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ALGEBRAIC GEOMETRY

Łojasiewicz Exponent of Overdetermined Mappings

by

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Summary. A mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ is called overdetermined if $m > n$. We prove that the calculations of both the local and global Łojasiewicz exponent of a real overdetermined polynomial mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ can be reduced to the case $m = n$.

1. Introduction and results. Let $\mathbb{K}$ be the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. By $F : (\mathbb{K}^n, a) \to (\mathbb{K}^m, 0)$, where $a \in \mathbb{K}^n$, we denote a mapping from a neighbourhood $U \subset \mathbb{K}^n$ of a to $\mathbb{K}^m$ such that $F(a) = 0$. In this paper we study the local Łojasiewicz exponent and the Łojasiewicz exponent at infinity of *overdetermined* mappings, i.e. mappings $f : \mathbb{K}^n \to \mathbb{K}^m$ with $m > n$.

If $F : (\mathbb{K}^n, a) \to (\mathbb{K}^m, 0)$ is an analytic mapping (real analytic for $\mathbb{K} = \mathbb{R}$, holomorphic for $\mathbb{K} = \mathbb{C}$), then there are positive constants C, η, ε such that the following *Łojasiewicz inequality* holds:

$$|F(x)| \geq C \operatorname{dist}(x, F^{-1}(0))^{\eta} \quad \text{if } |x - a| < \varepsilon, \tag{1}$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{K}^n$ and $\operatorname{dist}(x, V)$ is the distance from $x \in \mathbb{K}^n$ to the set V ($\operatorname{dist}(x, V) = 1$ if $V = \emptyset$). The smallest exponent η in (1) is called the *Łojasiewicz exponent* of F at a and is denoted by $\mathcal{L}_a^{\mathbb{K}}(F)$. It is known that $\mathcal{L}_a^{\mathbb{K}}(F)$ is a rational number and (1) holds for any $\eta \geq \mathcal{L}_a^{\mathbb{K}}(F)$ and some $C, \varepsilon > 0$. The exponent $\mathcal{L}_a^{\mathbb{K}}(F)$ is an important invariant and tool in singularity theory (for references see for instance [7]).

In the following we will say that a condition holds *for the generic* $x \in A$ if there exists an algebraic set V such that $A \setminus V$ is a dense subset of A and the condition holds for all $x \in A \setminus V$.

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We shall denote by $\mathbf{L}^{\mathbb{K}}(m,k)$ the set of all linear mappings $\mathbb{K}^m \to \mathbb{K}^k$ (where we identify $\mathbb{K}^0$ with $\{0\}$).

In Section 2 we will prove the following

THEOREM 1. *Let $F:(\mathbb{R}^n,a)\to(\mathbb{R}^m,0)$ be an analytic mapping having an isolated zero at a, and let $n\le k\le m$. Then for any $L\in\mathbf{L}^{\mathbb{R}}(m,k)$ such that a is an isolated zero of $L\circ F$ we have*

$$\mathcal{L}_a^{\mathbb{R}}(F)\le\mathcal{L}_a^{\mathbb{R}}(L\circ F). \tag{2}$$

Moreover, for the generic $L\in\mathbf{L}^{\mathbb{R}}(m,k)$, the point a is an isolated zero of $L\circ F$ and

$$\mathcal{L}_a^{\mathbb{R}}(F)=\mathcal{L}_a^{\mathbb{R}}(L\circ F). \tag{3}$$

The above theorem is a generalization of [10, Theorem 2.1] from the complex case to the real case. Note that Theorem 1 is not a direct consequence of [10, Theorem 2.1], since the complexification of F may have a nonisolated zero at a.

Let $m\ge k$. We denote by $\varDelta^{\mathbb{K}}(m,k)$ the set of all linear mappings $L=(L_1,\dots,L_k)\in\mathbf{L}^{\mathbb{K}}(m,k)$ of the form

$$L_i(y_1,\dots,y_m)=y_i+\sum_{j=k+1}^{m}\alpha_{i,j}y_j,\qquad i=1,\dots,k,$$

where $\alpha_{i,j}\in\mathbb{K}$.

