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Euler's Approximations of Solutions of Reflecting SDEs with Discontinuous Coefficients

by

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Summary. Let D be either a convex domain in $\mathbb{R}^d$ or a domain satisfying the conditions (A) and (B) considered by Lions and Sznitman (1984) and Saisho (1987). We investigate convergence in law as well as in L^p for the Euler and Euler–Peano schemes for stochastic differential equations in D with normal reflection at the boundary. The coefficients are measurable, continuous almost everywhere with respect to the Lebesgue measure, and the diffusion coefficient may degenerate on some subsets of the domain.

1. Introduction. In this paper we investigate solutions of d-dimensional stochastic differential equations (SDEs) on a domain $D \subseteq \mathbb{R}^d$ with reflecting boundary condition of the form

$$X_t = X_0 + \int_0^t \sigma(X_s)\,dW_s + \int_0^t b(X_s)\,ds + K_t, \qquad t \geq 0. \tag{1.1}$$

Here $X_0 = x_0 \in \overline{D} = D \cup \partial D$, X is a reflecting process on $\overline{D}$, K is a bounded variation process with variation $|K|$ increasing only when $X_t \in \partial D$, W is a d-dimensional standard Wiener process and $\sigma : \overline{D} \to \mathbb{R}^d \otimes \mathbb{R}^d$, $b : \overline{D} \to \mathbb{R}^d$ are measurable functions. Equation (1.1) is called the Skorokhod SDE by analogy to the one-dimensional case first investigated by Skorokhod (1961) for $\overline{D} = \mathbb{R}^+$. The problem of existence, uniqueness and approximation for solutions of the Skorokhod SDE attracted attention of many researchers who obtained many deep and important results. Equation (1.1) on a domain more general than a half-line or a half-space was first discussed by Tanaka (1979) for D any convex domain in $\mathbb{R}^d$, and then by Lions and Sznitman (1984),

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Saisho (1987), Constantini (1992), Dupuis and Ishii (1993), Storm (1995), Słomiński (1996) and many others for D satisfying some weaker conditions.

In this paper we assume that D is a convex subset of $\mathbb{R}^d$ or satisfies quite general conditions (A), (B) considered by Lions and Sznitman (1984) and Saisho (1987) and we discuss the problem of approximating solutions of (1.1) when the coefficients σ, b are possibly discontinuous. We investigate two approximations of X: discrete $\{\overline{X}^n\}_{n\in\mathbb{N}}$ and continuous $\{X^n\}_{n\in\mathbb{N}}$, defined to be the solutions of the SDEs with reflecting boundary conditions of the form

$$\overline{X}_t^n = X_0 + \int_0^t \sigma(\overline{X}_{s-}^n)\, dW_s^{\rho^n} + \int_0^t b(\overline{X}_{s-}^n)\, d\rho_s^n + \overline{K}_t^n, \qquad t \in \mathbb{R}^+, \tag{1.2}$$

and

$$X_t^n = X_0 + \int_0^t \sigma(X_{s-}^{n,\rho^n})\, dW_s + \int_0^t b(X_{s-}^{n,\rho^n})\, ds + K_t^n, \qquad t \in \mathbb{R}^+, \tag{1.3}$$

respectively, where $\rho_t^n = \max\{k/n : k \in \mathbb{N}\cup\{0\},\ k/n \le t\}$ and $W_t^{\rho^n}$ is a discretization of W, i.e. $W_t^{\rho^n} = W_{k/n}$ for $t \in [k/n,(k+1)/n)$, $k \in \mathbb{N}\cup\{0\}$, $n \in \mathbb{N}$. The sequences $\{\overline{X}^n\}_{n\in\mathbb{N}}$ and $\{X^n\}_{n\in\mathbb{N}}$ are called the *Euler* and *Euler–Peano schemes* for (1.1) respectively. We will see in Section 2 that $\{\overline{X}^n\}_{n\in\mathbb{N}}$ and in some cases also $\{X^n\}_{n\in\mathbb{N}}$ can be computed by simple recurrent formulas.

Note that (1.2) is a well known projection scheme considered earlier in Chitashvili and Lazrieva (1981), Słomiński (1994) and Pettersson (1995). The approximation (1.3) comes from Słomiński (1994) and Lépingle (1995). The papers cited above contain some results on L^p convergence of $\{\overline{X}^n\}_{n\in\mathbb{N}}$ and $\{X^n\}_{n\in\mathbb{N}}$ to the solution X of (1.1). However, these results are proved under rather restrictive conditions like boundedness of the domain, condition (β) introduced by Tanaka (1979) as well as boundedness and Lipschitz continuity of coefficients. A similar result for $\{\overline{X}^n\}_{n\in\mathbb{N}}$ under the assumptions that the coefficients σ, b of (1.1) are continuous and the SDE (1.1) has the pathwise uniqueness property can be found in Słomiński (2001).

It is worth mentioning some papers on approximations of solutions of nonreflecting SDEs with discontinuous coefficients. In the multidimensional case, convergence in probability of $\{X^n\}_{n\in\mathbb{N}}$ is proved in Gyöngy and Krylov (1996). Approximations of weak solutions, under the assumption that the coefficients are discontinuous on some sets of Lebesgue measure zero, were considered by Krylov and Liptser (2002), Yan (2002) and Yamada (1986). The results in the last paper are only one-dimensional.

To the best of our knowledge, first theorems on approximations for reflecting SDEs with discontinuous coefficients appear in Semrau (2007, 2009). The first paper establishes convergence in law of $\{\overline{X}^n\}_{n\in\mathbb{N}}$ and $\{X^n\}_{n\in\mathbb{N}}$ to a solution of the SDE (1.1) on a convex domain D under the assumptions that the coefficients $\sigma\sigma^*$, b are continuous almost everywhere with respect to the Lebesgue measure, satisfy the linear growth condition, $\sigma\sigma^*$ is uniformly elliptic in $\overline{D}$ and the SDE (1.1) has a unique weak solution. The same assumptions on the coefficients and the pathwise uniqueness property of (1.1) give L^p convergence of our schemes. This is announced in the latter paper.

In the present paper we strengthen and extend those results. First, we considerably weaken the assumption of uniform ellipticity of $\sigma\sigma^*$, and secondly, we give analogous theorems for D satisfying conditions (A) and (B).

Now we describe briefly the content of the paper. In Section 2 we investigate convergence in law of $\{\overline{X}^n\}_{n\in\mathbb{N}}$ and $\{X^n\}_{n\in\mathbb{N}}$ to a weak solution of (1.1) in a convex domain D. Here we assume that (1.1) has a unique weak solution and $\sigma\sigma^*$, b are measurable functions continuous almost everywhere with respect to the Lebesgue measure, i.e.

$$l(G) = 0, \quad G = D_{\sigma\sigma^*} \cup D_b, \tag{1.4}$$

where $D_{\sigma\sigma^*}$, D_b are the sets of discontinuity points of $\sigma\sigma^*$ and b respectively. Moreover, $\sigma\sigma^*$ and b have at most linear growth, i.e.

$$\|\sigma\sigma^*(x)\| + |b(x)|^2 \le L(1 + |x|^2), \qquad x \in \overline{D}, \tag{1.5}$$

for some constant $L > 0$ and $G \subset H$, where H is a closed set in $\overline{D}$ with an open neighborhood H_δ in $\overline{D}$, for which $\sigma\sigma^*$ is uniformly elliptic, i.e.

$$(\sigma\sigma^*(y)x, x) \ge \lambda|x|^2, \qquad y \in H_\delta,\ x \in \mathbb{R}^d, \tag{1.6}$$

for some constant $\lambda > 0$.

