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PARTIAL DIFFERENTIAL EQUATIONS

Global Attractors for a Class of Semilinear Degenerate Parabolic Equations on $\mathbb{R}^N$

by

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Summary. We prove the existence of global attractors for the following semilinear degenerate parabolic equation on $\mathbb{R}^N$:

$$\frac{\partial u}{\partial t} - \operatorname{div}(\sigma(x)\nabla u) + \lambda u + f(x,u) = g(x),$$

under a new condition concerning the variable nonnegative diffusivity $\sigma(\cdot)$ and for an arbitrary polynomial growth order of the nonlinearity f. To overcome some difficulties caused by the lack of compactness of the embeddings, these results are proved by combining the tail estimates method and the asymptotic *a priori* estimate method.

1. Introduction. In this paper we consider the following semilinear degenerate parabolic equation with a variable, nonnegative coefficient in $\mathbb{R}^N$, $N \geq 2$:

$$(1.1)\qquad \begin{aligned} \frac{\partial u}{\partial t} - \operatorname{div}(\sigma(x)\nabla u) + \lambda u + f(x,u) &= g(x), \qquad x \in \mathbb{R}^N, t > 0,\\ u(0,x) &= u_0(x), \quad x \in \mathbb{R}^N, \end{aligned}$$

where $\lambda > 0$, $u_0 \in L^2(\mathbb{R}^N)$ and $g \in L^2(\mathbb{R}^N)$ are given, and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ and $\sigma(\cdot)$ are functions satisfying some conditions specified later.

Problem (1.1) can be derived as a simple model for neutron diffusion (feedback control of nuclear reactor) (see [DL]). In this case u and σ stand for the neutron flux and neutron diffusion respectively. The degeneracy of problem (1.1) is considered in the sense that the measurable nonnegative

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diffusion coefficient $\sigma(\cdot)$ is allowed to have at most a finite number of (essential) zeroes at some points.

Problem (1.1) in a general (bounded or unbounded) domain $\Omega \subset \mathbb{R}^N$ was studied in [ABT, AT, KZ1, KZ2], in which the diffusivity $\sigma(\cdot)$ was assumed to satisfy the following conditions which ensure important compactness properties:

($\mathcal{H}_\alpha$) $\sigma \in L^1_{\mathrm{loc}}(\Omega)$ and for some $\alpha \in (0,2), \liminf_{x\to z} |x-z|^{-\alpha}\sigma(x) > 0$ for every $z \in \overline{\Omega}$, when the domain Ω is bounded;

($\mathcal{H}^\infty_{\alpha,\beta}$) σ satisfies condition ($\mathcal{H}_\alpha$) and $\liminf_{|x|\to\infty} |x|^{-\beta}\sigma(x) > 0$ for some $\beta > 2$, when the domain Ω is unbounded.

Both assumptions have a strong physical significance which is related to the existence of regions occupied by perfect insulators or perfect conductors [CM, DL, KZ1, KZ2]. The natural phase space for problem (1.1) in these cases involves $\mathcal{D}^1_0(\Omega,\sigma)$, which is defined as the closure of $C^\infty_0(\Omega)$ in the norm

$$\|u\|_{\mathcal{D}^1_0(\Omega,\sigma)} := \Big(\int_\Omega \sigma(x)|\nabla u|^2\,dx\Big)^{1/2}.$$

Under either assumption ($\mathcal{H}_\alpha$) or ($\mathcal{H}^\infty_{\alpha,\beta}$), the embedding $\mathcal{D}^1_0(\Omega,\sigma) \hookrightarrow L^2(\Omega)$ is compact and this property plays an essential role for the investigation in [ABT, AT, KZ1, KZ2]. Observe, however, that when Ω is unbounded, the function $\sigma(\cdot)$ must grow faster than quadratically at infinity for this property to hold (see [CM]). We also refer the reader to [AK] for some related results in the quasilinear case.

In this paper we would like to find a new condition concerning the diffusivity $\sigma(\cdot)$ which ensures the asymptotic compactness of the semigroup generated by problem (1.1) and as a result, the existence of global attractors, without restricting the limiting behavior of $\sigma(\cdot)$ at infinity. It turns out that such a condition can be found with careful tail estimates as in [W] (see the proof of Lemma 3.4 below).

In order to study problem (1.1), we make the following assumptions:

(**F**) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying

$$f(x,u)u \ge \alpha_1|u|^p - C_1(x), \tag{1.2}$$

$$|f(x,u)| \le \alpha_2|u|^{p-1} + C_2(x), \tag{1.3}$$

$$\frac{\partial f}{\partial u}(x,u) \ge -\alpha_3, \tag{1.4}$$

for some $p \ge 2$, where $\alpha_1,\alpha_2,\alpha_3$ are positive constants, $C_1(\cdot) \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ and $C_2(\cdot) \in L^{p'}(\mathbb{R}^N)$ with $1/p + 1/p' = 1$ are nonnegative functions. Denote $F(x,s) = \int_0^s f(x,\tau)\,d\tau$. Then we assume that F satisfies

$$-C_4(x)+\alpha_4|u|^p \le F(x,u) \le \alpha_5|u|^p + C_3(x), \tag{1.5}$$

where α_4, α_5 are positive constants, and $C_3(\cdot), C_4(\cdot) \in L^1(\mathbb{R}^N)$ are nonnegative functions.

($\mathcal{H}^\infty$) σ is a nonnegative measurable function such that $\sigma \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, and for some $\alpha \in (0,2)$,

$$\liminf_{x\to z} |x-z|^{-\alpha}\sigma(x) > 0 \quad \text{for every } z \in \mathbb{R}^N,$$

and σ satisfies one of the following two conditions:

(i) there exists $K_0 > 0$ such that

$$\sup_{k\ge K_0} \sup_{k\le|x|\le\sqrt{2}\,k} \sigma(x) < \infty;$$

(ii) there exists $K_0 > 0$ such that

$$\sup_{k\ge K_0} \int\limits_{k\le|x|\le\sqrt{2}\,k} |\sigma(x)|^{\frac{p-1}{p-2}}\,dx < \infty,$$

where p is given in assumption (**F**);

(**G**) $g \in L^2(\mathbb{R}^N)$.

Let us comment on the condition ($\mathcal{H}^\infty$). Observe that the absence of a specific limiting behavior at infinity for $\sigma(\cdot)$ (cf. condition ($\mathcal{H}^\infty_{\alpha,\beta}$)) is now compensated by a higher local integrability. A simple example in which ($\mathcal{H}^\infty$) is fulfilled but ($\mathcal{H}^\infty_{\alpha,\beta}$) is not, is provided by the function $\sigma \equiv 1$ (the nondegenerate case) or $\sigma(x) = e^{-|x|}(|x|^\alpha + |x|^\gamma)$ with $\alpha, \gamma \in (0,2)$.

For $\Omega \subset \mathbb{R}^N$, we define $\mathcal{H}^1_0(\Omega,\sigma)$ to be the closure of $C^\infty_0(\Omega)$ with respect to the norm

$$\|u\|^2_{\mathcal{H}^1_0(\Omega,\sigma)} := \int\limits_\Omega |u|^2\,dx + \int\limits_\Omega \sigma(x)|\nabla u|^2\,dx.$$

Notice that in the compact case, that is, when ($\mathcal{H}_\alpha$) or ($\mathcal{H}^\infty_{\alpha,\beta}$) holds, $\mathcal{H}^1_0(\Omega,\sigma) \equiv \mathcal{D}^1_0(\Omega,\sigma)$ since $\mathcal{D}^1_0(\Omega,\sigma) \hookrightarrow L^2(\Omega)$ (see [CM]). The natural energy space for problem (1.1) involves the space $\mathcal{H}^1_0(\mathbb{R}^N,\sigma)$ and its dual space $\mathcal{H}^{-1}(\mathbb{R}^N,\sigma)$.