From Theorem 1, as in [10, Proposition 2.1], one can deduce

COROLLARY 1. *Under the assumptions of Theorem 1, for the generic $L\in\varDelta^{\mathbb{R}}(m,n)$, the point a is an isolated zero of $L\circ F$ and $\mathcal{L}_a^{\mathbb{R}}(F)=\mathcal{L}_a^{\mathbb{R}}(L\circ F)$.*

If additionally F is a polynomial mapping then without the assumptions on the zeroes of F we will prove (in Section 3)

THEOREM 2. *Let $F:(\mathbb{K}^n,a)\to(\mathbb{K}^m,0)$ be a polynomial mapping, and let $n\le k\le m$. Then for any $L\in\mathbf{L}^{\mathbb{K}}(m,k)$ such that $F^{-1}(0)\cap U_L=(L\circ F)^{-1}(0)\cap U_L$ for some neighbourhood $U_L\subset\mathbb{K}^n$ of a, we have*

$$\mathcal{L}_a^{\mathbb{K}}(F)\le\mathcal{L}_a^{\mathbb{K}}(L\circ F). \tag{4}$$

Moreover, for the generic $L\in\mathbf{L}^{\mathbb{K}}(m,k)$, we have $F^{-1}(0)\cap U_L=(L\circ F)^{-1}(0)\cap U_L$ for some neighbourhood $U_L\subset\mathbb{K}^n$ of a and

$$\mathcal{L}_a^{\mathbb{K}}(F)=\mathcal{L}_a^{\mathbb{K}}(L\circ F). \tag{5}$$

Theorem 2 implies

COROLLARY 2. *Under the assumptions of Theorem 2, for the generic $L\in\varDelta^{\mathbb{R}}(m,n)$ we have $\mathcal{L}_a^{\mathbb{R}}(F)=\mathcal{L}_a^{\mathbb{R}}(L\circ F)$.*

By Corollary 2 and [2] (see also [3], [1]) we obtain

COROLLARY 3. *Let $F = (f_1, \dots, f_m) : (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$, $m > n$, be a polynomial mapping and let $d_j = \deg f_j$ for $j = 1, \dots, m$. If $d_1 \geq \dots \geq d_m > 0$, then $\mathcal{L}_0^{\mathbb{C}}(F) \leq d_1 \cdots d_n$.*

Indeed, by Corollary 2, for the generic $L = (L_1, \dots, L_n) \in \Delta^{\mathbb{K}}(m, n)$ we have $\mathcal{L}_a^{\mathbb{K}}(F) = \mathcal{L}_a^{\mathbb{K}}(L \circ F)$. Moreover $d_j = \deg L_j \circ F$ for $j = 1, \dots, n$. E. Cygan [2] proved that for an analytic sets $X, Y \subset \mathbb{C}^n$ the separation exponent of X and Y at a point $a \in X \cap Y$ is the intersection index of $X \times Y$ and the diagonal $\Delta_{\mathbb{C}}^n$ of $\mathbb{C}^n \times \mathbb{C}^n$ at (a, a). It is known that for $X = \operatorname{graph} L \circ F$ and $Y = \mathbb{C}^n \times \{0\} \subset \mathbb{C}^n \times \mathbb{C}^n$, the index does not exceed $d_1 \cdots d_n$ (see [11], [3]). Consequently, $\mathcal{L}_0^{\mathbb{C}}(F) \leq d_1 \cdots d_n$.

By the *Łojasiewicz exponent at infinity* of a mapping $F : \mathbb{K}^n \to \mathbb{K}^m$ we mean the supremum of the exponents ν in the following Łojasiewicz inequality:

$$|F(x)| \geq C|x|^{\nu} \quad \text{whenever} \quad |x| \geq R \tag{6}$$

for some positive constants C, R; we denote it by $\mathcal{L}_{\infty}^{\mathbb{K}}(F)$. The Łojasiewicz exponent at infinity of a mapping has been considered by many authors in the context of effective Nullstellensatz and properness of mappings (for references see for instance [6], [8]).