The conditions ensuring weak uniqueness of the SDE (1.1) in the case of discontinuous coefficients σ, b were considered by Stroock and Varadhan (1971) and Schmidt (1989). In the latter paper it is shown that if $d = 1$ and $b \equiv 0$, then (1.1) has a weak solution on $\overline{D} = [r_1, r_2]$ for every starting point $x_0 \in \overline{D}$ iff the set M of all $x \in \overline{D}$ such that $\int_{\overline{D}\cap U_x} \sigma^{-2}(y)\,dy = \infty$ for every open neighbourhood U_x of x is contained in the set N of zeros of σ. Therefore, if σ is merely bounded and measurable, some additional assumptions on boundedness of $(\sigma\sigma^*)^{-1}$ are indispensable. Schmidt also proved that in the above situation the solution of (1.1) is unique if $N \subseteq M$. In the multidimensional case, from Stroock and Varadhan (1971) we know that a sufficient condition for the uniqueness of a weak solution of (1.1) is that $\sigma\sigma^*$ is bounded, continuous and uniformly elliptic, b is bounded measurable and ∂D is of class $\mathcal{C}_b^2$. Such a D satisfies conditions (A) and (B).

Section 3 is devoted to the study of L^p convergence of $\{\overline{X}^n\}_{n\in\mathbb{N}}$ and $\{X^n\}_{n\in\mathbb{N}}$ under the same assumptions on the coefficients and D as in Section 2 and the pathwise uniqueness property for (1.1).

From the Yamada–Watanabe theorem (see e.g. Theorem 1.3 in Rong (2000)) it is well known that (1.5), (1.6) and the pathwise uniqueness property imply the existence of a unique strong solution of (1.1).

We do not know any more general conditions that guarantee the pathwise uniqueness property for multidimensional equations with discontinuous coefficients. In Semrau (2009) this property is shown for the one-dimensional reflecting SDEs in the half-line $\overline{D} = \mathbb{R}^+$. This result is proved under the assumptions that σ and b satisfy the linear growth condition (1.5) for $d = 1$, σ is uniformly positive and

$$(\sigma(x) - \sigma(y))^2 \leq |f(x) - f(y)|, \qquad x, y \geq 0,$$

for some increasing and bounded function $f : \mathbb{R}^+ \to \mathbb{R}$. This theorem generalizes the results of Nakao (1972) and Le Gall (1983) proved for one-dimensional nonreflecting SDEs and strengthens the result of Zhang (1994), who requires a stronger condition on σ, namely that $\sigma \in \mathcal{C}^1$.

Section 4 deals with the general case of domains satisfying conditions (A) and (B). Here it is additionally required that the coefficients are bounded, i.e.

$$\|\sigma\sigma^*(x)\| + |b(x)|^2 \leq L, \qquad x \in \overline{D}, \tag{1.7}$$

for some constant $L > 0$. We consider convergence in law and in probability of $\{\overline{X}^n\}_{n\in\mathbb{N}}$ and $\{X^n\}_{n\in\mathbb{N}}$.

An important point to note here is that some new generalized inequalities of Krylov's type for stochastic integrals are crucial tools used in the proofs of theorems on convergence of our approximations.

Let us now introduce some definitions and notations used further on. $\mathcal{D}(\mathbb{R}^+, \mathbb{R}^d)$ is the space of all mappings $x : \mathbb{R}^+ \to \mathbb{R}^d$ which are right continuous and admit left-hand limits, equipped with the Skorokhod topology J_1. Processes we consider have their trajectories in $\mathcal{D}(\mathbb{R}^+, \mathbb{R}^d)$. For a given process X we denote by ΔX_t the difference $X_t - X_{t-}$ and by X^{ρ^n} the discretization of X, i.e. $X_t^{\rho^n} = X_{k/n}$ for $t \in [k/n, (k+1)/n)$, $k \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$. If $X = (X^1, \ldots, X^d)$ is a local martingale then $[X]_t$ stands for $\sum_{i=1}^d [X^i]_t$, where $[X^i]$ is a quadratic variation process of X^i, $i = 1, \ldots, d$. If $K = (K^1, \ldots, K^d)$ is a process with locally finite variation then $|K|_t$ stands for $\sum_{i=1}^d |K^i|_t$, where $|K^i|_t$ is the total variation of K^i on $[0, t]$. Further, $L^p(Q)$, $p \geq 1$ is the usual L^p-space with the Lebesgue measure l on Q. Moreover, $\mathbb{R}^d \otimes \mathbb{R}^d$ is the space of $d \times d$-matrices with the norm $\|\sigma\| = (\operatorname{tr} \sigma\sigma^*)^{1/2}$, where σ^* is the matrix transpose to σ. We write $B(x, R) = \{y \in \mathbb{R}^d : |y - x| \leq R\}$, where $|\cdot|$ denotes the usual Euclidean norm on $\mathbb{R}^d$. Finally, "$\to_{\mathcal{D}}$" and "$\to_{\mathcal{P}}$" denote convergence in law and in probability respectively.

2. Weak convergence in convex domains. Let D be a convex domain in $\mathbb{R}^d$. Define the set $\mathcal{N}_x$ of inward normal unit vectors at $x \in \partial D$ by

$$\mathcal{N}_x = \{n \in \mathbb{R}^d : |n| = 1,\ \langle y - x, n\rangle \ge 0 \text{ for all } y \in \overline{D}\}.$$

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ be a filtration on $(\Omega, \mathcal{F}, P)$ satisfying the usual conditions. Let Y be an $(\mathcal{F}_t)$-adapted semimartingale with initial value in $\overline{D}$, i.e.

$$Y_t = Y_0 + M_t + A_t, \qquad t \in \mathbb{R}^+,$$

where $Y_0 \in \overline{D}, M$ is an $(\mathcal{F}_t)$-adapted local martingale, A is an $(\mathcal{F}_t)$-adapted process with locally bounded variation, and $M_0 = A_0 = 0$. Recall that a pair (X, K) of $(\mathcal{F}_t)$-adapted processes is called a *solution to the Skorokhod problem* associated with Y if:

(i) $X_t = Y_t + K_t$, $t \in \mathbb{R}^+$,
(ii) X is $\overline{D}$-valued,
(iii) K is a process with locally bounded variation such that $K_0 = 0$ and

$$K_t = \int_0^t n_s\, d|K|_s, \qquad |K|_t = \int_0^t \mathbf{1}_{\{X_s\in\partial D\}}\, d|K|_s, \qquad t \in \mathbb{R}^+,$$

where $n_s \in \mathcal{N}_{X_s}$ if $X_s \in \partial D$.

Recall also that the SDE (1.1) is said to have a *weak solution* if there exists a filtered probability space $(\overline{\Omega}, \overline{\mathcal{F}}, (\overline{\mathcal{F}}_t)_{t\in\mathbb{R}^+}, \overline{P})$, an $(\overline{\mathcal{F}}_t)$-adapted Wiener process $\overline{W}$ and a pair $(\overline{X}, \overline{K})$ of $(\overline{\mathcal{F}}_t)$-adapted processes that is a solution of the Skorokhod problem associated with

$$\overline{Y}_t = \overline{X}_0 + \int_0^t \sigma(\overline{X}_s)\, d\overline{W}_s + \int_0^t b(\overline{X}_s)\, ds.$$

If for any two weak solutions (X, K, W), $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, P)$ and $(\overline{X}, \overline{K}, \overline{W})$, $(\overline{\Omega}, \overline{\mathcal{F}}, (\overline{\mathcal{F}}_t)_{t\in\mathbb{R}^+}, \overline{P})$ with the same initial distribution, the laws of (X, K) and $(\overline{X}, \overline{K})$ are the same, then we say that *weak uniqueness* holds for the SDE (1.1).