The main aim of this paper is to prove the existence of a global attractor in the space $\mathcal{H}^1_0(\mathbb{R}^N,\sigma) \cap L^p(\mathbb{R}^N)$ for the semigroup generated by problem (1.1). First, we use the Galerkin method to prove the global existence of a weak solution and then construct the semigroup associated to problem (1.1). Next, we use *a priori* estimates to show the existence of a bounded absorbing set in $\mathcal{H}^1_0(\mathbb{R}^N,\sigma)\cap L^p(\mathbb{R}^N)$ for the semigroup. In the compact case [ABT], i.e. when the domain Ω satisfies ($\mathcal{H}_\alpha$) or ($\mathcal{H}^\infty_{\alpha,\beta}$), since the embedding $\mathcal{D}^1_0(\Omega,\sigma) \hookrightarrow L^2(\Omega)$ is compact, this immediately implies the asymptotic compactness in $L^2(\Omega)$. Here, because the embedding is no longer compact, the proof of the asymptotic compactness in $L^2(\mathbb{R}^N)$ is much more involved. To do this, we exploit the tail estimates method introduced in [W], and as a result, we obtain the existence of a global attractor in $L^2(\mathbb{R}^N)$. When proving the existence of global attractors in $L^p(\mathbb{R}^N)$ and in $\mathcal{H}_0^1(\mathbb{R}^N,\sigma)\cap L^p(\mathbb{R}^N)$, to overcome the difficulty arising from the lack of embedding results, we use the asymptotic *a priori* estimate method initiated in [MWZ, ZYS]. The main new feature of the paper is that we are able to prove the existence of global attractors for a class of semilinear degenerate parabolic equations in the noncompact case. As fas as we know, this is the first result for parabolic equations of type (1.1) in this case.

The paper is organized as follows. In Section 2, we prove the existence and uniqueness of a weak solution to problem (1.1) by using the Galerkin method. In Section 3, we show the existence of global attractors in various function spaces for the semigroup generated by the problem (1.1) by exploiting and combining the tail estimates method and the asymptotic *a priori* estimate method.

2. Existence and uniqueness of weak solutions. We first give the definition of a weak solution.

DEFINITION 2.1. A function $u:(0,\infty)\to\mathcal{H}_0^1(\mathbb{R}^N,\sigma)\cap L^p(\mathbb{R}^N)$ is said to be a *weak solution* of (1.1) if $u\in L^2(0,T;\mathcal{H}_0^1(\mathbb{R}^N,\sigma))\cap L^p(0,T;L^p(\mathbb{R}^N))\cap L^\infty(0,T;L^2(\mathbb{R}^N))$ for all $T>0$, and

$$(u(t),v)_{L^2(\mathbb{R}^N)}+\int_0^t\int_{\mathbb{R}^N}\sigma\nabla u\nabla v\,dx\,dt+\lambda\int_0^t(u,v)_{L^2(\mathbb{R}^N)}\,dt+\int_0^t\int_{\mathbb{R}^N}f(x,u)v\,dx\,dt$$
$$=(u_0,v)_{L^2(\mathbb{R}^N)}+\int_0^t(g,v)_{L^2(\mathbb{R}^N)}\,dt,\qquad \forall t>0,$$

for all $v\in\mathcal{H}_0^1(\mathbb{R}^N,\sigma))\cap L^p(\mathbb{R}^N)$.

One can check that this definition in fact coincides with the usual definition of (global) weak solutions (see e.g. [CV]). Thus, it follows from [CV, p. 285] that if u is a weak solution of (1.1), then $u\in C([0,T];L^2(\mathbb{R}^N))$, the function $t\mapsto\|u(t)\|^2_{L^2(\mathbb{R}^N)}$ is absolutely continuous on every interval $[0,T]$, and

$$\frac{d}{dt}\|u(t)\|^2_{L^2(\mathbb{R}^N)}=\left\langle\frac{du}{dt}(t),u(t)\right\rangle\quad\text{for a.e. } t\in[0,T].$$

We now prove the following theorem.

THEOREM 2.2. *Let* $(\mathcal{H}^\infty)$, $(\mathbf{F})$ *and* $(\mathbf{G})$ *hold. Then, for any given* $u_0\in L^2(\mathbb{R}^N)$, *problem* (1.1) *has a unique weak solution* u. *Moreover, the mapping* $u_0\mapsto u(t)$ *is continuous on* $L^2(\mathbb{R}^N)$.

Proof. The proof is similar to the proof of Theorem 3.2 in [ABT], except passing to the limit in the nonlinear term $f(\cdot,\cdot)$, so we only sketch it. For each $m \geq 1$, we denote

$$\Omega_m = \{x \in \mathbb{R}^N : |x|_{\mathbb{R}^N} < m\},$$

where $|\cdot|_{\mathbb{R}^N}$ denotes the Euclidean norm in $\mathbb{R}^N$. For each integer $n \geq 1$, we denote by

$$u_n(t) = \sum_{j=1}^{n} \gamma_{nj}(t)\omega_j$$

a solution of

$$\frac{d}{dt}(u_n(t),\omega_j) - (\operatorname{div}(\sigma\nabla u),\omega_j) + \lambda(u_n(t),\omega_j) + (f(x,u_n(t)),\omega_j) = (g,\omega_j), \quad t > 0,$$

$$(u_n(0),\omega_j) = (u_0,\omega_j), \quad j = 1,\ldots,n,$$

where $\{\omega_j : j \geq 1\} \subset \mathcal{H}_0^1(\mathbb{R}^N,\sigma) \cap L^p(\mathbb{R}^N)$ is a Hilbert basis of $L^2(\mathbb{R}^N)$ such that $\operatorname{span}\{\omega_j : j \geq 1\}$ is dense in $\mathcal{H}_0^1(\mathbb{R}^N,\sigma) \cap L^p(\mathbb{R}^N)$.

It is a standard matter to deduce that

(2.1)
- $\{u_n\}$ is bounded in $L^2(0,T;\mathcal{H}_0^1(\mathbb{R}^N,\sigma)) \cap L^p(0,T;L^p(\mathbb{R}^N)) \cap L^\infty(0,T;L^2(\mathbb{R}^N))$,
- $\{f(x,u_n)\}$ is bounded in $L^{p'}(0,T;L^{p'}(\mathbb{R}^N))$.

for all $T > 0$. Then there exists a subsequence $\{u_\mu\}$ such that

$$u_\mu \rightharpoonup^* u \quad \text{weakly-star in } L^\infty(0,T;L^2(\mathbb{R}^N)),$$

$$u_\mu \rightharpoonup u \quad \text{in } L^p(0,T;L^p(\mathbb{R}^N)),$$

$$u_\mu \rightharpoonup u \quad \text{in } L^2(0,T;\mathcal{H}_0^1(\mathbb{R}^N,\sigma)), \tag{2.2}$$

$$f(x,u_\mu) \rightharpoonup \chi \quad \text{in } L^{p'}(0,T;L^{p'}(\mathbb{R}^N)), \tag{2.3}$$

for all $T > 0$. Hence, (2.2) implies that

$$-\operatorname{div}(\sigma(x)\nabla u_\mu) + \lambda u_\mu \rightharpoonup -\operatorname{div}(\sigma(x)\nabla u) + \lambda u \quad \text{in } L^2(0,T;\mathcal{H}^{-1}(\mathbb{R}^N,\sigma)).$$

Now, to prove that $\chi(t) = f(\cdot,u(t))$, we argue similarly to [R]. Arguing as in [R, p. 75] we first deduce

$$\lim_{a\to 0}\sup_{\mu} \int_0^{T-a} \|u_\mu(t+a) - u_\mu(t)\|^2_{L^2(\mathbb{R}^N)}\,dt = 0. \tag{2.4}$$

