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MEASURE AND INTEGRATION

# Narrow Convergence in Spaces of Set-Valued Measures

by

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**Summary.** We prove an analogue of Topsøe's criterion for relative compactness of a family of probability measures which are regular with respect to a family sets. We consider measures whose values are compact convex sets in a locally convex linear topological space.

**Introduction.** Let $T$ be an abstract set, $\mathcal{K}$ a family of subsets of $T$, and $(E, F)$ a dual pair of real vector spaces, with $E$ endowed with the weak topology $\sigma(E, F)$. Let $\mathrm{cc}(E, F)$ be the set of all convex compact non-empty subsets of $E$, and $\widetilde{M}_+(T, \mathcal{K}, \mathrm{cc}(E, F))$ the set of $\mathcal{K}$-inner regular positive set-valued measures defined on a $\sigma$-field $\mathcal{B}$ of subsets of $T$ and with values in $\mathrm{cc}(E, F)$. We denote by $M_+(T, \mathcal{K})$ the set of $\mathcal{K}$-inner regular non-negative measures defined on $\mathcal{B}$ provided with the topology of weak convergence. Prokhorov [11] has proved that if $T$ is a Polish space and $\mathcal{B}$ the set of Borel subsets, then the relatively compact subsets of $M_+(T, \mathcal{K})$ are precisely the tight ones. But this result is not valid for all topological space (see e.g. [5], [10], [18]). In [16] Topsøe has characterized the relatively compact subsets of $M_+(T, \mathcal{K})$ in general situations. Before and after Topsøe's paper there were others (e.g. [1], [3], [18], [5]–[10]). In this paper we generalize to the space $\widetilde{M}_+(T, \mathcal{K}, \mathrm{cc}(E, F))$ the criterion of Topsøe (Theorem 2.1). In addition, we prove a result (Theorem 3.3) analogous to Theorem 8.1 in [17, p. 40].

## 1. Preliminaries

**1.1.** We denote by $T$ an abstract set; $\mathcal{G}$ and $\mathcal{K}$ are families of subsets of $T$. We let $\mathcal{B}$ denote the smallest $\sigma$-field containing every set $A \subseteq T$ for

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which $K \cap A \in \mathcal{K}$ for all $K \in \mathcal{K}$. The family $\mathcal{K}$ is said to be *semicompact* if every countable subfamily of $\mathcal{K}$ with the finite intersection property has a non-empty intersection. We shall say that $\mathcal{G}$ *separates the sets* in $\mathcal{K}$ if for any pair $K, K'$ of disjoint sets in $\mathcal{K}$ we can find a pair $G, G'$ of disjoint sets in $\mathcal{G}$ such that $K \subset G$ and $K' \subset G'$.

Let $\mathcal{G}'$ be a family of subsets of $T$ such that $\mathcal{G}' \subseteq \mathcal{G}$. We shall say that $\mathcal{G}'$ *dominates* $\mathcal{K}$ and write $\mathcal{G}' \succ \mathcal{K}$ if for any $K \in \mathcal{K}$ there exists $G' \in \mathcal{G}'$ such that $K \subseteq G'$.

**1.2.** *Nets on* $T$. Let $X$ be a non-empty subset of $T$ and $(x_i)_{i\in I}$ be a net on $T$. We say that $x_i \in X$ *eventually* if there exists $i \in I$ such that $x_j \in X$ for every $j \in I$ with $j \geq i$. A net $(x_i)_{i\in I}$ on $T$ is *universal* if, for every subset $X \subset T$ either $x_i \in X$ eventually or $x_i \in T \setminus X$ eventually.

