_l for $k \neq l$ and Z_i is not biholomorphic to Z_j for $i \neq j$. If $\{W_1, \ldots, W_p\} \cap \{Z_1, \ldots, Z_q\} = \emptyset$ then $b(X \times Y) = 1$; in this case we say that X and Y are relatively prime.*
If $\{W_1, \ldots, W_p\} \cap \{Z_1, \ldots, Z_q\} = \{T_1, \ldots, T_m\}$, then
$$b(X \times Y) = \prod_{k=1}^{m} \binom{r_k + s_k}{r_k}.$$

COROLLARY 2. *Let X and Y be relatively prime compact modest varieties. Then $a(X \times Y) = a(X)a(Y)$.*

COROLLARY 3. *Let X be a compact modest variety and let n be an arbitrary positive integer. Then $a(X^n) = n!a(X)^n$.*

From the above and from the well known result of Hurwitz we infer that if X is a compact complex curve of genus $g \geq 2$ then for any positive integer n we have the estimate $a(X^n) \leq n![84(\mathrm{gen}(X) - 1)]^n$, and equality holds if X is the Klein curve given by the equation $x_0^3 x_1 + x_1^3 x_2 + x_1^3 x_0 = 0$.

COROLLARY 4. *Let X be a primary compact variety such that $a(X) = 1$. Then for any positive integer n we have $a(X^n) = n!$, and $\mathrm{Aut}(X^n)$ is isomorphic to the permutation group on n symbols.*

PROPOSITION 5. *Let n be an integer such that $n \geq 3$ and $n + 2$ is a prime. Suppose that α is a primitive root of 1 of order $n + 2$.*

(1) *The mapping*
$$\mu_\alpha : Z_{n+2} \times \mathbb{P}_n \ni (k, [x]) \mapsto [\alpha^k \cdot x_1, \ldots, \alpha^{ks} \cdot x_s, \ldots, \alpha^{k(n+1)} \cdot x_{n+1}] \in \mathbb{P}_n$$
is a holomorphic action of the additive group of nonnegative integers modulo $n + 2$.

(2) *The Fermat hypersurface* $\mathbb{E} := \{[x] \in \mathbb{P}_n : x_1^{n+2} + \cdots + x_{n+1}^{n+2} = 0\}$ *is nonsingular and simply connected, and the restriction of* μ_α *to* $Z_{n+2} \times \mathbb{E}$ *is a well defined free holomorphic action of* Z_{n+2} *on* $\mathbb{E}$.

(3) *The canonical projection* $p : \mathbb{E} \to B := \mathbb{E}/Z_{n+2}$ *is a holomorphic universal covering of the restricted manifold* B. *Hence* $\Pi_1(B) \simeq Z_{n+2}$.

(4) *Any Godeaux manifold* (*i.e. a complex manifold biholomorphic to* B) *is very modest.*

Proof. From [8] we infer $\operatorname{kod}(\mathbb{E}) = \operatorname{kod}(B)$. We have $\deg \mathbb{E} - (n+1) > 0$, hence $\operatorname{kod}(\mathbb{E}) = \dim(\mathbb{E})$ and so Y is modest. ∎

The class of very modest manifolds is large as shown by

PROPOSITION 6. *Suppose that* Y *is a Godeaux manifold and let* m *be a positive integer. Then there exists a very modest restricted manifold* X *such that*

(1) $\operatorname{kod}(X) = \dim X$,

(2) $b_2(X) \geq b_2(Y) + m$, *where* $b_2(X)$ *denotes the second Betti number.*

Proof. Let $f : X \to Y$ be a holomorphic surjection such that X is a complex manifold and there exists a point $y \in Y$ such that $f^{-1}(y)$ is biholomorphic to $\mathbb{P}_{n-1}$ and $f|X \setminus f^{-1}(y) : X \setminus f^{-1}(y) \to Y \setminus \{y\}$ is biholomorphic. We call such a map a *dilatation* of Y. It is well known that X is a restricted manifold and from [6, Satz 1.15] we infer that $b_2(X) \geq b_2(Y) + 1$. By a succession of m dilatations we obtain a holomorphic modification $f : X \to Y$ such that $b_2(X) \geq b_2(Y) + m$.

Observe that $\operatorname{kod}(X) = \operatorname{kod}(Y)$. From [3] we know that $\Pi_1(X) = \Pi_1(Y)$. ∎

Now we prove that there exists a large class of modest manifolds for which $\operatorname{kod}(X) < \dim X$.

PROPOSITION 7. *Let* n *be a positive integer. Then for any positive integer* k *such that* $n = 2q + k$ *for some positive integer* q, *there exists a restricted manifold* Y *for which the following conditions are satisfied*:

(1) $\dim Y = n$,

(2) $\operatorname{kod}(Y) = k$,

(3) Y *is modest.*

Proof. Let B be a restricted modest surface such that $\operatorname{kod}(B) = 0$ (examples of such surfaces are given in [4] and [5]). Let F be a restricted manifold of dimension k such that $\operatorname{kod}(F) = k$. If we put $Y := B^q \times F$ then $\operatorname{kod}(Y) = q \cdot \operatorname{kod}(B) + \operatorname{kod}(F) = k$. From Corollary 1 we conclude that Y is modest. ∎

Modest manifolds constructed above are restricted, hence Kähler. Below we show that modest Kähler manifolds form a small subclass of modest manifolds.

Proposition 8. *Let n be an integer ≥ 3 and let k be a positive integer such that $n = 2q+k$ for some positive integer q. Then there exists a compact complex manifold Y for which the following conditions are satisfied*:

(1) $\dim Y = n$,
(2) $\mathrm{kod}(Y) = k$,
(3) *Y is modest*,
(4) *Y is complex complete algebraic*,
(5) *Y is non-Kähler.*

Proof. Let B be a modest restricted surface such that $\mathrm{kod}(B) = 0$. Let M be a restricted 3-dimensional manifold. Then by [7, Chapter VI, § 4, Exercise 6], there exists a complex complete algebraic manifold N which is bimeromorphic to M and moreover N is not restricted. Let $F := N \times S$, where S is a restricted manifold such that $\mathrm{kod}(S) = \dim S = k$, and put $Y := B^q \times F$. If q is such that $n = 2q+k$ then condition (1) holds. Hence $\mathrm{kod}(Y) = q\cdot\mathrm{kod}(B)+\mathrm{kod}(F) = k$. Applying Corollary 1 to the equality $Y = B^q \times F$ we infer that Y is modest. Now observe that B and F are complete algebraic, hence Y is complete algebraic. The fact that F is complete algebraic implies that F is Moishezon; but a Moishezon manifold admits a Kähler structure iff it is restricted. Hence we infer that F is non-Kähler.

A product of compact complex manifolds is Kähler iff each factor is Kähler, hence we conclude that Y is non-Kähler. ∎

Now by applying Corollary 1 we get

Corollary 9. *Let X be a compact modest manifold and let Y be as in Proposition* 8. *Then $X \times Y$ is a modest non-Kähler manifold.*

Corollary 9 shows that compact modest Kähler manifolds form a very small subclass of compact modest manifolds.

If $f : X \to S$ is a smooth holomorphic surjective map such that S and all fibres are compact with discrete automorphism group, then $\mathrm{Aut}(X)$ is discrete. The proof is in principle the same as in the locally trivial case in [1]. Hence the following problem arises:

Given a smooth holomorphic surjective map $f : X \to S$ such that S and all fibres are compact modest, is X also modest?

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Received June 7, 2004;
received in final form January 31, 2005 (7398)