Let $\phi \in C^1([0,\infty))$ be a function such that

$$0 \leq \phi(s) \leq 1,$$

$$\phi(s) = 1 \quad \forall s \in [0,1],$$

$$\phi(s) = 0 \quad \forall s \geq 2.$$

For each μ and $m \geq 1$, we define

(2.5) $$v_{\mu,m}(x,t) = \phi\Big(\frac{|x|^2_{\mathbb{R}^N}}{m^2}\Big)u_\mu(t), \quad \forall x \in \Omega_{2m}, \forall \mu, \forall m \geq 1.$$

We infer from (2.1) that, for all $m \geq 1$, the sequence $\{v_{\mu,m}\}_{\mu \geq 1}$ is bounded in $L^\infty(0,T;L^2(\Omega_{2m})) \cap L^p(0,T;L^p(\Omega_{2m})) \cap L^2(0,T;\mathcal{H}^1_0(\Omega_{2m},\sigma))$ for all $T > 0$. In particular, it follows that

$$\lim_{a\to 0}\sup_\mu \Big(\int_0^a \|v_{\mu,m}(x,t)\|^2_{L^2(\Omega_{2m})}\,dt + \int_{T-a}^T \|v_{\mu,m}(x,t)\|^2_{L^2(\Omega_{2m})}\,dt\Big) = 0.$$

On the other hand, from (2.4) we deduce that for all $m \geq 1$,

$$\lim_{a\to 0}\sup_\mu \Big(\int_0^{T-a} \|v_{\mu,m}(x,t+a) - v_{\mu,m}(x,t)\|^2_{L^2(\Omega_{2m})}\,dt\Big) = 0.$$

Moreover, as Ω_{2m} is a bounded set, $\mathcal{H}^1_0(\Omega_{2m},\sigma)$ is included in $L^2(\Omega_{2m})$ with compact injection. Then, by Theorem 13.3 and Remark 13.1 in [T1], it follows that

$$\{v_{\mu,m}\}_{\mu\geq 1} \text{ is relatively compact in } L^2(0,T;L^2(\Omega_{2m})),$$

and thus, taking into account that $v_{\mu,m}(x,t) = u_\mu(x,t)$ for all $x \in \Omega_m$, we deduce that, in particular, for all $m \geq 1$,

(2.6) $$\{u_\mu|_{\Omega_m}\} \text{ is pre-compact in } L^2(0,T;L^2(\Omega_m)).$$

Hence, by a diagonal procedure, one can conclude from (2.6) and (2.2) that there exists a subsequence $\{u^\mu_\mu\}_{\mu\geq 1} \subset \{u_\mu\}_{\mu \geq 1}$ such that

$$u^\mu_\mu \to u \quad \text{in } \Omega_m \times (0,\infty) \text{ as } n \to \infty, \forall m \geq 1.$$

Then, as $f(\cdot,\cdot)$ is continuous,

$$f(x,u^\mu_\mu) \to f(x,u) \quad \text{a.e. in } \Omega_m \times (0,\infty),$$

and as $\{f(x,u^\mu_\mu)\}$ is bounded in $L^{p'}(\Omega_m \times (0,T))$, by Lemma 1.3 in [L, Chapter 1], we obtain

$$f(x,u^\mu_\mu) \rightharpoonup f(x,u) \quad \text{in } L^{p'}(0,T;L^{p'}(\Omega_m)).$$

By the uniqueness of the weak limit, we have

$$\chi = f(x,u) \quad \text{a.e. in } \Omega_m \times (0,T) \quad \forall T > 0, \forall m \geq 1,$$

and thus, taking into account that $\bigcup_{m=1}^\infty \Omega_m = \mathbb{R}^N$, we obtain

(2.7) $$\chi = f(x,u) \quad \text{a.e. in } \mathbb{R}^N \times (0,\infty).$$

Then, (2.7) and (2.3) yield

$$f(x,u_\mu) \rightharpoonup f(x,u) \quad \text{in } L^{p'}(0,T;L^{p'}(\mathbb{R}^N)) \quad \forall T > 0.$$

Hence, it is standard to show that u is a weak solution to problem (1.1). The uniqueness and continuous dependence of the weak solutions on the initial data follows by the same arguments as in [ABT]. ∎

3. Existence of global attractors. Thanks to Theorem 2.2, we can define a continuous semigroup

$$S(t) : L^2(\mathbb{R}^N) \to \mathcal{H}_0^1(\mathbb{R}^N,\sigma) \cap L^p(\mathbb{R}^N),$$

where $S(t)u_0 := u(t)$ is the unique weak solution of (1.1) with u_0 as initial datum.

We first prove the existence of an absorbing set for $S(t)$ in $\mathcal{H}_0^1(\mathbb{R}^N,\sigma) \cap L^p(\mathbb{R}^N)$. For brevity, in the following lemmas, we only give some formal calculations; the rigorous proof uses Galerkin approximations and Lemma 11.2 in [R].

LEMMA 3.1. *Suppose* $(\mathcal{H}^\infty)$, $(\mathbf{F})$ *and* $(\mathbf{G})$ *hold. Then the semigroup* $S(t)$ *generated by* (1.1) *has a bounded absorbing set in* $\mathcal{H}_0^1(\mathbb{R}^N,\sigma) \cap L^p(\mathbb{R}^N)$, *that is, there exists a positive constant* ρ *such that for every bounded subset* B *in* $L^2(\mathbb{R}^N)$, *there is a number* $T = T(B) > 0$ *such that for all* $t \geq T$, $u_0 \in B$, *we have*

$$\|u(t)\|^2_{\mathcal{H}_0^1(\mathbb{R}^N,\sigma)} + \|u(t)\|^p_{L^p(\mathbb{R}^N)} \leq \rho.$$

Proof. Taking the inner product of (1.1) with u in $L^2(\mathbb{R}^N)$ we get

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} \sigma(x)|\nabla u|^2\,dx + \lambda\|u\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} f(x,u)u\,dx = (g,u). \tag{3.1}$$

By (1.2), we have

$$\int_{\mathbb{R}^N} f(x,u)u\,dx \geq \alpha_1 \int_{\mathbb{R}^N} |u|^p\,dx - \int_{\mathbb{R}^N} C_1(x)\,dx. \tag{3.2}$$

By Cauchy's inequality, the right-hand side of (3.1) is estimated as follows:

$$|(g,u)| \leq \|g\|_{L^2(\mathbb{R}^N)}\|u\|_{L^2(\mathbb{R}^N)} \leq \frac{\lambda}{2}\|u\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{2\lambda}\|g\|^2_{L^2(\mathbb{R}^N)}. \tag{3.3}$$

It follows from (3.1)–(3.3) that

$$\frac{d}{dt}\|u\|^2_{L^2(\mathbb{R}^N)} + 2\int_{\mathbb{R}^N} \sigma|\nabla u|^2\,dx + \lambda\|u\|^2_{L^2(\mathbb{R}^N)} + 2\alpha_1\int_{\mathbb{R}^N} |u|^p\,dx \leq C + \frac{1}{\lambda}\|g\|^2_{L^2(\mathbb{R}^N)}. \tag{3.4}$$

Hence, in particular, we have

$$\frac{d}{dt}\|u(t)\|^2_{L^2(\mathbb{R}^N)} + \lambda\|u(t)\|^2_{L^2(\mathbb{R}^N)} \leq C + \frac{1}{\lambda}\|g\|^2_{L^2(\mathbb{R}^N)}.$$

Using Gronwall's inequality, we obtain

$$\|u(t)\|^2_{L^2(\mathbb{R}^N)} \le e^{-\lambda t}\|u_0\|^2_{L^2(\mathbb{R}^N)} + \left(\frac{C}{\lambda} + \frac{1}{\lambda^2}\|g\|^2_{L^2(\mathbb{R}^N)}\right)(1 - e^{-\lambda t}). \tag{3.5}$$