**1.3.** *The space* $\mathrm{cc}(E, F)$. Let $(E, F)$ be a dual pair of real vector spaces, with $E$ and $F$ endowed with the weak topologies $\sigma(E, F)$ and $\sigma(F, E)$ respectively. If $X$ and $Y$ are subsets of $E$, we denote by $X + Y$ the subset of $E$ consisting of all elements of the form $x + y$, where $x \in X$ and $y \in Y$. The closed convex hull of $X$ is denoted by $\overline{\mathrm{co}}\, X$, the polar of $X$ by $\mathring{X}$, and the closure of $X$ by $\mathrm{cl}\, X$. The *support function* of $X$ is the map

$$\delta^*(\cdot|X) : F \to [-\infty, +\infty], \qquad y \mapsto \delta^*(y|X) = \sup\{y(x);\, x \in X\}.$$

We denote by $\mathrm{cc}(E, F)$ the set of all $\sigma(E, F)$-compact non-empty convex subsets of $E$. We equip $\mathrm{cc}(E, F)$ with the Hausdorff topology. Let $C \in \mathrm{cc}(E, F)$, $\beta(o)$ a base of neighbourhoods of $o$ in $E$, $V \in \beta(o)$, and $\varepsilon > 0$. The set

$$W_{(V,\varepsilon,C)} = \{C' \in \mathrm{cc}(E, F);\, \sup_{y\in\mathring{V}} |\delta^*(y|C) - \delta^*(y|C')| < \varepsilon\}$$

is a neighbourhood of $C$. The family $\{W_{(V,\varepsilon,C)};\, V \in \beta(o) \text{ and } \varepsilon > 0\}$ is a base of neighborhoods of $C$. The space $\mathrm{cc}(E, F)$ is a completely regular topological space ([2, Theorem II.19]).

**1.4.** *Set-valued measures.* Let $M$ be a map from $\mathcal{B}$ to $\mathrm{cc}(E, F)$. The map $M$ is called *additive* if $M(A \cup B) = M(A) + M(B)$ for any disjoint sets $A, B$ in $\mathcal{B}$; *monotone* if $M(\emptyset) = \{o\}$ and $M(A) \subseteq M(B)$ for all $A$ and $B$ in $\mathcal{B}$ such that $A \subseteq B$; and *positive* if $M(\emptyset) = \{o\}$ and $o \in M(A)$ for all $A \in \mathcal{B}$. We say that $M$ is a *weak set-valued measure* if $M$ is additive and for every $y \in F$ the map $A \mapsto \delta^*(y|M(A))$ from $\mathcal{B}$ to $\mathbb{R}$ is a $\sigma$-additive measure. A positive weak set-valued measure is $\mathcal{K}$-*inner regular* if for every $A \in \mathcal{B}$, $M(A) = \overline{\mathrm{co}} \bigcup\{M(K);\, K \subset A,\, K \in \mathcal{K}\}$. Note that a positive additive map $M : \mathcal{B} \to \mathrm{cc}(E, F)$ is monotone. Indeed, if $A, B \in \mathcal{B}$ and $A \subseteq B$, then $M(B) = M(A) + M(B \setminus A)$. Hence $M(A) = M(A) + \{o\} \subset M(B)$ since $o \in M(B \setminus A)$.

**1.5.** *Set-valued integral.* An integration theory for positive weak set-valued measures is developed in [15]. Let us only recall the following definitions and results. Assume that $M : \mathcal{B} \to \mathrm{cc}(E, F)$ is a positive weak set-valued measure. If $h$ is a positive simple function defined on $T$ (i.e. $h = \sum_{i=1}^{n} \alpha_i 1_{A_i}$ where $\alpha_i \geq 0$, $A_i \in \mathcal{B}$ and $\{A_1, \dots, A_n\}$ is a partition of $T$) then the integral of $h$ with respect to $M$ is defined by $\int hM = \sum_{i=1}^{n} \alpha_i M(A_i)$. If $f$ is a positive measurable function with respect to $\mathcal{B}$ and the Borel field of $\mathbb{R}$, there exists an increasing sequence $(h_n)$ of simple functions such that $f = \sup\{h_n;\, n \in \mathbb{N}\}$. The integral of $f$ is defined by $\int fM = \overline{\mathrm{co}} \bigcup \{\int h_n M;\, n \in \mathbb{N}\}$. We have $\delta^*(y | \int fM) = \int f \delta^*(y | M(\cdot))$ for every $y \in F$. If $f$ is bounded we have $\int fM \in \mathrm{cc}(E, F)$. If $f$ and $g$ are measurable functions and $f \leq g$ then $\int fM \subseteq \int gM$.

