Summary. Let $K$ be a field and let $L = K[\xi]$ be a finite field extension of $K$ of degree $m > 1$. If $f \in L[Z]$ is a polynomial, then there exist unique polynomials $u_0, \ldots, u_{m-1} \in K[X_0, \ldots, X_{m-1}]$ such that $f(\sum_{j=0}^{m-1} \xi^j X_j) = \sum_{j=0}^{m-1} \xi^j u_j$. A. Nowicki and S. Spodzieja proved that, if $K$ is a field of characteristic zero and $f \neq 0$, then $u_0, \ldots, u_{m-1}$ have no common divisor in $K[X_0, \ldots, X_{m-1}]$ of positive degree. We extend this result to the case when $L$ is a separable extension of a field $K$ of arbitrary characteristic. We also show that the same is true for a formal power series in several variables.

1. Introduction. Throughout the paper, $K$ is a field and $L = K[\xi]$ is a finite field extension of $K$ of degree $m > 1$. For $j = 1, \ldots, n$ let $X_j = (X_{j,0}, \ldots, X_{j,m-1})$ denote a system of variables and set

$$[X_j] = X_{j,0} + \xi X_{j,1} + \cdots + \xi^{m-1} X_{j,m-1}.$$  

If $n = 1$, then we write briefly $X = (X_0, \ldots, X_{m-1})$ instead of $X_1 = (X_{1,0}, \ldots, X_{1,m-1})$. If $f \in L[Z_1, \ldots, Z_n]$ is a polynomial, then there exist unique polynomials $u_0, \ldots, u_{m-1} \in K[X_1, \ldots, X_n]$ such that

$$f([X_1], \ldots, [X_n]) = u_0 + \xi u_1 + \cdots + \xi^{m-1} u_{m-1}.$$  

This representation is called the imaginary decomposition of $f$ relative to $\xi$, and the polynomials $u_0, \ldots, u_{m-1}$ are the imaginary parts of $f$ (see [1]).

Assume that

$$\phi(t) = t^m - a_{m-1} t^{m-1} - \cdots - a_1 t - a_0,$$

where $a_0, \ldots, a_{m-1} \in K$, is the minimal polynomial of $\xi$ over $K$ and let $u = (u_0, \ldots, u_{m-1})$ be a sequence of polynomials belonging to $K[X]$. Denote by $\overline{u} = (\overline{u}_0, \ldots, \overline{u}_{m-1})$
the sequence of polynomials defined by
\[ u_0 = a_0 u_{m-1}, \quad u_1 = a_1 u_{m-1} + u_0, \ldots, \quad u_{m-1} = a_{m-1} u_{m-1} + u_{m-2}. \]

We say that \( u \) is a \( \xi \)-sequence if \( u \) satisfies the following generalized Cauchy–Riemann equations introduced in [1]:
\[ \frac{\partial u}{\partial X_i} = \frac{\partial \pi}{\partial X_{i-1}}, \quad i = 1, \ldots, m-1. \]

In 2003, A. Nowicki and S. Spodzieja proved the following theorem.

**Theorem 1** ([1, Theorem 3.8]). Let \( K \) be a field of characteristic zero and let \( L = K[\xi] \) be a finite field extension of \( K \) of degree \( m > 1 \). The following two conditions are equivalent:

(i) \( u \) is a \( \xi \)-sequence.

(ii) There exists \( f \in L[Z] \) such that \( u_0, \ldots, u_{m-1} \) are the imaginary parts of \( f \).

As a consequence of Theorem 1, A. Nowicki and S. Spodzieja also proved the following curious theorem.

**Theorem 2** ([1, Theorem 5.3]). If under the assumptions of Theorem 1, \( u_0, \ldots, u_{m-1} \) are the imaginary parts of \( f \in L[Z_1, \ldots, Z_n] \setminus \{0\} \), then \( \gcd(u_0, \ldots, u_{m-1}) = 1 \).

The assumption that \( \text{char} \ K = 0 \) played an essential role in the proof of Theorem 2. The aim of this paper is to extend this theorem to the case when \( L \) is a separable extension of a field \( K \) of arbitrary characteristic. More precisely, our main result is the following.

**Theorem 3.** Let \( K \) be a field and let \( L = K[\xi] \) be a finite separable extension of \( K \) of degree \( m > 1 \). If \( u_0, \ldots, u_{m-1} \) are the imaginary parts of \( f \in L[Z_1, \ldots, Z_n] \setminus \{0\} \), then \( \gcd(u_0, \ldots, u_{m-1}) = 1 \).

Additionally, in Section 4 we generalize Theorems 1–3 to formal power series (Propositions 4–6, respectively).

2. **Some auxiliary results.** To prove Theorem 3 we need several known simple facts (see [1]).

**Proposition 1.** If \( u_0, \ldots, u_{m-1} \) are the imaginary parts of a homogeneous polynomial \( f \in L[Z_1, \ldots, Z_n] \) of degree \( s \), then \( u_i \) is zero or a homogeneous polynomial of degree \( s \) for \( i = 0, \ldots, m-1 \).