From (3.5) we deduce the existence of a bounded absorbing set in $L^2(\mathbb{R}^N)$: There is a constant R and a time $t_0(\|u_0\|_{L^2(\mathbb{R}^N)})$ such that for the solution $u(t) = S(t)u_0$,

$$\|u(t)\|_{L^2(\mathbb{R}^N)} \le R \quad \text{for all } t \ge t_0(\|u_0\|_{L^2(\mathbb{R}^N)}).$$

Integrating (3.4) on $(t, t+1)$, $t \ge t_0(\|u_0\|_{L^2(\mathbb{R}^N)})$, and using (1.5), we find that

$$\int_t^{t+1} \Big(\int_{\mathbb{R}^N} \sigma(x)|\nabla u(s)|^2\,dx + \lambda\|u(s)\|^2_{L^2(\mathbb{R}^N)} + 2\int_{\mathbb{R}^N} F(x,u(s))\,dx\Big)\,ds \tag{3.6}$$
$$\le C(\|u(t)\|^2_{L^2(\mathbb{R}^N)} + 1 + \|g\|^2_{L^2(\mathbb{R}^N)}) \le C(R^2 + 1 + \|g\|^2_{L^2(\mathbb{R}^N)}).$$

Multiplying (1.1) by $u_t(s)$ and integrating over $\mathbb{R}^N$, we obtain

$$\|u_t(s)\|^2_{L^2(\mathbb{R}^N)} \tag{3.7}$$
$$+ \frac{1}{2}\frac{d}{ds}\Big(\int_{\mathbb{R}^N} \sigma(x)|\nabla u(s)|^2\,dx + \lambda\|u(s)\|^2_{L^2(\mathbb{R}^N)} + 2\int_{\mathbb{R}^N} F(x,u(s))\,dx\Big)$$
$$= \int_{\mathbb{R}^N} g u_t(s)\,dx \le \frac{1}{2}\|g\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{2}\|u_t(s)\|^2_{L^2(\mathbb{R}^N)}.$$

Hence,

$$\frac{d}{ds}\Big(\int_{\mathbb{R}^N} \sigma(x)|\nabla u(s)|^2\,dx + \lambda\|u(s)\|^2_{L^2(\mathbb{R}^N)} + 2\int_{\mathbb{R}^N} F(x,u(s))\,dx\Big) \tag{3.8}$$
$$\le \|g\|^2_{L^2(\mathbb{R}^N)}.$$

Combining (3.6), (3.8), and using the uniform Gronwall inequality, we have

$$\int_{\mathbb{R}^N} \sigma(x)|\nabla u(t)|^2\,dx + \lambda\|u(t)\|^2_{L^2(\mathbb{R}^N)} + 2\int_{\mathbb{R}^N} F(x,u(t))\,dx \tag{3.9}$$
$$\le C(R^2 + 1 + \|g\|^2_{L^2(\mathbb{R}^N)}).$$

Using (1.5) once again, we finish the proof. ■

We now derive uniform estimates of the derivatives of solutions in time.

LEMMA 3.2. *Suppose* $(\mathcal{H}^\infty)$, $(\mathbf{F})$ *and* $(\mathbf{G})$ *hold. Then for every bounded subset* B *in* $L^2(\mathbb{R}^N)$, *there exists a constant* $T = T(B) > 0$ *such that*

$$\|u_t(s)\|^2_{L^2(\mathbb{R}^N)} \le \rho_1 \quad \textit{for all } u_0 \in B \textit{ and } s \ge T,$$

where $u_t(s) = \frac{d}{dt}(S(t)u_0)\big|_{t=s}$ *and* ρ_1 *is a positive constant independent of* B.

Proof. By differentiating (1.1) in time and denoting $v = u_t$, we get

$$\frac{\partial v}{\partial t} - \operatorname{div}(\sigma(x)\nabla v) + \lambda v + \frac{\partial f}{\partial u}(x,u)v = 0.$$

Taking the inner product of the above equality with v in $L^2(\mathbb{R}^N)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} \sigma(x)|\nabla v|^2\,dx + \lambda\|v\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} \frac{\partial f}{\partial u}(x,u)|v|^2\,dx = 0. \tag{3.10}$$

By (1.4), it follows from (3.10) that

$$\frac{d}{dt}\|v\|^2_{L^2(\mathbb{R}^N)} \le 2\alpha_3\|v\|^2_{L^2(\mathbb{R}^N)}. \tag{3.11}$$

On the other hand, integrating (3.7) from t to $t+1$ and using (3.9), we obtain

$$\int_t^{t+1} \|u_t(s)\|^2_{L^2(\mathbb{R}^N)}\,ds \le C(\rho, \|g\|^2_{L^2(\mathbb{R}^N)}) \tag{3.12}$$

for t large enough. Combining (3.11) with (3.12), and using the uniform Gronwall inequality, we have

$$\|u_t(s)\|^2_{L^2(\mathbb{R}^N)} \le C(\rho, \|g\|^2_{L^2(\mathbb{R}^N)}).$$

The proof is complete. ∎

LEMMA 3.3. *Suppose* $(\mathcal{H}^\infty)$, $(\mathbf{F})$ *and* $(\mathbf{G})$ *hold. Then the semigroup* $\{S(t)\}_{t\ge 0}$ *has a bounded absorbing set in* $L^{2p-2}(\mathbb{R}^N)$, *i.e., there exists a positive constant* ρ_{2p-2} *such that for any bounded subset* $B \subset L^2(\mathbb{R}^N)$, *there is a number* $T = T(B) > 0$ *such that*

$$\|u(t)\|_{L^{2p-2}(\mathbb{R}^N)} \le \rho_{2p-2} \quad \text{for any } t \ge T \text{ and } u_0 \in B.$$

Proof. Taking $|u|^{p-2}u$ as a test function, we obtain

$$\int_{\mathbb{R}^N} |u|^{p-2}u\cdot u_t\,dx + \int_{\mathbb{R}^N} \sigma(x)|\nabla u|^2|u|^{p-2}\,dx + \lambda\int_{\mathbb{R}^N} |u|^p\,dx + \int_{\mathbb{R}^N} f(x,u)|u|^{p-2}u\,dx = \int_{\mathbb{R}^N} g|u|^{p-2}u\,dx.$$

Hence, using (1.2) and Cauchy's inequality, we obtain

$$\int_{\mathbb{R}^N} \sigma(x)|\nabla u|^2|u|^{p-2}\,dx + \lambda\int_{\mathbb{R}^N} |u|^p\,dx + \alpha_1\int_{\mathbb{R}^N} |u|^{2p-2}\,dx \le \int_{\mathbb{R}^N} C_1(x)|u|^{p-1}\,dx + \frac{1}{\alpha_1}\int_{\mathbb{R}^N} |g|^2\,dx + \frac{\alpha_1}{2}\int_{\mathbb{R}^N} |u|^{2p-2}\,dx + \frac{1}{\alpha_1}\int_{\mathbb{R}^N} |u_t|^2\,dx.$$

Using Cauchy's inequality once again, we arrive at

$$\frac{\alpha_1}{4}\int_{\mathbb{R}^N}|u|^{2p-2}\,dx \le \frac{1}{\alpha_1}\|g\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{\alpha_1}\int_{\mathbb{R}^N}|u_t|^2\,dx + \frac{1}{\alpha_1}\int_{\mathbb{R}^N}|C_1(x)|^2\,dx.$$

Hence, by Lemma 3.2, there exists $T = T(B)$ such that

$$\|u(t)\|_{L^{2p-2}} \le \rho_{2p-2} \quad \text{for any } t \ge T \text{ and } u_0 \in B,$$

where ρ_{2p-2} depends only on $\|g\|_{L^2(\mathbb{R}^N)}$. ∎

3.1. Existence of a global attractor in $L^2(\mathbb{R}^N)$

LEMMA 3.4. *Suppose* $(\mathcal{H}^\infty)$, $(\mathbf{F})$ *and* $(\mathbf{G})$ *hold. Then for any* $\eta > 0$ *and any bounded subset* $B \subset L^2(\mathbb{R}^N)$, *there exist* $T = T(\eta, B) > 0$ *and* $K = K(\eta, B) > 0$ *such that for all* $t \ge T$ *and* $k \ge K$,

$$\int_{|x|\ge k}|u(x,t)|^2\,dx \le \eta,$$

where u *is the weak solution of* (1.1) *subject to the initial condition* $u(0) = u_0 \in B$.