**1.6.** *Topologies on* $\widetilde{M}_+(T, \mathrm{cc}(E, F))$. We denote by $\widetilde{M}_+(T, \mathrm{cc}(E, F))$ the set of all positive weak set-valued measures from $\mathcal{B}$ to $\mathrm{cc}(E, F)$ and by $\widetilde{M}_+(T, \mathcal{K}, \mathrm{cc}(E, F))$ the subset of $\widetilde{M}_+(T, \mathrm{cc}(E, F))$ consisting of all $\mathcal{K}$-inner regular elements. In $\widetilde{M}_+(T, \mathrm{cc}(E, F))$ we define the following topologies.

The *weak narrow topology* ($wn$-topology) on $\widetilde{M}_+(T, \mathrm{cc}(E, F))$ is the weakest topology for which the map $M \mapsto M(T)$ is continuous and all maps $M \mapsto \delta^*(y | M(G))$ are lower semicontinuous for every $G \in \mathcal{G}$ and $y \in F$.

The *strong narrow topology* ($sn$-topology) on $\widetilde{M}_+(T, \mathrm{cc}(E, F))$ is the weakest topology for which the map $M \mapsto M(T)$ is continuous and all maps $M \mapsto M(G)$ are lower semicontinuous for every $G \in \mathcal{G}$.

Let $M \in \widetilde{M}_+(T, \mathrm{cc}(E, F))$ and let $(M_i)_{i \in I}$ be a net on $\widetilde{M}_+(T, \mathrm{cc}(E, F)))$. Then $(M_i)$ converges to $M$ in the $wn$-topology if and only if $(M_i(T))$ converges to $M(T)$ in $\mathrm{cc}(E, F)$ and $\liminf_i \delta^*(y | M_i(G)) \geq \delta^*(y | M(G))$ for all $y \in F$ and $G \in \mathcal{G}$; and $(M_i)$ converges to $M$ in the $sn$-topology if and only if $(M_i(T))$ converges to $M(T)$ in $\mathrm{cc}(E, F)$ and for every $G \in \mathcal{G}$ and every open subset $O$ of $E$ such that $M(G) \cap O \neq \emptyset$ there exists $i_0 \in I$ such that $M_i(G) \cap O \neq \emptyset$ for every $i \in I$ with $i \geq i_0$. The subset $\widetilde{M}_+(T, \mathcal{K}, \mathrm{cc}(E, F))$ will be considered as a subspace of $\widetilde{M}_+(T, \mathrm{cc}(E, F))$.

Consider now the following axioms on $\mathcal{K}$ and $\mathcal{G}$, introduced by Topsøe [16].

(i) $\mathcal{K}$ is closed under finite unions and countable intersections, and $\emptyset \in \mathcal{K}$.
(ii) $\mathcal{G}$ is closed under finite unions and finite intersections, and $\emptyset \in \mathcal{G}$.
(iii) For every $K \in \mathcal{K}$ and every $G \in \mathcal{G}$, $K \setminus G \in \mathcal{K}$.
(iv) $\mathcal{G}$ separates the sets in $\mathcal{K}$.
(v) $\mathcal{K}$ is semicompact.

Note that (i) and (iv) imply that $\mathcal{G}$ dominates $\mathcal{K}$.

**1.7.** *Topological case.* Assume now that $T$ is a Hausdorff topological space. We then denote by $\mathcal{K}(T)$, $\mathcal{G}(T)$ and $\mathcal{B}(T)$ the families of compact subsets, open subsets, and Borel subsets of $T$, respectively. Now $\widetilde{M}_+(T,\mathrm{cc}(E,F))$ denotes the set of positive weak set-valued measures defined on $\mathcal{B}(T)$. The $wn$-topology and $sn$-topology are defined by means of $\mathcal{G}(T)$. Generally $\mathcal{K}(T)$, $\mathcal{G}(T)$, $\mathcal{B}(T)$ replace $\mathcal{K}$, $\mathcal{G}$ and $\mathcal{B}$ respectively. We denote by $C_+(T)$ the set of non-negative bounded continuous functions defined on $T$. In view of [17, Theorem 8.1 p. 40] if $T$ is a completely regular space then a net $(M_i)$ on $\widetilde{M}_+(T,\mathrm{cc}(E,F))$ converges in the $wn$-topology to $M$ if and only if $(M_i(T))$ converges to $M(T)$ in $\mathrm{cc}(E,F)$ and for every $y \in F$ and every $f \in C_+(T)$, $(\int f\delta^*(y|M_i(\cdot)))$ converges to $\int f\delta^*(y|M(\cdot))$. It follows that if $T$ is a completely regular space, the $wn$-topology in $\widetilde{M}_+(T,\mathcal{K}(T),\mathrm{cc}(E,F))$ is a uniform topology. The uniformity is generated by the families of pseudometrics $\{p_V;\ V \in \beta(o)\}$ and $\{p_{f,y};\ y \in F,\ f \in C_+(T)\}$, defined as follows: for every $M$ and $M'$ in $\widetilde{M}_+(T,\mathcal{K}(T),\mathrm{cc}(E,F))$