In Section 4 we will prove the following generalization of [9, Theorem 2.1] from the complex case to the real case.

THEOREM 3. *Let $F = (f_1, \dots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ be a polynomial mapping having a compact set of zeros, and let $n \leq k \leq m$. Then for any $L \in \mathbf{L}^{\mathbb{R}}(m, k)$ such that $(L \circ F)^{-1}(0)$ is compact we have*

$$\mathcal{L}_{\infty}^{\mathbb{R}}(F) \geq \mathcal{L}_{\infty}^{\mathbb{R}}(L \circ F). \tag{7}$$

Moreover, for the generic $L \in \mathbf{L}^{\mathbb{K}}(m, k)$, the set $(L \circ F)^{-1}(0)$ is compact and

$$\mathcal{L}_{\infty}^{\mathbb{R}}(F) = \mathcal{L}_{\infty}^{\mathbb{R}}(L \circ F). \tag{8}$$

From the above theorem one can deduce (cf. [9] in the complex case)

COROLLARY 4. *Under the assumptions of Theorem 3, for the generic $L = (L_1, \dots, L_n) \in \Delta^{\mathbb{R}}(m, n)$, the set $(L \circ F)^{-1}(0)$ is compact and*

$$\mathcal{L}_{\infty}^{\mathbb{R}}(F) = \mathcal{L}_{\infty}^{\mathbb{R}}(L \circ F).$$

Moreover, if $d_j = \deg f_j$ and $d_1 \geq \dots \geq d_m$, then $\deg(L_j \circ F) = d_j$ for $j = 1, \dots, n$.

2. Proof of Theorem 1. It suffices to prove Theorem 1 for $a = 0$.

For a polynomial mapping $G : \mathbb{R}^n \to \mathbb{R}^m$, we denote by $G_{\mathbb{C}} : \mathbb{C}^n \to \mathbb{C}^m$ the complexification of G.

Let $G : \mathbb{K}^n \to \mathbb{K}^m$, where $m \geq n$, be a polynomial mapping, and let $k \in \mathbb{Z}$, $n \leq k \leq m$. Let $Y = \overline{G(\mathbb{C}^n)}$ if $\mathbb{K} = \mathbb{C}$ or $Y = \overline{G_{\mathbb{C}}(\mathbb{C}^n)}$ if $\mathbb{K} = \mathbb{R}$. The set Y is algebraic and $\dim_{\mathbb{C}} Y \le n$. Assume that $0 \in Y$, and let $C_0(Y)$ be the tangent cone to Y at 0 in the sense of Whitney [12, p. 510]. It is known that $C_0(Y)$ is an algebraic set and $\dim_{\mathbb{C}} C_0(Y) = \dim_{\mathbb{C}} Y \le n$. So, we have

LEMMA 1. *For the generic* $L \in \mathbf{L}^{\mathbb{K}}(m,k)$,

$$L^{-1}(0) \cap C_0(Y) \subset \{0\}.$$

In the proofs of Theorems 1–3 we will need

LEMMA 2. *If* $L \in \mathbf{L}^{\mathbb{K}}(m,k)$ *satisfies* $L^{-1}(0) \cap C_0(Y) \subset \{0\}$, *then there exist* $\varepsilon, C_1, C_2 > 0$ *such that for all* $x \in \mathbb{K}^n$ *with* $|G(x)| < \varepsilon$ *we have*

$$C_1|G(x)| \le |L(G(x))| \le C_2|G(x)|. \tag{9}$$

Proof. It is obvious that for $C_2 = \|L\|$ we obtain $|L(G(x))| \le C_2|G(x)|$ for all $x \in \mathbb{K}^n$. This gives the right-hand inequality in (9).