Proof. We use a cut-off technique to establish the estimates on the tails of solutions. Let θ be a smooth function satisfying $0 \le \theta(s) \le 1$ for $s \in \mathbb{R}^+$, and

$$\theta(s) = 0 \quad \text{for } 0 \le s \le 1, \qquad \theta(s) = 1 \quad \text{for } s \ge 2.$$

Then there exists a constant C such that $|\theta'(s)| \le C$ for all $s \in \mathbb{R}^+$. Taking the inner product of (1.1) with $\theta(|x|^2/k^2)u$ in $L^2(\mathbb{R}^N)$, we get

$$(3.13)\qquad \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}\theta\Big(\frac{|x|^2}{k^2}\Big)|u|^2\,dx - \int_{\mathbb{R}^N}\theta\Big(\frac{|x|^2}{k^2}\Big)u\,\mathrm{div}(\sigma(x)\nabla u)\,dx$$
$$+\lambda\int_{\mathbb{R}^N}\theta\Big(\frac{|x|^2}{k^2}\Big)|u|^2\,dx + \int_{\mathbb{R}^N}\theta\Big(\frac{|x|^2}{k^2}\Big)f(x,u)u\,dx = \int_{\mathbb{R}^N}\theta\Big(\frac{|x|^2}{k^2}\Big)g(x)u(x,t)\,dx.$$

For the right-hand side of (3.13) we find that

$$(3.14)\qquad \int_{\mathbb{R}^N}\theta\Big(\frac{|x|^2}{k^2}\Big)g(x)u(x,t)\,dx = \int_{|x|\ge k}\theta\Big(\frac{|x|^2}{k^2}\Big)g(x)u(x,t)\,dx$$
$$\le \frac{\lambda}{2}\int_{|x|\ge k}\theta^2\Big(\frac{|x|^2}{k^2}\Big)|u|^2\,dx + \frac{1}{2\lambda}\int_{|x|\ge k}|g(x)|^2\,dx$$
$$\le \frac{\lambda}{2}\int_{\mathbb{R}^N}\theta^2\Big(\frac{|x|^2}{k^2}\Big)|u|^2\,dx + \frac{1}{2\lambda}\int_{|x|\ge k}|g(x)|^2\,dx.$$

We now estimate the last term of the left-hand side of (3.13) as follows:

$$(3.15)\qquad \int_{\mathbb{R}^N} \theta\Big(\frac{|x|^2}{k^2}\Big) f(x,u)\,dx \geq \alpha_1 \int_{\mathbb{R}^N} \theta\Big(\frac{|x|^2}{k^2}\Big)|u|^p\,dx - \int_{\mathbb{R}^N} \theta\Big(\frac{|x|^2}{k^2}\Big) C_1(x)\,dx \geq -\int_{|x|\geq k} C_1(x)\,dx.$$

For the second term on the left-hand side of (3.13), by integrating by parts, we have

$$(3.16)\qquad -\int_{\mathbb{R}^N} \theta\Big(\frac{|x|^2}{k^2}\Big) u \operatorname{div}(\sigma(x)\nabla u)\,dx = \int_{\mathbb{R}^N} \theta\Big(\frac{|x|^2}{k^2}\Big)\sigma(x)|\nabla u|^2\,dx + \int_{\mathbb{R}^N} \theta'\Big(\frac{|x|^2}{k^2}\Big)\Big(\frac{2x}{k^2}\cdot\sigma(x)\nabla u\Big) u\,dx \geq \int_{k\leq|x|\leq\sqrt{2}\,k} \theta'\Big(\frac{|x|^2}{k^2}\Big)\Big(\frac{2x}{k^2}\cdot\sigma(x)\nabla u\Big) u\,dx.$$

It follows from (3.13)–(3.16) that

$$(3.17)\qquad \frac{d}{dt}\int_{\mathbb{R}^N} \theta\Big(\frac{|x|^2}{k^2}\Big)|u|^2\,dx + \lambda\int_{\mathbb{R}^N} \theta\Big(\frac{|x|^2}{k^2}\Big)|u|^2\,dx \leq 2\int_{|x|\geq k} |C_1(x)|\,dx + \frac{1}{\lambda}\int_{|x|\geq k} |g(x)|^2\,dx + 2\int_{k\leq|x|\leq\sqrt{2}\,k} \Big|\theta'\Big(\frac{|x|^2}{k^2}\Big)\Big|\frac{2|x|}{k^2}\cdot\sigma(x)|\nabla u|\,|u|\,dx.$$

We have

$$(3.18)\qquad \int_{k\leq|x|\leq\sqrt{2}\,k} \Big|\theta'\Big(\frac{|x|^2}{k^2}\Big)\Big|\frac{2|x|}{k^2}\cdot\sigma(x)|\nabla u|\,|u|\,dx \leq \frac{C}{k}\int_{k\leq|x|\leq\sqrt{2}\,k} \sigma(x)|\nabla u|\,|u|\,dx \leq \frac{C}{k}\Big(\int_{k\leq|x|\leq\sqrt{2}\,k} \sigma(x)|u|^2\,dx\Big)^{1/2}\Big(\int_{k\leq|x|\leq\sqrt{2}\,k} \sigma(x)|\nabla u|^2\,dx\Big)^{1/2},$$

where C is independent of k.

We now estimate the term $\int_{k\leq|x|\leq\sqrt{2}\,k}\sigma(x)|u|^2\,dx$.

CASE 1: *σ satisfies condition* (i) *in* $(\mathcal{H}^\infty)$. We have for all $k\geq K_0$,

$$\int_{k\leq|x|\leq\sqrt{2}\,k} \sigma(x)|u|^2\,dx \leq C\int_{k\leq|x|\leq\sqrt{2}\,k} |u|^2\,dx.$$

Hence, by Lemma 3.1, we deduce that for all $k \geq K_0$ and $t \geq T_0$,

$$\begin{aligned}(3.19)\qquad &\int_{k\leq|x|\leq\sqrt{2}\,k}\left|\theta'\Big(\frac{|x|^2}{k^2}\Big)\right|\frac{2|x|}{k^2}\cdot\sigma(x)|\nabla u|\,|u|\,dx\\
&\leq \frac{C}{k}\Big(\int_{k\leq|x|\leq\sqrt{2}\,k}|u|^2\,dx\Big)^{1/2}\Big(\int_{k\leq|x|\leq\sqrt{2}\,k}\sigma(x)|\nabla u|^2\,dx\Big)^{1/2}\\
&\leq \frac{C}{k}\Big(\int_{k\leq|x|\leq\sqrt{2}\,k}\sigma(x)|\nabla u|^2\,dx\Big)^{1/2}.\end{aligned}$$

Case 2: *σ satisfies condition* (ii) *in* $(\mathcal{H}^\infty)$. By Hölder's inequality, we obtain

$$\begin{aligned}(3.20)\qquad &\int_{k\leq|x|\leq\sqrt{2}\,k}\sigma(x)|u|^2\,dx\\
&\leq\Big(\int_{k\leq|x|\leq\sqrt{2}\,k}\sigma(x)^{\frac{p-1}{p-2}}\,dx\Big)^{\frac{p-2}{p-1}}\Big(\int_{k\leq|x|\leq\sqrt{2}\,k}|u|^{2p-2}\,dx\Big)^{\frac{1}{p-1}}.\end{aligned}$$