$$p_V(M,M') = \sup_{y\in \overset{\circ}{V}} |\delta^*(y|M(T)) - \delta^*(y|M'(T))|,$$

$$p_{f,y}(M,M') = \Big|\int f\delta^*(y|M(\cdot)) - \int f\delta^*(y|M'(\cdot))\Big|.$$

It is evident that $\widetilde{M}_+(T,\mathcal{K}(T),\mathrm{cc}(E,F))$ endowed with this uniform topology is a Hausdorff space.

Let us introduce another topology. The *simple topology* ($s$-topology) on $\widetilde{M}_+(T,\mathrm{cc}(E,F))$ is the weakest topology for which all maps $M \mapsto M(f)$ are continuous for every $f \in C_+(T)$.

## 2. Main results.

The following theorem was proved by Topsøe ([16, Theorem 4, p. 202]) for non-negative scalar measures.

Theorem 2.1. *Let $\mathcal{G}$ and $\mathcal{K}$ be families of subsets of a set $T$ which satisfy axioms* (i)–(v) *and let $H$ be a subset of $\widetilde{M}_+(T,\mathcal{K},\mathrm{cc}(E,F))$ endowed with the $wn$-topology. Then the following conditions* (1) *and* (2) *are equivalent*:

(1) *Every net on $H$ has a convergent subnet in $\widetilde{M}_+(T,\mathcal{K},\mathrm{cc}(E,F))$.*

(2) (a) *The set $\{M(T); M \in H\}$ is relatively compact in* $\mathrm{cc}(E,F)$.

(b) *For every $y \in F$, every subclass $\mathcal{G}'$ of $\mathcal{G}$ which dominates $\mathcal{K}$, and every $\varepsilon > 0$ there exists a finite subclass $\mathcal{G}''$ of $\mathcal{G}'$ such that*

$$\sup_{M\in H}\inf_{G\in\mathcal{G}''}\delta^*(y|M(T\setminus G)) < \varepsilon.$$

*Proof.* Assume that (1) is satisfied. It is obvious that (a) holds. If (b) failed we would find $y \in F$, $\varepsilon > 0$, $\mathcal{G}' \subseteq \mathcal{G}$ with $\mathcal{G}' \succ \mathcal{K}$ such that for any finite subfamily $\mathcal{G}''$ of $\mathcal{G}'$ there exists $M_{\mathcal{G}''} \in H$ such that $\inf\{\delta^*(y|M_{\mathcal{G}''}(T\setminus G));\ G \in \mathcal{G}''\} \geq \varepsilon$. We then obtain a net $(M_{\mathcal{G}''})_{\mathcal{G}''\subset\mathcal{G}'}$, where the family of all finite subsets of $\mathcal{G}'$ is directed by $\supset$. According to (1), the net $M_{\mathcal{G}''}$ has a subnet convergent in $\widetilde{M}_+(T,\mathcal{K},\mathrm{cc}(E,F))$. We denote this subnet again by $(M_{\mathcal{G}''})$; let $M$ be its limit. Then