Now, we show the left-hand inequality. Assume to the contrary that for any $\varepsilon, C_1 > 0$ there exists $x \in \mathbb{K}^n$ such that

$$C_1|G(x)| > |L(G(x))| \quad \text{and} \quad |G(x)| < \varepsilon.$$

In particular for $\nu \in \mathbb{N}$, $C_1 = 1/\nu$, $\varepsilon = 1/\nu$ there exists $x_\nu \in \mathbb{K}^n$ such that

$$\frac{1}{\nu}|G(x_\nu)| > |L(G(x_\nu))| \quad \text{and} \quad |G(x_\nu)| < \frac{1}{\nu}.$$

Thus $|G(x_\nu)| > 0$ and

$$\frac{1}{\nu} > \frac{1}{|G(x_\nu)|}|L(G(x_\nu))| = \left|L\left(\frac{1}{|G(x_\nu)|}G(x_\nu)\right)\right|. \tag{10}$$

Let $\lambda_\nu = 1/|G(x_\nu)|$ for $\nu \in \mathbb{N}$. Then $|\lambda_\nu G(x_\nu)| = 1$. Choosing a subsequence if necessary, we may assume that $\lambda_\nu G(x_\nu) \to v$ as $\nu \to \infty$, where $v \in \mathbb{K}^n$, $|v| = 1$ and $G(x_\nu) \to 0$ as $\nu \to \infty$. Thus $v \in C_0(Y)$ and $v \ne 0$. Moreover, by (10), we have $L(v) = 0$. So $v \in L^{-1}(0) \cap C_0(Y) \subset \{0\}$. This contradicts the assumption and ends the proof. ∎

We will also need the following lemma (cf. [5], [10] in the complex case).

LEMMA 3. *Let* $F, G : (\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$ *be analytic mappings such that* $\operatorname{ord}_0(F - G) > \mathcal{L}_0^{\mathbb{R}}(F)$. *If* 0 *is an isolated zero of* F *then it is an isolated zero of* G *and for some positive constants* ε, C_1, C_2,

$$C_1|F(x)| \le |G(x)| \le C_2|F(x)| \quad \text{for } x \in \mathbb{R}^n \text{ with } |x| < \varepsilon. \tag{11}$$

In particular $\mathcal{L}_0^{\mathbb{R}}(F) = \mathcal{L}_0^{\mathbb{R}}(G)$.

Proof. Since 0 is an isolated zero of F, we have $1 \le \mathcal{L}_0^{\mathbb{R}}(F) < \infty$ and for some positive constants ε_0, C,

$$|F(x)| \ge C|x|^{\mathcal{L}_0^{\mathbb{R}}(F)} \quad \text{for } x \in \mathbb{R}^n \text{ with } |x| < \varepsilon_0. \tag{12}$$

Since $\operatorname{ord}_0(F - G) > \mathcal{L}_0^{\mathbb{R}}(F)$, there exist $\eta \in \mathbb{R}$, $\eta > \mathcal{L}_0^{\mathbb{R}}(F)$ and $\varepsilon_1 > 0$ such that $\big||F(x)| - |G(x)|\big| \le |x|^\eta$ for all $x \in \mathbb{R}^n$ with $|x| < \varepsilon_1$. Assume that (11) fails. Then for some sequence $x_\nu \in \mathbb{R}^n$ such that $x_\nu \to 0$ as $\nu \to \infty$, we have either $(1/\nu)|F(x_\nu)| > |G(x_\nu)|$ for all $\nu \in \mathbb{N}$, or $(1/\nu)|G(x_\nu)| > |F(x_\nu)|$ for all $\nu \in \mathbb{N}$. In both cases, by (12) for $\nu \geq 2$, we have

$$\frac{C}{2}|x_\nu|^{\mathcal{L}_0^{\mathbb{R}}(F)} \leq \frac{1}{2}|F(x_\nu)| < |F(x_\nu) - G(x_\nu)| \leq |x_\nu|^\eta,$$

which is impossible. The equality $\mathcal{L}_0^{\mathbb{R}}(F) = \mathcal{L}_0^{\mathbb{R}}(G)$ follows from (11). ∎

Proof of Theorem 1. By the argument in the proof of [10, Theorem 2.1] we obtain (2). We now prove (3).