**Proposition 2.** If the polynomials \( u_0, \ldots, u_{m-1} \in K[X_1, \ldots, X_n] \) are not relatively prime, then their homogeneous components of the highest degree are also not relatively prime.
Let \( d, n \in \mathbb{Z}, d, n \geq 2 \). Consider the Kronecker substitution (cf. [2, 1.6, Definition 5]), i.e. the \( L \)-automorphism \( \kappa_d \) of \( L[Z_1, \ldots, Z_n] \) defined by
\[
\kappa_d(Z_j) = \begin{cases} 
Z_1 & \text{if } j = 1, \\
Z_j + Z_1^{d-1} & \text{if } j = 2, \ldots, n.
\end{cases}
\]

**Proposition 3** ([1, Proposition 5.1]). Let \( f \in L[Z_1, \ldots, Z_n] \), and let \( d > \max_{j=1, \ldots, n} \deg_Z f > 0 \). Then
\[
\kappa_d(f) = aZ_1^N + \text{terms of degrees lower than } N, \quad N \geq 1, \ a \in L \setminus \{0\}.
\]
Let \( P_j = \kappa_d(Z_j) \in L[Z_1, \ldots, Z_n] \) for \( j = 1, \ldots, n \) and
\[
P_j([X_1], \ldots, [X_n]) = v_{j,0} + \xi v_{j,1} + \cdots + \xi^{m-1}v_{j,m-1}, \quad v_{j,i} \in K[X_1, \ldots, X_n].
\]
Let \( \gamma : K[X_1, \ldots, X_n] \rightarrow K[X_1, \ldots, X_n] \) be the homomorphism such that
\[
\gamma(X_{j,i}) = v_{j,i}.
\]

**Lemma 1** ([1, Lemma 5.2]). \( \gamma \) is a \( K \)-automorphism of \( K[X_1, \ldots, X_n] \).

**3. Proof of Theorem 3.** A crucial role in the proof is played by the following lemma.

**Lemma 2.** If under the assumptions of Theorem 3, \( u_0, \ldots, u_{m-1} \) are the imaginary parts of \( f(Z) = a_0Z^s, \ a_0 \in L \setminus \{0\} \), then \( \gcd(u_0, \ldots, u_{m-1}) = 1 \).

*Proof.* Let \( \phi \) be the minimal polynomial of \( \xi \) over \( K \) and let \( M \) be a decomposition field of \( \phi \). Then \( K[\xi] = K(\xi) \subset M \) and \( \deg \phi = m > 1 \). Consequently, since \( \xi \) is a simple root of \( \phi \), there exists \( b \in M, \ b \neq \xi, \) such that \( \phi(b) = 0 \). There is a \( K \)-isomorphism \( \varphi : K(\xi) \rightarrow K(b) \) such that \( \varphi(\xi) = b \).

Suppose that there is a polynomial \( v \in K[X] \) of positive degree which is a common divisor of \( u_0, \ldots, u_{m-1} \) in \( K[X] \), and so also in \( L[X] \). Since \( L[X] \) is a UFD and \( X_0 + \xi X_1 + \cdots + \xi^{m-1}X_{m-1} \) is irreducible in \( L[X] \), there exist \( l \in \mathbb{Z}, \ l \geq 1 \), and \( a \in L \setminus \{0\} \) such that
\[
v(X_0, \ldots, X_{m-1}) = a(X_0 + \xi X_1 + \cdots + \xi^{m-1}X_{m-1})^l.
\]
Then \( v(-\xi, 1, 0, \ldots, 0) = 0 \), and so, since \( v \in K[X] \), we get
\[
a(-b + \xi)^l = v(-b, 1, 0, \ldots, 0) = \varphi(v(-\xi, 1, 0, \ldots, 0)) = 0,
\]
a contradiction. \( \blacksquare \)

Using the facts in Section 2 we will extend Lemma 2 so as to obtain Theorem 3.

*Proof of Theorem 3.* Suppose that \( u_0, \ldots, u_{m-1} \) have a common divisor in \( K[X_1, \ldots, X_n] \) of positive degree. Denote by \( f(s) \) the homogeneous part of the highest degree of \( f \) and let \( u_0^{(s)}, \ldots, u_{m-1}^{(s)} \) be the homogeneous parts of the highest degree of \( u_0, \ldots, u_{m-1} \), respectively. By Proposition 3 and Lemma 1 one can assume that \( f(s)(Z_1, \ldots, Z_n) = a_0Z_1^s, \ a_0 \in L \setminus \{0\} \), and
so \( f^{(s)} \in L[Z_1] \). By Propositions \( 1 \) and \( 2 \), \( u_0^{(s)}, \ldots, u_{m-1}^{(s)} \) are the imaginary parts of \( f^{(s)} \) and they are not relatively prime. This contradicts Lemma 2 and ends the proof. \( \blacksquare \)

The following example, due to the referee, shows that the assumption of Theorem 3 concerning separability of the extension \( L \) of \( K \) is necessary.