$$\begin{aligned}
&\lim_{\mathcal{G}''}\delta^*(y|M_{\mathcal{G}''}(T))\\
&\qquad= \delta^*(y|M(T)) = \sup_{K\in\mathcal{K}}\delta^*(y|M(K)) \le \sup_{K\in\mathcal{K}}\inf_{G\in\mathcal{G}',\,K\subset G}\delta^*(y|M(G))\\
&\qquad\le \sup_{K\in\mathcal{K}}\inf_{G\in\mathcal{G}',\,K\subset G}\liminf_{\mathcal{G}''}\delta^*(y|M_{\mathcal{G}''}(G))\\
&\qquad= \sup_{K\in\mathcal{K}}\inf_{K\subseteq G,\,G\in\mathcal{G}'}\liminf_{\mathcal{G}''}[\delta^*(y|M_{\mathcal{G}''}(T)) - \delta^*(y|M_{\mathcal{G}''}(T\setminus G))]\\
&\qquad= \sup_{K\in\mathcal{K}}\inf_{K\subseteq G,\,G\in\mathcal{G}'}[\lim_{\mathcal{G}''}\delta^*(y|M_{\mathcal{G}''}(T)) - \limsup_{\mathcal{G}''}\delta^*(y|M_{\mathcal{G}''}(T\setminus G))]\\
&\qquad\le \lim_{\mathcal{G}''}\delta^*(y|M_{\mathcal{G}''}(T)) - \varepsilon,
\end{aligned}$$

a contradiction.

Let us now prove the converse. Assume that (2) is satisfied. It suffices to prove that every universal net $(M_i)_{i\in I}$ on $H$ converges in $\widetilde{M}_+(T,\mathcal{K},\mathrm{cc}(E,F))$. Because of (2a), $(M_i(T))_{i\in I}$ is convergent in $\mathrm{cc}(E,F)$. Put $\lim_i M_i(T) = C$. The set $\bigcup\{M(T);\, M\in H\}$ is bounded in $E$. So is $\bigcup\{M(A);\, M\in H\}$ for every $A\in\mathcal{B}$ because $M(A)\subset M(T)$. Then for each $y\in F$ and $A\in\mathcal{B}$ the universal net $(\delta^*(y|M_i(A)))_{i\in I}$ is convergent in $\mathbb{R}$. Put $p_y(A) = \lim_i \delta^*(y|M_i(A))$. Let $G\in\mathcal{G}$. Define $S_G : F\to\mathbb{R}$ by $S_G(y) = p_y(G)$. One has $S_G(y+y')\le S_G(y)+S_G(y')$ and $S_G(\alpha y) = \alpha S_G(y)$ for all $\alpha\ge 0$ and $y,y'\in F$, and $|S_G(y)|\le\delta^*(y|\widetilde{C})$ where $\widetilde{C}$ is the absolutely convex hull of $C$. We have $\widetilde{C}\in\mathrm{cc}(E,F)$ ([7, p. 242]). This proves that $S_G$ is $\sigma(F,E)$-continuous. By the Hahn–Banach theorem ([4, p. 62]) we have $S_G(y) = \sup\{l_G(y);\, l_G : F\to\mathbb{R}$ linear and $l_G\le S_G\}$. The relation $l_G\le S_G$ shows that $l_G$ is also $\sigma(F,E)$-continuous. Hence we may put $l_G(y) = y(x_G)$ where $x_G\in E$. Denote by $\mathrm{cf}(E,F)$ the set of all convex closed non-empty subsets of $E$ and consider the map

$$M:\mathcal{G}\to\mathrm{cf}(E),\quad G\mapsto M(G) = \mathrm{cl}\{x_G;\, x_G\in E,\,\forall y\in F\ y(x_G)\le S_G(y)\}.$$

One has $S_G(y) = \delta^*(y|M(G))$. Since $S_G(y)\le\delta^*(y|\widetilde{C})$ for every $y\in F$ we have $M(G)\in\mathrm{cc}(E,F)$. Moreover, $M$ is positive, monotone and subadditive. In view of [12, Theorem 2] the map $\widetilde{M}$ from $\mathcal{B}$ to $\mathrm{cf}(E,F)$ defined by

$$\widetilde{M}(A) = \overline{\mathrm{co}}\bigcup_{K\subseteq A}\bigcap_{G\supseteq K}M(G)$$

is a positive weak set-valued measure. It is $\mathcal{K}$-inner regular and $\widetilde{M}(G)\subseteq M(G)$ for every $G\in\mathcal{G}$. Since $M(G)\subseteq\widetilde{C}$ for all $G\in\mathcal{G}$, we have $\widetilde{M}(A)\in\mathrm{cc}(E,F)$ for all $A\in\mathcal{B}$.