We first consider the discrete Euler approximation $\{\overline{X}^n\}_{n\in\mathbb{N}}$ given by formula (1.2). Let $(\mathcal{F}_t^{\rho^n})_{t\in\mathbb{R}^+}$ denote the discretization of $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$, i.e. $\mathcal{F}_t^{\rho^n} = \mathcal{F}_{k/n}$ for $t \in [k/n, (k+1)/n)$, $k \in \mathbb{N}\cup\{0\}$, $n \in \mathbb{N}$. We say that the SDE (1.2) has a *strong solution* if there exists a pair $(\overline{X}^n, \overline{K}^n)$ of $(\mathcal{F}_t^{\rho^n})$-adapted processes such that $(\overline{X}^n, \overline{K}^n)$ is a solution of the Skorokhod problem associated with

$$\overline{Y}_t^n = X_0 + \int_0^t \sigma(\overline{X}_{s-}^n)\, dW_s^{\rho^n} + \int_0^t b(\overline{X}_{s-}^n)\, d\rho_s^n. \tag{2.1}$$

It is easy to prove that the solution $\overline{X}^n$ of the SDE (1.2) is given by the recurrent formula

$$\overline{X}^n_0 = X_0,$$
$$\overline{X}^n_{(k+1)/n} = \Pi\Big(\overline{X}^n_{k/n} + b(\overline{X}^n_{k/n})\frac{1}{n} + \sigma(\overline{X}^n_{k/n})(W_{(k+1)/n} - W_{k/n})\Big)$$

and $\overline{X}^n_t = \overline{X}^n_{k/n}$ for $t \in [k/n,(k+1)/n)$, $k \in \mathbb{N}\cup\{0\}$, $n \in \mathbb{N}$, where $\Pi(x)$ is a projection of x on $\overline{D}$.

We can now formulate our main result.

THEOREM 2.1. *Let* $\{(\overline{X}^n, \overline{K}^n)\}_{n\in\mathbb{N}}$ *be a sequence of solutions of the SDEs* (1.2) *with coefficients* σ, b *satisfying* (1.4)–(1.6). *If the SDE* (1.1) *has a unique weak solution* (X,K) *then* $(\overline{X}^n, \overline{K}^n) \to_{\mathcal{D}} (X,K)$ *in* $\mathcal{D}(\mathbb{R}^+, \mathbb{R}^{2d})$.

Proof. From the proof of Theorem 2.1 in Semrau (2007) we know that for every $T \in \mathbb{R}^+$ the sequences $\{\sup_{t\le T}|\overline{X}^n_t|\}_{n\in\mathbb{N}}$, $\{|\overline{K}^n|_T\}_{n\in\mathbb{N}}$ are bounded in probability and there exists a subsequence $(n') \subset (n)$ and processes $\overline{X}, \overline{K}, \overline{W}$ such that

$$(\overline{X}^{n'}, \overline{K}^{n'}, W^{\rho^{n'}}) \underset{\mathcal{D}}{\to} (\overline{X}, \overline{K}, \overline{W}) \tag{2.2}$$

in $\mathcal{D}(\mathbb{R}^+, \mathbb{R}^{3d})$, where $\overline{W}$ is a Wiener process with respect to the natural filtration $(\mathcal{F}_t^{\overline{X},\overline{K},\overline{W}})_{t\in\mathbb{R}^+}$.

In the further part of the proof we need an inequality of Krylov's type for $\{\overline{X}^n\}_{n\in\mathbb{N}}$.

Let

$$\psi(x) = \frac{\operatorname{dist}(x, H^c_\delta)}{\operatorname{dist}(x,H) + \operatorname{dist}(x, H^c_\delta)}, \qquad x \in \overline{D}. \tag{2.3}$$

It is easy to see that

(i) $\psi : \overline{D} \to [0,1]$ is continuous,
(ii) $\psi(x) = 1$ for $x \in H$,
(iii) $\psi(x) = 0$ for $x \in H^c_\delta$.

LEMMA 2.2. *Let* $\tau^R_n = \inf\{t : |\overline{X}^n_t| > R\}, R \in \mathbb{R}^+$, $n \in \mathbb{N}$. *Then for all bounded measurable functions* $f : \overline{D} \to \mathbb{R}^+$ *such that* $l(D_f) = 0$,

$$\limsup_{n\to\infty} E \int_0^{T\wedge\tau^R_n} (\psi f)(\overline{X}^n_{s-})\, ds \le C\|\psi f\|_{L^d(\overline{B(0,R)}\cap\overline{D})}, \tag{2.4}$$

where C *is a positive constant depending only on* d, T, R *and* λ.

Proof. For every $n \in \mathbb{N}$ set

$$\widehat{Y}^n_t = X_0 + \int_0^t \sigma(\overline{X}^n_{s-})\, dW_s + \int_0^t b(\overline{X}^n_{s-})\, ds$$

and let $(\widehat{X}^n, \widehat{K}^n)$ be a solution of the Skorokhod problem associated with $\widehat{Y}^n$. In the proof of Lemma 2.2 in Semrau (2007) it has been shown that for every $T \in \mathbb{R}^+$,

$$\lim_{n\to\infty} E \sup_{t\le T} |\widehat{X}^n_t - \overline{X}^n_t|^2 = 0. \tag{2.5}$$

The semimartingale $\widehat{X}^n$ is of the form

$$\widehat{X}^n_t = X_0 + \widehat{A}^n_t + \widehat{M}^n_t + \widehat{K}^n_t,$$

where $\widehat{A}^n_t = \int_0^t b(\overline{X}^n_{s-})\,ds$ and $\widehat{M}^n_t = \int_0^t \sigma(\overline{X}^n_{s-})\,dW_s$. For every $n \in \mathbb{N}$ let $\widehat{\tau}^{R'}_n = \inf\{t : |\widehat{X}^n_t| > R'\}$, where $R' \in \mathbb{R}^+$ and $R' > R$. According to (2.5),

$$\lim_{n\to\infty} P(T \ge \tau^R_n > \widehat{\tau}^{R'}_n) = 0. \tag{2.6}$$

We also know that for every $T \in \mathbb{R}^+$,

$$\sup_{n\in\mathbb{N}} E|\widehat{A}^n|_{T\wedge\widehat{\tau}^{R'}_n} < \infty \quad \text{and} \quad \sup_{n\in\mathbb{N}} E|\widehat{K}^n|_T < \infty.$$