Hence, by Lemma 3.3, we deduce that for all $k \geq K_0$ and $t \geq T_0$,

$$\begin{aligned}(3.21)\qquad &\int_{k\leq|x|\leq\sqrt{2}\,k}\left|\theta'\Big(\frac{|x|^2}{k^2}\Big)\right|\frac{2|x|}{k^2}\cdot\sigma(x)|\nabla u|\,|u|\,dx\\
&\leq \frac{C}{k}\|\sigma\|^{1/2}_{L^{\frac{p-1}{p-2}}(k\leq|x|\leq\sqrt{2}\,k)}\Big(\int_{k\leq|x|\leq\sqrt{2}\,k}|u|^{2p-2}\,dx\Big)^{\frac{1}{2p-2}}\\
&\quad\times\Big(\int_{k\leq|x|\leq\sqrt{2}\,k}\sigma(x)|\nabla u|^2\,dx\Big)^{1/2}\\
&\leq \frac{C}{k}\rho_{2p-2}\|\sigma\|^{1/2}_{L^{\frac{p-1}{p-2}}(k\leq|x|\leq\sqrt{2}\,k)}\Big(\int_{k\leq|x|\leq\sqrt{2}\,k}\sigma(x)|\nabla u|^2\,dx\Big)^{1/2}\\
&\leq \frac{C}{k}\Big(\int_{k\leq|x|\leq\sqrt{2}\,k}\sigma(x)|\nabla u|^2\,dx\Big)^{1/2}.\end{aligned}$$

It follows from (3.17), (3.19), and (3.21) that

$$\begin{aligned}(3.22)\qquad &\frac{d}{dt}\int_{\mathbb{R}^N}\theta\Big(\frac{|x|^2}{k^2}\Big)|u|^2\,dx+\lambda\int_{\mathbb{R}^N}\theta\Big(\frac{|x|^2}{k^2}\Big)|u|^2\,dx\\
&\leq 2\int_{|x|\geq k}|C_1(x)|\,dx+\frac{1}{\lambda}\int_{|x|\geq k}|g(x)|^2\,dx+2\frac{C}{k}\Big(\int_{k\leq|x|\leq\sqrt{2}\,k}\sigma(x)|\nabla u|^2\,dx\Big)^{1/2}.\end{aligned}$$

Multiplying (3.22) by $e^{\lambda t}$ and then integrating over (T_0, t), we obtain

$$\int_{\mathbb{R}^N} \theta\Big(\frac{|x|^2}{k^2}\Big)|u(t)|^2\,dx \le e^{-\lambda t}\int_{\mathbb{R}^N} \theta\Big(\frac{|x|^2}{k^2}\Big)|u_0|^2\,dx \tag{3.23}$$
$$+ 2e^{-\lambda t}\int_{T_0}^{t}\int_{|x|\ge k} e^{\lambda\xi}|C_1(x)|\,dx\,d\xi + \frac{1}{\lambda}e^{-\lambda t}\int_{T_0}^{t} e^{\lambda\xi}\int_{|x|\ge k}|g(x)|^2\,dx\,d\xi$$
$$+ 2\frac{C}{k}e^{-\lambda t}\int_{T_0}^{t} e^{\lambda\xi}\Big(\int_{k\le|x|\le\sqrt{2}\,k}\sigma(x)|\nabla u(\xi)|^2\,dx\Big)^{1/2}d\xi$$
$$\le e^{-\lambda t}\|u(T_0)\|^2_{L^2(\mathbb{R}^N)} + 2e^{-\lambda t}\int_{T_0}^{t}\int_{|x|\ge k} e^{\lambda\xi}|C_1(x)|\,dx\,d\xi$$
$$+ \frac{1}{\lambda^2}\int_{|x|\ge k}|g(x)|^2\,dx + 2\frac{C}{k}e^{-\lambda t}\int_{T_0}^{t} e^{\lambda\xi}\Big(\int_{\mathbb{R}^N}\sigma(x)|\nabla u(\xi)|^2\,dx\Big)^{1/2}d\xi.$$

Note that for given $\eta > 0$, there is $T_1 = T_1(\eta) > 0$ such that for all $t \ge T_1$,

$$e^{-\lambda t}\|u(T_0)\|^2_{L^2(\mathbb{R}^N)} \le \frac{\eta}{4}. \tag{3.24}$$

Since $C_1(\cdot) \in L^1(\mathbb{R}^N)$, there exists $K_1 = K_1(\eta) > K_0$ such that for all $k \ge K_1$,

$$2e^{-\lambda t}\int_{T_0}^{t}\int_{|x|\ge k} e^{\lambda\xi}|C_1(x)|\,dx\,d\xi \le \frac{\eta}{4}. \tag{3.25}$$

On the other hand, since $g \in L^2(\mathbb{R}^N)$, there is $K_2 = K_2(\eta) > K_1$ such that for all $k \ge K_2$,

$$\frac{1}{\lambda^2}\int_{|x|\ge k}|g(x)|^2\,dx \le \frac{\eta}{4}. \tag{3.26}$$

For the last term on the right-hand side of (3.23), it follows from Lemma 3.1 that there is $T_2 > 0$ such that for all $\xi \ge T_2$,

$$\int_{\mathbb{R}^N}\sigma(x)|\nabla u(\xi)|^2\,dx \le \rho.$$

Therefore, there is $K_3 = K_3(\eta) > K_2$ such that for all $k \ge K_3$ and $t \ge T_2$,

$$\frac{C}{k}e^{-\lambda t}\int_0^t e^{\lambda\xi}\Big(\int_{\mathbb{R}^N}\sigma(x)|\nabla u(\xi)|^2\,dx\Big)^{1/2}d\xi \le \frac{\eta}{4}. \tag{3.27}$$

Let $T = \max\{T_0, T_1, T_2\}$. Then by (3.23)–(3.27) we find that for all $k \ge K_3$

and $t \geq T$,

$$\int_{\mathbb{R}^N} \theta\left(\frac{|x|^2}{k^2}\right)|u(t)|^2\,dx \leq \eta,$$

and hence for all $k \geq K_3$ and $t \geq T$,

$$\int_{|x|\geq\sqrt{2}\,k} |u(t)|^2\,dx \leq \int_{\mathbb{R}^N} \theta\left(\frac{|x|^2}{k^2}\right)|u(t)|^2\,dx \leq \eta,$$

which completes the proof. ∎

Now, we show the asymptotic compactness of $S(t)$ in $L^2(\mathbb{R}^N)$.

LEMMA 3.5. *Suppose* $(\mathcal{H}^\infty)$, $(\mathbf{F})$ *and* $(\mathbf{G})$ *hold. Then* $S(t)$ *is asymptotically compact in* $L^2(\mathbb{R}^N)$, *that is, for any bounded sequence* $\{x_n\}_{n=1}^\infty \subset L^2(\mathbb{R}^N)$ *and any sequence* $t_n \geq 0$ *with* $t_n \to \infty$, $\{S(t_n)x_n\}_{n=1}^\infty$ *has a convergent subsequence with respect to the topology of* $L^2(\mathbb{R}^N)$.