Let $G = (g_1, \ldots, g_m) : (\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$ be a polynomial mapping such that $\mathrm{ord}_0^{\mathbb{R}}(F - G) > \mathcal{L}_0^{\mathbb{R}}(F)$. Obviously, such a mapping exists. By Lemma 3, $\mathcal{L}_0^{\mathbb{R}}(F) = \mathcal{L}_0^{\mathbb{R}}(G)$ and 0 is an isolated zero of G. By Lemmas 1 and 2 for the generic $L \in \mathbf{L}^{\mathbb{R}}(m, k)$ the mapping $L \circ G$ has an isolated zero at $0 \in \mathbb{R}^n$, $\mathcal{L}_0^{\mathbb{R}}(G) = \mathcal{L}_0^{\mathbb{R}}(L \circ G)$, and

$$\begin{aligned} \mathrm{ord}_0(L \circ G - L \circ F) = {} & \mathrm{ord}_0\, L \circ (G - F) \geq \mathrm{ord}_0(G - F) \\ & > \mathcal{L}_0^{\mathbb{R}}(F) = \mathcal{L}_0^{\mathbb{R}}(G) = \mathcal{L}_0^{\mathbb{R}}(L \circ G), \end{aligned}$$

so, by Lemma 3, $\mathcal{L}_0^{\mathbb{R}}(L \circ F) = \mathcal{L}_0^{\mathbb{R}}(L \circ G) = \mathcal{L}_0^{\mathbb{R}}(F)$. This gives the assertion. ∎

3. Proof of Theorem 2. From [9, Proposition 1.1] we immediately obtain

Proposition 1. *Let $G = (g_1, \ldots, g_m) : \mathbb{K}^n \to \mathbb{K}^m$ be a polynomial mapping with $\deg g_j > 0$ for $j = 1, \ldots, m$, where $m \geq n \geq 1$, and let $k \in \mathbb{Z}$, $n \leq k \leq m$.*

(i) *For the generic $L \in \mathbf{L}^{\mathbb{K}}(m, k)$,*

$$\#[(L \circ G)^{-1}(0) \setminus G^{-1}(0)] < \infty. \tag{13}$$

(ii) *The condition* (13) *also holds for the generic $L \in \Delta^{\mathbb{K}}(m, k)$.*

Proof. Let us first consider the case $n = k$. Let

$$W = \{L \in \mathbf{L}^{\mathbb{C}}(m, n) : \#[(L \circ F_{\mathbb{C}})^{-1}(0) \setminus F_{\mathbb{C}}^{-1}(0)] < \infty\}.$$

By [9, Proposition 1.1], W contains a nonempty Zariski open subset of $\mathbf{L}^{\mathbb{C}}(m, n)$. Then W contains a dense Zariski open subset of $\mathbf{L}^{\mathbb{R}}(m, n)$. This gives the assertion in the case $n = k$.

Now assume $k > n$. Since for $L = (L_1, \ldots, L_k) \in \mathbf{L}^{\mathbb{K}}(m, k)$,

$$(L \circ G)^{-1}(0) \subset ((L_1, \ldots, L_n) \circ G)^{-1}(0),$$

we deduce the assertion from the previous case. ∎

Proof of Theorem 2. It suffices to prove Theorem 2 for $a = 0$. Without loss of generality we may assume that $F \neq 0$. By definition, there exist