Moreover, by (1.6), for every $t \le \widehat{\tau}^{R'}_n$ the matrix Q^n_t defined as

$$Q^n_t = \left(\frac{(\sigma\sigma^*)^{ij}(\overline{X}^n_{t-})\,dt}{\operatorname{tr}\sigma\sigma^*(\overline{X}^n_{t-})\,dt}\right)_{i,j=1}^d$$

is uniformly elliptic in H_δ, $n \in \mathbb{N}$. Hence by the continuity of ψ, (2.5) and Theorem 6(i) in Melnikov (1983), for every measurable f,

$$\begin{aligned}
&\limsup_{n\to\infty} E \int_0^{T\wedge\widehat{\tau}^{R'}_n} \psi(\overline{X}^n_{s-}) f(\widehat{X}^n_s)\,d[\widehat{M}^n]_s \\
&\qquad \le C_1 \limsup_{n\to\infty} E \int_0^{T\wedge\widehat{\tau}^{R'}_n} (\det Q^n_s)^{1/d} \psi(\overline{X}^n_{s-}) f(\widehat{X}^n_s)\,d[\widehat{M}^n]_s \\
&\qquad = C_1 \limsup_{n\to\infty} E \int_0^{T\wedge\widehat{\tau}^{R'}_n} (\det Q^n_s)^{1/d} (\psi f)(\widehat{X}^n_s)\,d[\widehat{M}^n]_s \\
&\qquad \le C_2 \|\psi f\|_{L^d(\overline{B(0,R')}\cap\overline{D})},
\end{aligned}$$

where C_2 is a constant depending only on d, T, R'. If we assume now that f is continuous and bounded then by the above and (2.5),

$$\begin{aligned}
&\limsup_{n\to\infty} E \int_0^{T\wedge\widehat{\tau}^{R'}_n} (\psi f)(\overline{X}^n_{s-})\,d[\widehat{M}^n]_s \\
&\qquad = \limsup_{n\to\infty} E \int_0^{T\wedge\widehat{\tau}^{R'}_n} \psi(\overline{X}^n_{s-}) f(\widehat{X}^n_s)\,d[\widehat{M}^n]_s \\
&\qquad \le C_2 \|\psi f\|_{L^d(\overline{B(0,R')}\cap\overline{D})}.
\end{aligned}$$

As in the proof of Lemma 2.2 in Semrau (2007), we obtain the inequality

$$\limsup_{n\to\infty} E \int_0^{T\wedge\hat{\tau}_n^{R'}} (\psi f)(\overline{X}^n_{s-})\, ds \le C_3 \|\psi f\|_{L^d(\overline{B(0,R')}\cap\overline{D})},$$

where C_3 is a constant depending only on d, T, R' and λ. From this and (2.6), the same reasoning as in the above mentioned lemma gives the inequality

$$\limsup_{n\to\infty} E \int_0^{T\wedge\tau_n^{R}} (\psi f)(\overline{X}^n_{s-})\, ds \le C_3 \|\psi f\|_{L^d(\overline{B(0,R)}\cap\overline{D})}.$$

It remains to prove that the above inequality is true for an arbitrary nonnegative bounded function f such that $l(D_f) = 0$. For such an f and for every $\epsilon > 0$, there exists a continuous and bounded function f_ϵ^+ such that $f \le f_\epsilon^+$ and $\|f_\epsilon^+ - f\|_{L^d(\overline{B(0,R)}\cap\overline{D})} < \epsilon$. Since for f_ϵ^+ the inequality (2.4) is true,

$$\begin{aligned}\limsup_{n\to\infty} E \int_0^{T\wedge\tau_n^{R}} (\psi f)(\overline{X}^n_{s-})\, ds &\le \limsup_{n\to\infty} E \int_0^{T\wedge\tau_n^{R}} (\psi f_\epsilon^+)(\overline{X}^n_{s-})\, ds \\ &\le C\|\psi f_\epsilon^+\|_{L^d(\overline{B(0,R)}\cap\overline{D})}.\end{aligned}$$

As

$$\|\psi f_\epsilon^+\|_{L^d(K_R^d\cap D)} \le \|\psi f\|_{L^d(\overline{B(0,R)}\cap D)} + \epsilon,$$

we have the desired conclusion. ∎

For every $n \in \mathbb{N}$, let $\overline{\tau}_n^R = \inf\{t \in \mathbb{R}^+ : |\overline{X}_t^n| \ge R \text{ or } |\overline{X}_{t-}^n| \ge R\}$ and $\overline{\tau}^R = \inf\{t \in \mathbb{R}^+ : |\overline{X}_t| \ge R \text{ or } |\overline{X}_{t-}| \ge R\}$, $R \in \mathbb{R}^+$. In view of (2.2) and Proposition VI.2.12 from Jacod and Shiryaev (2003), there exists a sequence $\{R_k\}_{k\in\mathbb{N}}$ with $R_k \uparrow \infty$ such that for every $k \in \mathbb{N}$,

$$\Big(\overline{\tau}_{n'}^{R_k}, \overline{X}^{n',\overline{\tau}_{n'}^{R_k}}, \overline{K}^{n',\overline{\tau}_{n'}^{R_k}}, W^{\rho^{n'},\overline{\tau}_{n'}^{R_k}}\Big) \underset{\mathcal{D}}{\to} \Big(\overline{\tau}^{R_k}, \overline{X}^{\overline{\tau}^{R_k}}, \overline{K}^{\overline{\tau}^{R_k}}, \overline{W}^{\overline{\tau}^{R_k}}\Big) \tag{2.7}$$

in $\mathbb{R}\times\mathcal{D}(\mathbb{R}^+,\mathbb{R}^{3d})$. We next prove that

$$\begin{aligned}&\Big(\overline{\tau}_{n'}^{R_k}, \overline{X}^{n',\overline{\tau}_{n'}^{R_k}}, \int_0^{\cdot\wedge\overline{\tau}_{n'}^{R_k}} \sigma(\overline{X}^{n'}_{s-})\, dW_s^{\rho^{n'}}, \int_0^{\cdot\wedge\overline{\tau}_{n'}^{R_k}} b(\overline{X}^{n'}_{s-})\, d\rho_s^{n'}, \overline{K}^{n',\overline{\tau}_{n'}^{R_k}}\Big) \\ &\quad\underset{\mathcal{D}}{\to} \Big(\overline{\tau}^{R_k}, \overline{X}^{\overline{\tau}^{R_k}}, \int_0^{\cdot\wedge\overline{\tau}^{R_k}} \sigma(\overline{X}_s)\, d\overline{W}_s, \int_0^{\cdot\wedge\overline{\tau}^{R_k}} b(\overline{X}_s)\, ds, \overline{K}^{\overline{\tau}^{R_k}}\Big)\end{aligned} \tag{2.8}$$

in $\mathbb{R}\times\mathcal{D}(\mathbb{R}^+,\mathbb{R}^{4d})$. To see this, it is convenient to define functions

$$\hat{\sigma}(x) = (\psi\sigma)(x), \quad \tilde{\sigma}(x) = ((1-\psi)\sigma)(x),$$
$$\hat{b}(x) = (\psi b)(x), \quad \tilde{b}(x) = ((1-\psi)b)(x).$$

Since $\tilde\sigma$ and $\tilde b$ are continuous, from (2.7) and Theorem 2.6 of Jakubowski, Mémin and Pages (1989), it follows that for every $k\in\mathbb{N}$,

$$(2.9)\quad V^{n'}=\Big(\overline{\tau}^{R_k}_{n'},\overline{X}^{n',\overline{\tau}^{R_k}_{n'}},\int_0^{\cdot\wedge\overline{\tau}^{R_k}_{n'}}\tilde\sigma(\overline{X}^{n'}_{s-})\,dW^{\rho^{n'}}_s,\int_0^{\cdot\wedge\overline{\tau}^{R_k}_{n'}}\tilde b(\overline{X}^{n'}_{s-})\,d\rho^{n'}_s,\overline{K}^{n',\overline{\tau}^{R_k}_{n'}}\Big)$$

$$\underset{\mathcal{D}}{\to} V=\Big(\overline{\tau}^{R_k},\overline{X}^{\overline{\tau}^{R_k}},\int_0^{\cdot\wedge\overline{\tau}^{R_k}}\tilde\sigma(\overline{X}_s)\,d\overline{W}_s,\int_0^{\cdot\wedge\overline{\tau}^{R_k}}\tilde b(\overline{X}_s)\,ds,\overline{K}^{\overline{\tau}^{R_k}}\Big)$$

in $\mathbb{R}\times\mathcal{D}(\mathbb{R}^+,\mathbb{R}^{4d})$.