Proof. We use the uniform estimates on the tails of solutions to establish the precompactness of $\{u_n(t_n) := S(t_n)x_n\}$, that is, we prove that for every $\eta > 0$, the sequence $\{u_n(t_n)\}$ has a finite covering of balls of radii less than η. Given $K > 0$, denote

$$\Omega_K = \{x : |x| \leq K\} \quad \text{and} \quad \Omega_K^c = \{x : |x| > K\}.$$

Then by Lemma 3.4, for the given $\eta > 0$, there exist $K = K(\eta) > 0$ and $T = T(\eta) > 0$ such that for $t \geq T$,

$$\|u_n(t)\|_{L^2(\Omega_K^c)} \leq \eta.$$

Since $t_n \to \infty$, there is $N_1 = N_1(\eta) > 0$ such that $t_n \geq T$ for all $n \geq N_1$, and hence we obtain, for all $n \geq N_1$,

$$\|u_n(t_n)\|_{L^2(\Omega_K^c)} \leq \eta. \tag{3.28}$$

Let $\zeta(\cdot) \in C^\infty(\mathbb{R}^N)$ be a function such that $0 \leq \zeta(s) \leq 1$ for any $s \geq 0$, and

$$\zeta(s) = 1 \quad \text{for } 0 \leq s \leq 1, \qquad \zeta(s) = 0 \quad \text{for } s \geq 2.$$

Furthermore, define $\zeta_k(x) = \zeta(|x|^2/k^2)$. Then $\{\zeta_K u_n(t_n)\}$ belongs to $\mathcal{H}_0^1(\Omega_{\sqrt{2}\,k}, \sigma)$. By Lemma 3.1, there exist $C > 0$ and $N_2 > 0$ such that for all $n \geq N_2$,

$$\|\zeta_K u_n(t_n)\|_{\mathcal{H}_0^1(\Omega_{\sqrt{2}\,k}, \sigma)} \leq C.$$

By the compactness of the embedding $\mathcal{H}_0^1(\Omega_{\sqrt{2}\,k}, \sigma) \equiv \mathcal{D}_0^1(\Omega_{\sqrt{2}\,k}, \sigma) \hookrightarrow L^2(\Omega_{\sqrt{2}\,k})$ (see [CM]), the sequence $\{\zeta_K u_n(t_n)\}$ is precompact in $L^2(\Omega_{\sqrt{2}\,k})$. This in particular implies that $\{u_n(t_n)\}$ is precompact in $L^2(\Omega_K)$. Therefore, for the given $\eta > 0$, $\{u_n(t_n)\}$ has a finite covering in $L^2(\Omega_K)$ of balls of radii less than η, which along with (3.28) shows that $\{u_n(t_n)\}$ has a finite covering in $L^2(\mathbb{R}^N)$ of balls of radii less than η, and thus $\{u_n(t_n)\}$ is precompact in $L^2(\mathbb{R}^N)$. ■

We are now ready to prove the existence of a global attractor for $S(t)$ in $L^2(\mathbb{R}^N)$.

THEOREM 3.6. *Suppose* $(\mathcal{H}^\infty)$, $(\mathbf{F})$ *and* $(\mathbf{G})$ *hold. Then the semigroup* $S(t)$ *generated by problem* (1.1) *has a global attractor* $\mathcal{A}_{L^2}$ *in* $L^2(\mathbb{R}^N)$.

Proof. Denote

$$B = \{u : \|u\|_{L^2(\mathbb{R}^N)} \le R\},$$

where R is the positive constant in the proof of Lemma 3.1. Then B is a bounded absorbing set for $S(t)$ in $L^2(\mathbb{R}^N)$. In addition, $S(t)$ is asymptotically compact in $L^2(\mathbb{R}^N)$ since Lemma 3.5. Thus, there exists a global attractor $\mathcal{A}_{L^2}$ for $S(t)$ in $L^2(\mathbb{R}^N)$. ■

3.2. Existence of a global attractor in $L^p(\mathbb{R}^N)$. First, from Lemma 3.1, one can see that $S(t)$ maps compact subsets of $\mathcal{H}_0^1(\mathbb{R}^N,\sigma) \cap L^p(\mathbb{R}^N)$ to bounded subsets of $\mathcal{H}_0^1(\mathbb{R}^N,\sigma) \cap L^p(\mathbb{R}^N)$. Hence, by Theorem 3.2 in [ZYS], we see that $S(t)$ is norm-to-weak continuous on $\mathcal{H}_0^1(\mathbb{R}^N,\sigma) \cap L^p(\mathbb{R}^N)$.

To obtain the existence of a global attractor in $L^p(\mathbb{R}^N)$, we need the following lemma, whose proof is very similar to the proof of Corollary 5.7 in [ZYS], so we omit it.

LEMMA 3.7. *Let* $\{S(t)\}_{t\ge 0}$ *be a norm-to-weak continuous semigroup on* $L^p(\mathbb{R}^N)$, *and suppose it is continuous or weakly continuous on* $L^2(\mathbb{R}^N)$, *and has a global attractor in* $L^2(\mathbb{R}^N)$. *Then* $\{S(t)\}_{t\ge 0}$ *has a global attractor in* $L^p(\mathbb{R}^N)$ *if and only if*

(i) $\{S(t)\}_{t\ge 0}$ *has a bounded absorbing set in* $L^p(\mathbb{R}^N)$;
(ii) *for any* $\epsilon > 0$ *and any bounded subset* B *of* $L^p(\mathbb{R}^N)$, *there exist positive constants* $M = M(\epsilon, B)$ *and* $T = T(\epsilon, B)$ *such that for any* $u_0 \in B$ *and* $t \ge T$,

$$\int\limits_{\mathbb{R}^N(|S(t)u_0|\ge M)} |S(t)u_0|^p\,dx < \epsilon, \tag{3.29}$$

where $\mathbb{R}^N(|S(t)u_0| \ge M) := \{x \in \mathbb{R}^N : |(S(t)u_0)(x)| \ge M\}$.

THEOREM 3.8. *Assume* $(\mathcal{H}^\infty)$, $(\mathbf{F})$ *and* $(\mathbf{G})$ *hold. Then the semigroup* $S(t)$ *generated by problem* (1.1) *has a global attractor* $\mathcal{A}_{L^p}$ *in* $L^p(\mathbb{R}^N)$, *that is,* $\mathcal{A}_{L^p}$ *is compact, invariant in* $L^p(\mathbb{R}^N)$ *and attracts every bounded subset of* $L^2(\mathbb{R}^N)$ *in the topology of* $L^p(\mathbb{R}^N)$.

Proof. We only need to show that $\{S(t)\}$ satisfies condition (ii) in Lemma 3.7. Take M large enough such that $\alpha_1|u|^{p-1} \le f(x,u)$ in

$$\mathbb{R}^N(u \ge M) := \{x \in \mathbb{R}^N : u(x,t) \ge M\},$$

and denote

$$(u-M)_+ = \begin{cases} u-M, & u \ge M \\ 0, & u \le M. \end{cases}$$

First, for any fixed $\epsilon > 0$, there exists $\delta > 0$ such that for any $e \subset \mathbb{R}^N$ with $m(e) \le \delta$, we have

$$\int_e |g|^2\,dx < \epsilon. \tag{3.30}$$

In $\mathbb{R}^N(u \ge M)$ we see that

$$\begin{aligned} g(u-M)^{p-1} &\le \frac{\alpha_1}{2}(u-M)^{2p-2} + \frac{1}{2\alpha_1}|g|^2 \\ &\le \frac{\alpha_1}{2}(u-M)^{p-1}|u|^{p-1} + \frac{1}{2\alpha_1}|g|^2, \end{aligned} \tag{3.31}$$

and

$$\begin{aligned} f(x,u)(u-M)^{p-1} &\ge \alpha_1|u|^{p-1}(u-M)^{p-1} \\ &\ge \frac{\alpha_1}{2}(u-M)^{p-1}|u|^{p-1} + \frac{\alpha_1 M^{p-2}}{2}(u-M)^p. \end{aligned} \tag{3.32}$$

Multiplying equation (1.1) by $|(u-M)_+|^{p-1}$ and using (3.31), (3.32), we deduce that

$$\begin{aligned} \frac{2}{p}\frac{d}{dt}\|u-M\|^p_{L^p(\mathbb{R}^N(u\ge M))} &+ (p-1)\int\limits_{\mathbb{R}^N(u\ge M)} \sigma(x)|\nabla(u-M)|^2(u-M)^{p-2}\,dx \\ &+ \lambda \int\limits_{\mathbb{R}^N(u\ge M)} (u-M)^{2p-2}\,dx + \alpha_1 M^{p-2}\int\limits_{\mathbb{R}^N(u\ge M)} (u-M)^p\,dx \\ &\le \frac{1}{\alpha_1}\int\limits_{\mathbb{R}^N(u\ge M)} |g|^2\,dx. \end{aligned}$$