$C, \varepsilon > 0$ such that for all $x \in \mathbb{K}^n$ with $|x| < \varepsilon$ we have

$$(14) \qquad |F(x)| \geq C \operatorname{dist}(x, F^{-1}(0))^{\mathcal{L}_0^{\mathbb{K}}(F)},$$

and $\mathcal{L}_0^{\mathbb{K}}(F)$ is the smallest exponent for which the inequality holds. Let $L \in \mathbf{L}^{\mathbb{K}}(m,k)$ be such that $F^{-1}(0) \cap U_L = (L \circ F)^{-1}(0) \cap U_L$ for some neighbourhood $U_L \subset \mathbb{K}^n$ of 0. Diminishing ε and the neighbourhood U_L if necessary, we may assume that $\operatorname{dist}(x, F^{-1}(0)) = \operatorname{dist}(x, F^{-1}(0) \cap U_L)$ for all $x \in \mathbb{K}^n$ with $|x| < \varepsilon$. Obviously $L \neq 0$, so $\|L\| > 0$ and $|F(x)| \geq \frac{1}{\|L\|}|L(F(x))|$. Then by (14) we obtain $\mathcal{L}_a^{\mathbb{K}}(F) \leq \mathcal{L}_a^{\mathbb{K}}(L \circ F)$ and (4) is proved.

By Proposition 1 and Lemmas 1 and 2, for the generic $L \in \mathbf{L}^{\mathbb{K}}(m,k)$ we have $F^{-1}(0) \cap U_L = (L \circ F)^{-1}(0) \cap U_L$ for some neighbourhood $U_L \subset \mathbb{K}^n$ of 0 and there exist $\varepsilon, C_1, C_2 > 0$ such that for all $x \in \mathbb{K}^n$ with $|x| < \varepsilon$,

$$(15) \qquad C_1|F(x)| \leq |L(F(x))| \leq C_2|F(x)|.$$

Together with (14), this gives (5) and ends the proof of Theorem 2. ∎

4. Proof of Theorem 3. We recall Lemma 2.2 from [9]:

Lemma 4. *Let $F : \mathbb{C}^n \to \mathbb{C}^m$ with $m \geq n$ be a polynomial mapping. Then there exists a Zariski open and dense subset $W \subset \mathbf{L}^{\mathbb{C}}(m,n)$ such that for any $L \in W$ and any $\varepsilon > 0$ there exist $\delta > 0$ and $r > 0$ such that for any $x \in \mathbb{C}^n$,*

$$|x| > r \wedge |L \circ F(x)| < \delta \ \Rightarrow\ |F(x)| < \varepsilon.$$

In the proof of Theorem 3 we will need the following version of the above lemma in the real case.

Lemma 5. *Let $F : \mathbb{R}^n \to \mathbb{R}^m$ with $m \geq n$ be a polynomial mapping and let $k \in \mathbb{Z}$ with $n \leq k \leq m$. Then there exists a Zariski open and dense subset $W \subset \mathbf{L}^{\mathbb{R}}(m,k)$ such that for any $L \in W$ and any $\varepsilon > 0$ there exist $\delta > 0$ and $r > 0$ such that for any $x \in \mathbb{R}^n$,*

$$|x| > r \wedge |L \circ F(x)| < \delta \ \Rightarrow\ |F(x)| < \varepsilon.$$

Proof. For $k = n$ the assertion immediately follows from Lemma 4. Let W_1 be a Zariski open and dense subset of $\mathbf{L}^{\mathbb{R}}(m,n)$ for which the assertion holds with $k = n$. Let $k > n$ and

$$W = \{L = (L_1, \ldots, L_k) \in \mathbf{L}^{\mathbb{R}}(m,k) : (L_1, \ldots, L_n) \in W_1\}.$$

Then for any $L = (L_1, \ldots, L_k) \in W$ and $x \in \mathbb{R}^n$ we have

$$|(L_1, \ldots, L_n) \circ F(x)| \leq |L \circ F(x)|,$$

so the assertion immediately follows from the previous case. ∎

Proof of Theorem 3. Since for nonzero $L \in \mathbf{L}^{\mathbb{R}}(m,k)$ we have $|L \circ F(x)| \leq \|L\|\,|F(x)|$ and $\|L\| > 0$, the definition of the Łojasiewicz exponent at infinity yields the first part of the assertion. We now prove the second part.