To show (2.8), we use mollification. We construct sequences $\{\sigma_i\}_{i\in\mathbb{N}}$ and $\{b_i\}_{i\in\mathbb{N}}$ of continuous functions $\sigma_i:\overline{D}\to\mathbb{R}^d\otimes\mathbb{R}^d$, $b_i:\overline{D}\to\mathbb{R}^d$ such that $\|\sigma_i\sigma_i^*(x)\|+|b_i(x)|^2\le L(1+|x|^2)$, $x\in\mathbb{R}^d$, and $\sigma_i^{k,j}\to\sigma^{k,j}$ and $b_i^l\to b^l$ in $L^d(K)$, $k,j,l=1,\dots,d$, for every compact subset $K\subset\overline{D}$.

Let $\hat\sigma_i:=\psi\sigma_i$, $\hat b_i:=\psi b_i$, $i\in\mathbb{N}$. By (2.7), (2.9) and Theorem 2.6 from Jakubowski, Mémin and Pages (1989), for every $i\in\mathbb{N}$,

$$(2.10)\qquad \Big(V^{n'},\int_0^{\cdot\wedge\overline{\tau}^{R_k}_{n'}}\hat\sigma_i(\overline{X}^{n'}_{s-})\,dW^{\rho^{n'}}_s,\int_0^{\cdot\wedge\overline{\tau}^{R_k}_{n'}}\hat b_i(\overline{X}^{n'}_{s-})\,d\rho^{n'}_s\Big)$$

$$\underset{\mathcal{D}}{\to}\Big(V,\int_0^{\cdot\wedge\overline{\tau}^{R_k}}\hat\sigma_i(\overline{X}_s)\,d\overline{W}_s,\int_0^{\cdot\wedge\overline{\tau}^{R_k}}\hat b_i(\overline{X}_s)\,ds\Big)$$

in $\mathbb{R}\times\mathcal{D}(\mathbb{R}^+,\mathbb{R}^{6d})$.

Since $\overline{\tau}^R_n\le\tau^R_n$ and $\sigma_i\to\sigma$ in L^d_{loc}, Lemma 2.2 and the Lebesgue dominated convergence theorem show that for every $k\in\mathbb{N}$ and every $T\in\mathbb{R}^+$,

$$\lim_{i\to\infty}\limsup_{n'\to\infty}E\Big[\int_0^{\cdot}(\hat\sigma_i-\hat\sigma)(\overline{X}^{n'}_{s-})\,dW^{\rho^{n'}}_s\Big]_{T\wedge\overline{\tau}^{R_k}_{n'}}$$

$$\le\lim_{i\to\infty}\limsup_{n'\to\infty}E\int_0^{T\wedge\overline{\tau}^{R_k}_{n'}}\|(\hat\sigma_i-\hat\sigma)(\hat\sigma_i-\hat\sigma)^*(\overline{X}^{n'}_{s-})\|^2\,ds$$

$$=\lim_{i\to\infty}\limsup_{n'\to\infty}E\int_0^{T\wedge\overline{\tau}^{R_k}_{n'}}\psi^2(\overline{X}^{n'}_{s-})\|(\sigma_i-\sigma)(\sigma_i-\sigma)^*(\overline{X}^{n'}_{s-})\|^2\,ds$$

$$\le C\lim_{i\to\infty}\big\|\psi^2\|(\sigma_i-\sigma)(\sigma_i-\sigma)^*\|^2\big\|_{L^d(\overline{B(0,R_k)}\cap\overline{D})}=0.$$

Similarly, by the definition of variation,

$$\lim_{i\to\infty}\limsup_{n'\to\infty} E\Big|\int_0^{\cdot}(\hat{b}_i-\hat{b})(\overline{X}^{n'}_{s-})\,d\rho^{n'}_s\Big|_{T\wedge\overline{\tau}^{R_k}_{n'}}$$
$$\le d^{1/2}\lim_{i\to\infty}\limsup_{n'\to\infty} E\int_0^{T\wedge\overline{\tau}^{R_k}_{n'}}|(\hat{b}_i-\hat{b})(\overline{X}^{n'}_{s-})|\,ds$$
$$= d^{1/2}\lim_{i\to\infty}\limsup_{n'\to\infty} E\int_0^{T\wedge\overline{\tau}^{R_k}_{n'}}\psi(\overline{X}^{n'}_{s-})|(b_i-b)(\overline{X}^{n'}_{s-})|\,ds$$
$$\le d^{1/2}C\lim_{i\to\infty}\big\|\psi|b_i-b|\big\|_{L^d(\overline{B(0,R_k)}\cap\overline{D})}=0.$$

Hence, for every $k\in\mathbb{N}$ and every $\epsilon>0$,

(2.11)
$$\lim_{i\to\infty}\limsup_{n'\to\infty} P\Big(\sup_{t\le T}\Big|\int_0^{t\wedge\overline{\tau}^{R_k}_{n'}}\hat{\sigma}_i(\overline{X}^{n'}_{s-})\,dW^{\rho^{n'}}_s-\int_0^{t\wedge\overline{\tau}^{R_k}_{n'}}\hat{\sigma}(\overline{X}^{n'}_{s-})\,dW^{\rho^{n'}}_s\Big|\ge\epsilon\Big)=0$$

and

(2.12)
$$\lim_{i\to\infty}\limsup_{n'\to\infty} P\Big(\sup_{t\le T}\Big|\int_0^{t\wedge\overline{\tau}^{R_k}_{n'}}\hat{b}_i(\overline{X}^{n'}_{s-})\,d\rho^{n'}_s-\int_0^{t\wedge\overline{\tau}^{R_k}_{n'}}\hat{b}(\overline{X}^{n'}_{s-})\,d\rho^{n'}_s\Big|\ge\epsilon\Big)=0.$$

Using (2.4), in a similar way to the proof of Theorem 2.1 in Semrau (2007), we can show that for every measurable function $f:\overline{D}\to\mathbb{R}^+$,

$$E\int_0^{T\wedge\overline{\tau}^{R_k}}(\psi f)(\overline{X}_s)\,ds\le C\|\psi f\|_{L^d(\overline{B(0,R)}\cap\overline{D})}. \tag{2.13}$$

By the above, in much the same way as for $\overline{X}^{n'}$ and $W^{\rho^{n'}}$, we show that for every $T\in\mathbb{R}^+$,

$$\lim_{i\to\infty} E\Big[\int_0^{\cdot}(\hat{\sigma}_i-\hat{\sigma})(\overline{X}_s)\,d\overline{W}_s\Big]_{T\wedge\overline{\tau}^{R_k}}=0,$$
$$\lim_{i\to\infty} E\Big|\int_0^{\cdot}(\hat{b}_i-\hat{b})(\overline{X}_s)\,ds\Big|_{T\wedge\overline{\tau}^{R_k}}=0.$$