Therefore,

$$\frac{d}{dt}\|u-M\|^p_{L^p(\mathbb{R}^N(u\ge M))} + CM^{p-2}\|u-M\|^p_{L^p(\mathbb{R}^N(u\ge M))} \le C\|g\|^2_{L^2(\mathbb{R}^N(u\ge M))}.$$

By Gronwall's inequality, we have for all $M \ge M_1$ and $t \ge T_1$,

$$\int\limits_{\mathbb{R}^N(u\ge M)} (u-M)^p\,dx \le \varepsilon. \tag{3.33}$$

Repeating the same step above, just taking $(u+M)_-$ instead of $(u-M)_+$, where

$$(u+M)_- = \begin{cases} u+M, & u \le -M, \\ 0, & u \ge -M, \end{cases}$$

we deduce that there exist $M_2 > 0$ and $T_2 > 0$ such that for any $t > T_2$ and any $M \ge M_2$, we have

$$\int_{\mathbb{R}^N(u\le -M)} |(u+M)|^p\,dx \le \varepsilon. \tag{3.34}$$

Letting $M_0 = \max\{M_1, M_2\}$ and $T = \max\{T_1, T_2\}$, we obtain

$$\int_{\mathbb{R}^N(|u|\ge M)} (|u|-M)^p\,dx \le \varepsilon \quad \text{for } t \ge T \text{ and } M \ge M_0.$$

Using (3.33) and (3.34), we have

$$\begin{aligned}
\int_{\mathbb{R}^N(|u|\ge 2M)} |u|^p\,dx &= \int_{\mathbb{R}^N(|u|\ge 2M)} ((|u|-M)+M)^p\,dx \\
&\le 2^p\Big(\int_{\mathbb{R}^N(|u|\ge 2M)} (|u|-M)^p\,dx + \int_{\mathbb{R}^N(|u|\ge 2M)} M^p\,dx\Big) \\
&\le 2^p\Big(\int_{\mathbb{R}^N(|u|\ge 2M)} (|u|-M)^p\,dx + \int_{\mathbb{R}^N(|u|\ge 2M)} (|u|-M)^p\,dx\Big) \\
&\le 2^{p+1}\varepsilon.
\end{aligned}$$

This completes the proof. ∎

3.3. Existence of a global attractor in $\mathcal{H}_0^1(\mathbb{R}^N,\sigma)\cap L^p(\mathbb{R}^N)$

LEMMA 3.9. *Suppose* $(\mathcal{H}^\infty)$, $(\mathbf{F})$ *and* $(\mathbf{G})$ *hold. Then the semigroup* $S(t)$ *is asymptotically compact in* $\mathcal{H}_0^1(\mathbb{R}^N,\sigma)\cap L^p(\mathbb{R}^N)$.

Proof. Let B be a bounded subset in $L^2(\mathbb{R}^N)$. We will show that for any $\{u_{0n}\} \subset B$ and $t_n \to \infty$, $\{u_n(t_n)\} := \{S(t_n)u_{0n}\}$ is precompact in $\mathcal{H}_0^1(\mathbb{R}^N,\sigma)\cap L^p(\mathbb{R}^N)$. Thanks to Theorem 3.8, we only need to show that the sequence $\{u_n(t_n)\}$ is precompact in $\mathcal{H}_0^1(\mathbb{R}^N,\sigma)$. By Lemma 3.5, we can assume that $\{u_n(t_n)\}$ is a Cauchy sequence in $L^2(\mathbb{R}^N)$. For any $n, m \ge 1$, it follows from (1.1) that

$$\begin{aligned}
&-\operatorname{div}(\sigma(x)\nabla(u_n(t_n)-u_m(t_m))) + \lambda(u_n(t_n)-u_m(t_m)) \\
&\quad + f(x,u_n(t_n)) - f(x,u_m(t_m)) = -\frac{d}{dt}u_n(t_n) + \frac{d}{dt}u_m(t_m).
\end{aligned} \tag{3.35}$$

Multiplying (3.35) by $u_n(t_n) - u_m(t_m)$ and using (1.4) we get

$$\int_{\mathbb{R}^N} \sigma(x)|\nabla u_n(t_n) - u_m(t_m)|^2\,dx + \lambda\|u_n(t_n) - u_m(t_m)\|^2_{L^2(\mathbb{R}^N)} \le \|u_{nt}(t_n) - u_{mt}(t_m)\|_{L^2(\mathbb{R}^N)}\|u_n(t_n) - u_m(t_m)\|_{L^2(\mathbb{R}^N)} + \alpha_3\|u_n(t_n) - u_m(t_m)\|^2_{L^2(\mathbb{R}^N)}. \tag{3.36}$$

By Lemma 3.2, for any bounded subset B in $L^2(\mathbb{R}^N)$, there exists $T = T(B)$ such that for all $t_n \ge T$,

$$\|u_{nt}(t_n)\|_{L^2(\mathbb{R}^N)} \le C,$$

which along with (3.36) shows that, for all $n, m \ge N$,

$$\int_{\mathbb{R}^N} \sigma(x)|\nabla u_n(t_n) - u_m(t_m)|^2\,dx + \lambda\|u_n(t_n) - u_m(t_m)\|^2_{L^2(\mathbb{R}^N)} \le 2C\|u_n(t_n) - u_m(t_m)\|_{L^2(\mathbb{R}^N)} + \alpha_3\|u_n(t_n) - u_m(t_m)\|^2_{L^2(\mathbb{R}^N)}.$$

Hence, $\{u_n(t_n)\}$ is a Cauchy sequence in $\mathcal{H}^1_0(\mathbb{R}^N, \sigma)$. ∎

THEOREM 3.10. *Suppose* $(\mathcal{H}^\infty)$, $(\mathbf{F})$ *and* $(\mathbf{G})$ *hold. Then the semigroup* $S(t)$ *generated by problem* (1.1) *has a global attractor* $\mathcal{A}_{\mathcal{H}^1_0\cap L^p}$ *in* $\mathcal{H}^1_0(\mathbb{R}^N, \sigma)\cap L^p(\mathbb{R}^N)$, *that is,* $\mathcal{A}_{\mathcal{H}^1_0\cap L^p}$ *is compact, invariant in* $\mathcal{H}^1_0(\mathbb{R}^N, \sigma) \cap L^p(\mathbb{R}^N)$ *and attracts every bounded subset of* $L^2(\mathbb{R}^N)$ *in the topology of* $\mathcal{H}^1_0(\mathbb{R}^N, \sigma) \cap L^p(\mathbb{R}^N)$.

Proof. By Lemma 3.1, there exists a bounded absorbing set for $S(t)$ in $\mathcal{H}^1_0(\mathbb{R}^N, \sigma) \cap L^p(\mathbb{R}^N)$. In addition, $S(t)$ is asymptotically compact in $\mathcal{H}^1_0(\mathbb{R}^N, \sigma) \cap L^p(\mathbb{R}^N)$ since Lemma 3.9. Thus, there exists a global attractor for $S(t)$ in $\mathcal{H}^1_0(\mathbb{R}^N, \sigma) \cap L^p(\mathbb{R}^N)$. ∎

REMARK 3.1. The global attractors $\mathcal{A}_{L^2}$, $\mathcal{A}_{L^p}$ and $\mathcal{A}_{\mathcal{H}^1_0\cap L^p}$ obtained in Theorems 3.6, 3.8 and 3.10 are of course the same object, say $\mathcal{A}$. In particular, $\mathcal{A}$ is a compact connected set in $\mathcal{H}^1_0(\mathbb{R}^N, \sigma) \cap L^p(\mathbb{R}^N)$.

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