Since $F^{-1}(0)$ is a compact set by Proposition 1, there exists a dense Zariski open subset W_1 of $\mathbf{L}^{\mathbb{R}}(m,k)$ such that

$$W_1 \subset \{L \in \mathbf{L}^{\mathbb{R}}(m,k) : \#(L \circ F)^{-1}(0) < \infty\},$$

so for the generic $L \in \mathbf{L}^{\mathbb{R}}(m,k)$ we have $\#(L \circ F)^{-1}(0) < \infty$.

If $\mathcal{L}^{\mathbb{R}}_\infty(F) < 0$, the assertion (8) follows from Lemmas 1, 2 and 5.

Assume that $\mathcal{L}^{\mathbb{R}}_\infty(F) = 0$. Then there exist $C, R > 0$ such that $|F(x)| \geq C$ whenever $|x| \geq R$. Moreover, there exists a sequence $x_\nu \in \mathbb{R}^n$ such that $|x_\nu| \to \infty$ as $\nu \to \infty$ and $|F(x_\nu)|$ is a bounded sequence. So by Lemma 5 for the generic $L \in \mathbf{L}^{\mathbb{R}}(m,k)$ and $\varepsilon = C$ there exist $r, \delta > 0$ such that $|L \circ F(x)| \geq \delta$ if $|x| > r$, hence $\mathcal{L}^{\mathbb{R}}_\infty(L \circ F) \geq 0$. Since $|L \circ F(x_\nu)|$ is a bounded sequence, we have $\mathcal{L}^{\mathbb{R}}_\infty(L \circ F) \leq 0$. Summing up, $\mathcal{L}^{\mathbb{R}}_\infty(L \circ F) = \mathcal{L}^{\mathbb{R}}_\infty(F)$ in this case.

Now we prove the assertion in the case $\mathcal{L}^{\mathbb{R}}_\infty(F) > 0$. Let $Y = \overline{F_{\mathbb{C}}(\mathbb{C}^n)}$. Then $\dim Y \leq n$. From Sadullaev's Theorem ([4, VII, 7.1]) there exists a Zariski open and dense subset $W_2 \subset \mathbf{L}^{\mathbb{C}}(m,k)$ such that for any $L \in W_2$ there exist $r > 0$ and $M \in \mathbf{L}^{\mathbb{C}}(m, m-k)$ for which $(L, M) \in \mathbf{L}^{\mathbb{C}}(m,m)$ is a linear automorphism and for any $y \in Y$,

$$|y| \geq r \;\Rightarrow\; |M(y)| \leq |L(y)|.$$

Moreover, we may assume that $L \in W_2$ is a nonsingular linear mapping. So

$$|y| > r \;\Rightarrow\; |(L, M)(y)| = |L(y)|. \tag{16}$$

Obviously $W_1 \cap W_2$ contains a set W_3 which is Zariski open and dense in $\mathbf{L}^{\mathbb{R}}(m,k)$. Let $L \in W_3$ and $M \in \mathbf{L}^{\mathbb{R}}(m, m-k)$ be as above. Since $\mathcal{L}^{\mathbb{R}}_\infty(F) > 0$, there exists $R_1 > 0$ such that for any $x \in \mathbb{C}^n$ with $|x| > R_1$ we have $|F(x)| > r$. Then, from (16),

$$|x| > R_1 \;\Rightarrow\; |(L, M) \circ f(x)| = |L \circ f(x)|.$$

Thus, $\mathcal{L}^{\mathbb{R}}_\infty(L \circ F) = \mathcal{L}^{\mathbb{R}}_\infty((L, M) \circ F)$. Since (L, M) is a linear automorphism, we have $\mathcal{L}^{\mathbb{R}}_\infty((L, M) \circ F) = \mathcal{L}^{\mathbb{R}}_\infty(F)$, so (8) is proved in this case. ∎

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