Therefore, in view of (2.10)–(2.12) and Theorem 3.2 from Billingsley (1999),

$$\Big(V^{n'},\int_0^{\cdot\wedge\overline{\tau}^{R_k}_{n'}}\hat{\sigma}(\overline{X}^{n'}_{s-})\,dW^{\rho^{n'}}_s,\int_0^{\cdot\wedge\overline{\tau}^{R_k}_{n'}}\hat{b}(\overline{X}^{n'}_{s-})\,d\rho^{n'}_s\Big) \tag{2.14}$$
$$\underset{\mathcal{D}}{\rightarrow}\Big(V,\int_0^{\cdot\wedge\overline{\tau}^{R_k}}\hat{\sigma}(\overline{X}_s)\,d\overline{W}_s,\int_0^{\cdot\wedge\overline{\tau}^{R_k}}\hat{b}(\overline{X}_s)\,ds\Big)$$

in $\mathbb{R} \times \mathcal{D}(\mathbb{R}^+, \mathbb{R}^{6d})$. But $\sigma = \tilde{\sigma} + \hat{\sigma}$ and $b = \tilde{b} + \hat{b}$, therefore (2.8) follows from (2.14) and Proposition VI.2.2 in Jacod and Shiryaev (2003).

The rest of the proof runs as in the proof of Theorem 2.1 in Semrau (2007). ∎

REMARK 2.3. Theorem 2.1 is a generalization of Theorem 2.1 in Semrau (2007).

Consider now the continuous approximation $\{X^n\}_{n\in\mathbb{N}}$ of X defined as the solutions of the SDEs (1.3). We say that the SDE (1.3) have a *strong solution* if there exists a pair (X^n, K^n) of $(\mathcal{F}_t)$-adapted processes such that (X^n, K^n) is a solution of the Skorokhod problem associated with

$$Y_t^n = X_0 + \int_0^t \sigma(X_{s-}^{n,\rho^n})\, dW_s + \int_0^t b(X_{s-}^{n,\rho^n})\, ds, \qquad t \in \mathbb{R}^+. \tag{2.15}$$

It is worth pointing out that we can give a simple simulation scheme associated with this approximation in the case $\overline{D} = \mathbb{R}^{d-1} \times \mathbb{R}^+$ (see Lépingle (1995)).

For such approximation one can obtain a similar result as for $\{\overline{X}^n\}_{n\in\mathbb{N}}$. It is described in the following theorem.

THEOREM 2.4. *Let* $\{(X^n, K^n)\}_{n\in\mathbb{N}}$ *be a sequence of solutions of the SDEs* (1.3) *with coefficients* σ, b *satisfying* (1.4)–(1.6). *If the SDE* (1.1) *has a unique weak solution* (X, K) *then* $(X^n, K^n) \to_{\mathcal{D}} (X, K)$ *in* $\mathcal{D}(\mathbb{R}^+, \mathbb{R}^{2d})$. ∎

This result may be proved in much the same way as Theorem 2.1 because Lemma 2.2 holds true with $\overline{X}^n$ replaced by X^{n,ρ^n}.

REMARK 2.5. Theorem 2.4 is a generalization of Theorem 3.1 in Semrau (2007).

3. L^p convergence in convex domains. This section contains some results on approximations of the strong solution of (1.1).

Let W be an $(\mathcal{F}_t)$-adapted Wiener process. Recall that the SDE (1.1) has a *strong solution* if there exists a pair (X, K) of $(\mathcal{F}_t)$-adapted processes that is a solution of the Skorokhod problem associated with

$$Y_t = X_0 + \int_0^t \sigma(X_s)\, dW_s + \int_0^t b(X_s)\, ds, \qquad t \geq 0.$$

In the proofs of the results on L^p convergence for the Euler and Euler–Peano schemes for the SDE (1.1) in Semrau (2009) we used techniques from the proofs of the respective theorems on weak convergence in Semrau (2007). The same methods work for the results mentioned below. Using some steps from the proofs in the previous section, in a similar way to Theorems 2.1 and 3.1 from Semrau (2009) we can obtain the following theorems.

THEOREM 3.1. *Let* $\{(\overline{X}^n, \overline{K}^n)\}_{n\in\mathbb{N}}$ *be a sequence of solutions of the SDEs* (1.2) *with coefficients* σ, b *satisfying* (1.4)–(1.6). *If the solution to the SDE* (1.1) *is pathwise unique, then for every* $p \in \mathbb{N}$,

$$E \sup_{t\le T} |\overline{X}^n_t - X_t|^{2p} \to 0, \qquad T \ge 0. \ ■$$

THEOREM 3.2. *Let* $\{(X^n, K^n)\}_{n\in\mathbb{N}}$ *be a sequence of solutions of the SDEs* (1.3) *with coefficients* σ, b *satisfying* (1.4)–(1.6). *If the solution to the SDE* (1.1) *is pathwise unique, then for every* $p \in \mathbb{N}$,

$$E \sup_{t\le T} |X^n_t - X_t|^{2p} \to 0, \qquad T \ge 0. \ ■$$

REMARK 3.3. The above theorems generalize Theorems 2.1 and 3.1 in Semrau (2009).

4. General domains. In this section we discuss the case of a domain satisfying the following conditions:

(A) There exists a constant $r_0 > 0$ such that $\mathcal{N}_x = \mathcal{N}_{x,r_0} \neq \emptyset$ for every $x \in \partial D$, where

$$\mathcal{N}_{x,r} = \{n \in \mathbb{R}^d : |n| = 1, B(x - rn, r) \cap D = \emptyset\},\ \mathcal{N}_{x,\infty} = \bigcap_{r>0} \mathcal{N}_{x,r}.$$

(B) There exist constants $\delta > 0$ and $\beta \ge 1$ such that for every $x \in \partial D$ there is a unit vector l_x with

$$\langle l_x, n\rangle \ge \frac{1}{\beta} \quad \text{for every } n \in \bigcup_{y\in B(x,\delta)\cap\partial D} \mathcal{N}_y,$$

where $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbb{R}^d$.

REMARK 4.1. (i) If condition (A) is satisfied and $\operatorname{dist}(x, \overline{D}) < r_0$, $x \notin \overline{D}$, then there exists a unique $\Pi(x) \in \overline{D}$ such that $|x - \Pi(x)| = \operatorname{dist}(x, \overline{D})$ and moreover $(\Pi(x) - x)/|\Pi(x) - x| \in \mathcal{N}_{\Pi(x)}$.

(ii) If D is a convex domain in $\mathbb{R}^d$ with nonempty interior then $r_0 = \infty$ and assumptions (A), (B) are satisfied for $d = 1, 2$. For $d > 2$ there exists a sequence $\{D_k\}_{n\in\mathbb{N}}$ of bounded convex sets satisfying conditions (A), (B) such that $D_k \uparrow D$ (e.g. $D_k = B(0, k) \cap D, k \in \mathbb{N}$).

In the case when D satisfies the above conditions, if $r_0 < \infty$, the random variables X_t, $\overline{X}^n_t$, X^n_t are not necessarily integrable. In order to get convergence results, it is necessary to put some restriction on the coefficients. It is additionally required that they are bounded.

First we will look more closely at weak convergence of our schemes.