**Example 1.** Let \( K = \mathbb{F}_2(t^2) \), \( L = \mathbb{F}_2(t) \) and let \( \xi = t \). Consider the polynomial \( f(Z) = Z^2 \). Then
\[
f(X_0 + \xi X_1) = X_0^2 + t^2 X_1^2 \in K[X_0, X_1].
\]
Hence \( u_0 = X_0^2 + t^2 X_1^2 \) and \( u_1 = 0 \) are the imaginary parts of \( f \) and they are not relatively prime.

**4. Generalizations to formal power series.** In this section we generalize Theorems 1–3 to formal power series.

Let \( f \in L[[Z_1, \ldots, Z_n]] \) be a formal power series of the form \( f = \sum_{r=d}^{\infty} f^{(r)} \), where \( f^{(r)} \) is zero or a homogeneous polynomial of degree \( r \) for \( r \geq d \), and let \( u_0, \ldots, u_{m-1} \in K[[X_1, \ldots, X_n]] \) be formal power series of the form \( u_j = \sum_{r=d}^{\infty} u_j^{(r)} \), where \( u_j^{(r)} \) is zero or a homogeneous polynomial of degree \( r \) for \( r \geq d \), \( j = 0, \ldots, m-1 \). By Proposition 1 we get immediately

**Corollary 1.** \( u_0^{(r)}, \ldots, u_{m-1}^{(r)} \) are the imaginary parts of \( f^{(r)} \) for \( r \geq d \) if and only if
\[
f([X_1], \ldots, [X_n]) = u_0 + \xi u_1 + \cdots + \xi^{m-1} u_{m-1}.
\]

We call this representation the **imaginary decomposition** of \( f \) relative to \( \xi \), and the power series \( u_0, \ldots, u_{m-1} \) the **imaginary parts** of \( f \).

Similarly to Lemma 3.5 in [1] we obtain a version of that lemma for power series.

**Lemma 3.** \((u_0, \ldots, u_{m-1})\) is a \( \xi \)-sequence if and only if \((u_0^{(r)}, \ldots, u_{m-1}^{(r)})\) is a \( \xi \)-sequence for \( r \geq d \).

Now we show the following generalizations of Theorems 1 and 2.

**Proposition 4.** Under the assumptions of Theorem 1 on \( K \) and \( L \), if \( u_0, \ldots, u_{m-1} \in K[[X]] \) are power series, then the following two conditions are equivalent:

(i) \((u_0, \ldots, u_{m-1})\) is a \( \xi \)-sequence.

(ii) There exists \( f \in L[[Z]] \) such that \( u_0, \ldots, u_{m-1} \) are the imaginary parts of \( f \).

**Proof.** By Lemma 3 and Theorem 1, \((u_0, \ldots, u_{m-1})\) is a \( \xi \)-sequence if and only if there exist \( f^{(d)}, f^{(d+1)}, \ldots \in L[Z] \) such that \( u_0^{(r)}, \ldots, u_{m-1}^{(r)} \) are the imaginary parts of \( f^{(r)} \) for \( r \geq d \). By Corollary 1 this is equivalent to
the fact that \( u_0, \ldots, u_{m-1} \) are the imaginary parts of \( f := \sum_{r=d}^{\infty} f^{(r)} \). Thus, the proof is finished. ■

**Proposition 5.** Under the assumptions of Theorem 1 on \( K \) and \( L \), if the power series \( u_0, \ldots, u_{m-1} \) are the imaginary parts of \( f \in L[[Z_1, \ldots, Z_n]]\{0\} \), then \( \gcd(u_0, \ldots, u_{m-1}) = 1 \).

**Proof.** If \( u_0, \ldots, u_{m-1} \) have a common divisor in \( K[[X_1, \ldots, X_n]] \) of positive order, then by Corollary 1, \( u_0^{(d)}, \ldots, u_{m-1}^{(d)} \) are the imaginary parts of \( f^{(d)} \) and they have a common divisor in \( K[X_1, \ldots, X_n] \) of positive degree. This contradicts Theorem 2 and ends the proof. ■

Analogously we obtain the following generalization of Theorem 3.

**Proposition 6.** Under the assumptions of Theorem 3 on \( K \) and \( L \), if the power series \( u_0, \ldots, u_{m-1} \) are the imaginary parts of \( f \in L[[Z_1, \ldots, Z_n]]\{0\} \), then \( \gcd(u_0, \ldots, u_{m-1}) = 1 \).

**Acknowledgments.** I would like to thank the anonymous referee for his remarks improving the paper, and Professors Andrzej Schinzel and Stanisław Spodzieja for their valuable comments and advice. I am also grateful to my colleague Krzysztof Kamiński for pointing out a few mistakes in the paper.

**References**


Adam Grygiel
Faculty of Mathematics and Computer Science
University of Łódź
Banacha 22
90-238 Łódź, Poland
E-mail: adamgry@op.pl

Received March 15, 2008;
received in final form March 21, 2008 (7652)