THEOREM 4.2. *Let* $\{(\overline{X}^n, \overline{K}^n)\}_{n\in\mathbb{N}}$ *be a sequence of solutions of the SDEs* (1.2) *with coefficients* σ, b *satisfying* (1.4), (1.6) *and* (1.7). *If the SDE*

(1.1) *has a unique weak solution* (X,K) *then* $(\overline{X}^n,\overline{K}^n)\to_{\mathcal{D}}(X,K)$ *in* $\mathcal{D}(\mathbb{R}^+,\mathbb{R}^{2d})$.

Proof. Our proof starts with the observation that $\{(\overline{X}^n,\overline{K}^n,W^{\varrho^n})\}_{n\in\mathbb{N}}$ is tight in $\mathcal{D}(\mathbb{R}^+,\mathbb{R}^{3d})$. Indeed, let

$$\overline{\tau}_n^k=\inf\Big\{t\in\mathbb{R}^+:|\overline{K}^n|_t>k\ \vee\ |\Delta W_t^{\varrho^n}|+\frac{1}{n}\geq\frac{r_0}{4L}\Big\},\qquad k,n\in\mathbb{N}.$$

On account of Theorem 7.2 in Słomiński (1996), for every $T\in\mathbb{R}^+$ the sequence $\{|\overline{K}^n|_T\}_{n\in\mathbb{N}}$ is bounded in probability. Moreover,

$$\sup_{t\leq T}|\Delta W_t^{\varrho^n}|\to 0\quad\text{a.s.},\qquad T\in\mathbb{R}^+,$$

and consequently

$$\lim_{k\to\infty}\limsup_{n\to\infty}P(\overline{\tau}_n^k\leq T)=0,\qquad T\in\mathbb{R}^+.\tag{4.1}$$

Let $\{\gamma_n\}_{n\in\mathbb{N}}$ be a sequence of $(\mathcal{F}_t^{\rho^n})$ stopping times and let $\{\delta_n\}_{n\in\mathbb{N}}$ be a sequence of positive constants such that $\delta_n\downarrow 0$ and $\gamma_n+\delta_n\leq T$, $n\in\mathbb{N}$, $T\in\mathbb{R}^+$. In order to get tightness of $\{\overline{X}^n\}_{n\in\mathbb{N}}$, it will be necessary to check that

$$\overline{X}^n_{\gamma_n+\delta_n}-\overline{X}^n_{\gamma_n}\underset{\mathcal{P}}{\to}0.$$

Having (4.1), it is sufficient to show that for every $k\in\mathbb{N}$,

$$\lim_{n\to\infty}E|\overline{X}^{n,\overline{\tau}_n^k-}_{\gamma_n+\delta_n}-\overline{X}^{n,\overline{\tau}_n^k-}_{\gamma_n}|^2=0.$$

According to Theorem 1 from Słomiński (1994), Schwarz's inequality and (1.7),

$$\begin{aligned}
E|\overline{X}^{n,\overline{\tau}_n^k-}_{\gamma_n+\delta_n}-\overline{X}^{n,\overline{\tau}_n^k-}_{\gamma_n}|^2&=E|\overline{X}^n_{(\gamma_n+\delta_n)\wedge\overline{\tau}_n^k-}-\overline{X}^n_{\gamma_n\wedge\overline{\tau}_n^k-}|^2\\
&\leq C_1E\Big(\Big[\int_0^{\cdot}\sigma(\overline{X}^n_{s-})\,dW_s^{\rho^n}\Big]_{\gamma_n\wedge\overline{\tau}_n^k-}^{(\gamma_n+\delta_n)\wedge\overline{\tau}_n^k-}+\Big\langle\int_0^{\cdot}\sigma(\overline{X}^n_{s-})\,dW_s^{\rho^n}\Big\rangle_{\gamma_n\wedge\overline{\tau}_n^k-}^{(\gamma_n+\delta_n)\wedge\overline{\tau}_n^k-}\\
&\quad+\Big(\Big|\int_0^{\cdot}b(\overline{X}^n_{s-})\,d\rho_s^n\Big|_{\gamma_n\wedge\overline{\tau}_n^k-}^{(\gamma_n+\delta_n)\wedge\overline{\tau}_n^k-}\Big)^2\Big)\\
&\leq C_1E\Big(2d\int_{\gamma_n\wedge\overline{\tau}_n^k-}^{(\gamma_n+\delta_n)\wedge\overline{\tau}_n^k-}\|\sigma\sigma^*(\overline{X}^n_{s-})\|\,ds+d\delta_n\int_{\gamma_n\wedge\overline{\tau}_n^k-}^{(\gamma_n+\delta_n)\wedge\overline{\tau}_n^k-}|b(\overline{X}^n_{s-})|^2\,ds\Big)\\
&\leq dLC_1(2+\delta_n)\delta_n\xrightarrow[n\to\infty]{}0.
\end{aligned}$$

It can be similarly shown that

$$\lim_{n\to\infty}E|\overline{K}^{n,\overline{\tau}_n^k-}_{\gamma_n+\delta_n}-\overline{K}^{n,\overline{\tau}_n^k-}_{\gamma_n}|^2=0.$$

Consequently, by the criterion given by Aldous (1978), $\{(\overline{X}^n, \overline{K}^n, W^{\varrho^n})\}_{n\in\mathbb{N}}$ is tight in $\mathcal{D}(\mathbb{R}^+, \mathbb{R}^{3d})$.

Our next goal is again to obtain a suitable Krylov type inequality for $\{\overline{X}^n\}_{n\in\mathbb{N}}$, which is necessary to finish this proof.

LEMMA 4.3. *Let $\tau_n^R = \inf\{t : |\overline{X}_t^n| > R\}$, $R \in \mathbb{R}^+$, $n \in \mathbb{N}$. Then for all bounded measurable functions $f : \overline{D} \to \mathbb{R}^+$ such that $l(D_f) = 0$,*

$$\limsup_{n\to\infty} E \int_0^{T\wedge\tau_n^R} (\psi f)(\overline{X}_{s-}^n)\, ds \le C\|\psi f\|_{L^d(\overline{B(0,R)}\cap\overline{D})}, \tag{4.2}$$

where C is a positive constant depending only on d, T, R and λ.

Proof. We proceed analogously to the proof of Lemma 2.2, but in the different manner we show that for every $T \in \mathbb{R}$,

$$\sup_{t\le T} |\widehat{X}_t^n - \overline{X}_t^n|^2 \underset{\mathcal{P}}{\to} 0. \tag{4.3}$$

First note that similar arguments to that used in the proof of Lemma 2.2 in Semrau (2007) can be applied to get the estimates

$$\begin{aligned} E\sup_{t\le T} |\widehat{Y}_t^n - \overline{Y}_t^n|^2 &= E\sup_{t\le T} |\sigma(\overline{X}_{\rho_t^n-}^n)(W_t - W_{\rho_t^n}) + b(\overline{X}_{\rho_t^n-}^n)(t-\rho_t^n)|^2 \\ &\le 2LE\Big\{d\sup_{t\le T}|W_t - W_{\rho_t^n}|^2 + \sup_{t\le T}|t-\rho_t^n|^2\Big\} \\ &\le 2L\Big\{dE\Big\{\omega_W\Big(\frac{1}{n}, T\Big)\Big\}^2 + \Big(\frac{1}{n}\Big)^2\Big\} \le C_3\frac{\ln n}{n}, \qquad n > 1. \end{aligned} \tag{4.4}$$

Let

$$\tau_n^k = \inf\Big\{t \in \mathbb{R}^+ : |\overline{K}^n|_t + |\widehat{K}^n|_t > k \ \vee\ |\varDelta W_t^{\varrho^n}| + \frac{1}{n} \ge \frac{r_0}{4L}\Big\}, \qquad k, n \in \mathbb{N}.$$

Since $\sup_{t\le T} |\varDelta W_t^{\varrho^n}| \to 0$ a.s. and $\{|\overline{K}^n|_T\}_{n\in\mathbb{N}}$, $\{|\widehat{K}^n|_T\}_{n\in\mathbb{N}}$ are bounded in probability, it follows that for every $T \in \mathbb{R}^+$,

$$\lim_{k\to\infty}\limsup_{n\to\infty} P(\tau_n^k \le T) = 0. \tag{4.5}$$

On account of Lemma 2.3 in Saisho (1987), for every $t \le T$,

$$\begin{aligned} |\widehat{X}_t^{n,\tau_n^k} - \overline{X}_t^{n,\tau_n^k-}|^2 &\le |\widehat{Y}_t^{n,\tau_n^k} - \overline{Y}_t^{n,\tau_n^k-}|^2 \\ &\quad + \frac{1}{r_0}\int_0^t |\widehat{X}_s^{n,\tau_n^k} - \overline{X}_s^{n,\tau_n^k-}|^2\, d(|\widehat{K}^{n,\tau_n^k}|_s + |\overline{K}^{n,\tau_n^k-}|_s) \\ &\quad + 2\int_0^t (\widehat{Y}_t^{n,\tau_n^k} - \overline{Y}_t^{n,\tau_n^k-} - \widehat{Y}_s^{n,\tau_n^k} + \overline{Y}_s^{n,\tau_n^k-})\, d(\widehat{K}_s^{n,\tau_n^k} - \overline{K}_s^{n,\tau_n^k-}). \end{aligned}$$

Moreover, for every stopping time τ, we have $|\Delta\overline{K}^{n,\tau_n^k-}_\tau| \le |\Delta\overline{Y}^{n,\tau_n^k-}_\tau| \le r_0/4$. Consequently, using the estimates from the proof of Lemma 2.2 in Semrau (2007) and (4.4), we get

$$
\begin{aligned}
E\sup_{t\le T\wedge\tau}|\widehat{X}^{n,\tau_n^k}_t-\overline{X}^{n,\tau_n^k-}_t|^2 &\le E\sup_{t\le T\wedge\tau}|\widehat{Y}^{n,\tau_n^k}_t-\overline{Y}^{n,\tau_n^k-}_t|^2\\
&\quad+4\sqrt{2}\Big(E\sup_{t\le T\wedge\tau}|\widehat{Y}^{n,\tau_n^k}_t-\overline{Y}^{n,\tau_n^k-}_t|^2\Big)^{1/2}(E|\widehat{K}^{n,\tau_n^k}|^2_T+E|\overline{K}^{n,\tau_n^k-}|^2_T)^{1/2}\\
&\quad+\frac{1}{r_0}E\int_0^{\tau-}\sup_{u\le T\wedge s}|\widehat{X}^{n,\tau_n^k}_u-\overline{X}^{n,\tau_n^k-}_u|^2\,d(|\widehat{K}^{n,\tau_n^k}|_s+|\overline{K}^{n,\tau_n^k-}|_s)\\
&\quad+\frac{1}{r_0}E\sup_{t\le T\wedge\tau}|\widehat{X}^{n,\tau_n^k}_t-\overline{X}^{n,\tau_n^k-}_t|^2(|\Delta\widehat{K}^{n,\tau_n^k}_\tau|+|\Delta\overline{K}^{n,\tau_n^k-}_\tau|)\\
&\le C_3\frac{\ln n}{n}+8k\Big(C_3\frac{\ln n}{n}\Big)^{1/2}\\
&\quad+\frac{1}{r_0}E\int_0^{\tau-}\sup_{u\le T\wedge s}|\widehat{X}^{n,\tau_n^k}_u-\overline{X}^{n,\tau_n^k-}_u|^2\,d(|\widehat{K}^{n,\tau_n^k}|_s+|\overline{K}^{n,\tau_n^k-}|_s)\\
&\quad+\frac{1}{r_0}\cdot\frac{r_0}{4}E\sup_{t\le T\wedge\tau}|\widehat{X}^{n,\tau_n^k}_t-\overline{X}^{n,\tau_n^k-}_t|^2,\qquad n>1.
\end{aligned}
$$

By Lemma C.1 of Słomiński (1996), for every $T\in\mathbb{R}^+$ and every $k\in\mathbb{N}$,

$$
E\sup_{t\le T}|\widehat{X}^{n,\tau_n^k}_t-\overline{X}^{n,\tau_n^k-}_t|^2\le C_2\Big(\frac{\ln n}{n}\Big)^{1/2},\qquad n>1.
$$

Combining this inequality with (4.5) gives (4.3). This finishes the proof. ∎

The rest of the proof runs as in the proof of Theorem 2.1. ∎

We can now state the analogue of Theorem 4.2 for the approximation $\{X^n\}_{n\in\mathbb{N}}$. It may be proved in much the same way.

THEOREM 4.4. *Let* $\{(X^n,K^n)\}_{n\in\mathbb{N}}$ *be a sequence of solutions of the SDEs* (1.3) *with coefficients* σ, b *satisfying* (1.4), (1.6) *and* (1.7). *If the SDE* (1.1) *has a unique weak solution* (X,K) *then* $(X^n,K^n)\to_{\mathcal{D}}(X,K)$ *in* $\mathcal{D}(\mathbb{R}^+,\mathbb{R}^{2d})$. ∎

The remainder of this section will be devoted to theorems on convergence of $\{\overline{X}^n\}_{n\in\mathbb{N}}$ and $\{X^n\}_{n\in\mathbb{N}}$ to a strong solution of SDE (1.1). Since the random variables X_t, $\overline{X}^n_t$ and X^n_t are not necessarily integrable, the results are weaker than their analogues in Section 3. Here only convergence in probability is obtained.

THEOREM 4.5. *Let $\{(\overline{X}^n, \overline{K}^n)\}_{n\in\mathbb{N}}$ be a sequence of solutions of the SDEs* (1.2) *with coefficients σ, b satisfying* (1.4), (1.6) *and* (1.7). *If the solution to the SDE* (1.1) *is pathwise unique, then for every $p \in \mathbb{N}$,*

$$\sup_{t\le T} |\overline{X}_t^n - X_t| \underset{\mathcal{P}}{\to} 0, \qquad T \in \mathbb{R}^+.$$

Proof. This can be shown by making slight changes in the proof of Theorem 2.1 in Semrau (2009). We use results from the proof of Theorem 4.2 instead of the respective results from the proof of Theorem 2.1 in Semrau (2007). ■

In a similar way we can obtain the following result.

THEOREM 4.6. *Let $\{(X^n, K^n)\}_{n\in\mathbb{N}}$ be a sequence of solutions of the SDEs* (1.3) *with coefficients σ, b satisfying* (1.4), (1.6) *and* (1.7). *If the solution to the SDE* (1.1) *is pathwise unique, then for every $p \in \mathbb{N}$,*

$$\sup_{t\le T} |X_t^n - X_t| \underset{\mathcal{P}}{\to} 0, \qquad T \in \mathbb{R}^+. \ ■$$

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