Let us prove that $(M_i)_{i\in I}$ converges to $\widetilde{M}$. By the definition of $M$, we have $\lim_i \delta^*(y|M_i(G)) = \delta^*(y|M(G))$. Since $\widetilde{M}(G) \subseteq M(G)$,

$$\forall y \in F, \forall G \in \mathcal{G}, \quad \lim_i \delta^*(y|M_i(G)) \geq \delta^*(y|\widetilde{M}(G)).$$

It remains to show that $\lim_i M_i(T) = \widetilde{M}(T)$. First let us prove that $\lim_i \delta^*(y|M_i(T)) = \delta^*(y|\widetilde{M}(T))$ for all $y \in F$. Note that

$$\delta^*(y|\widetilde{M}(T)) = \sup_{K\in\mathcal{K}} \inf_{G\supseteq K} \delta^*(y|M(G))$$

([12, Lemmas 1–3]). Therefore we have to prove that

$$\forall y \in F \quad \inf_{K\in\mathcal{K}} \sup_{K\subseteq G} \lim_i \delta^*(y|M_i(T \setminus G)) = 0.$$

If this were not so we would find $\varepsilon > 0$ and $y \in F$ such that for every $K \in \mathcal{K}$ there exists $G_K \in \mathcal{G}$ with $G_K \supset K$ and $\delta^*(y|M_i(T \setminus G_K)) > \varepsilon$ eventually. Put $\mathcal{G}' = \{G_K;\, K \in \mathcal{K}\}$; then $\mathcal{G}'$ dominates $\mathcal{K}$ and for every finite subfamily $\mathcal{G}''$ of $\mathcal{G}'$ we have $\inf\{\delta^*(y|M_i(T \setminus G_K));\, G_K \in \mathcal{G}''\} > \varepsilon$ eventually. This contradicts condition (b) of (2). Therefore $\lim_i \delta^*(y|M_i(T)) = \delta^*(y|\widetilde{M}(T))$ for all $y \in F$.

On the other hand, the net $(M_i(T))$ converges to $C$ in $\mathrm{cc}(E, F)$. It follows that $\delta^*(y|C) = \delta^*(y|\widetilde{M}(T))$ for all $y \in F$ and therefore $\widetilde{M}(T) = C$.

Remark. If in condition (2)(b) we only take subclasses $\mathcal{G}''$ of $\mathcal{G}'$ consisting of one set then we obtain the following condition:

(3) For all $y \in F$, all $\mathcal{G}' \subset \mathcal{G}$ with $\mathcal{G}' \succ \mathcal{K}$ and all $\varepsilon > 0$ there exists $G \in \mathcal{G}'$ such that

$$\sup\{\delta^*(y|M(T \setminus G));\, M \in H\} < \varepsilon.$$

In view of [14, Lemma 7] this condition is equivalent to the following:

(4) For all $y \in F$ and $\varepsilon > 0$ there exists $K \in \mathcal{K}$ such that

$$\sup\{\delta^*(y|M(T \setminus K));\, M \in H\} < \varepsilon.$$

Definition. A subset of $\widetilde{M}_+(T, \mathcal{K}, \mathrm{cc}(E, F))$ which satisfies condition (4) is said to be *uniformly tight.*

Corollary 2.1. *Let $T$ be an abstract set, and let $\mathcal{G}$ and $\mathcal{K}$ be families of subsets of $T$ which satisfy axioms* (i)–(v). *Let $H \subset \widetilde{M}_+(T, \mathcal{K}, \mathrm{cc}(E, F))$ be such that $\{M(T);\, M \in H\}$ is relatively compact in* $\mathrm{cc}(E, F)$. *If $H$ is uniformly tight, then every net on $H$ has a convergent subnet.*

The results of the next corollary have been proved in [6] for scalar-valued measures. For non-negative measures they have been proved separately by several authors (e.g. [3], [1], [8], [18], [9], [5]). We have generalized them to set-valued measures [13].

Corollary 2.2. *Assume that $T$ is a locally compact space or a complete metric space or else a hemicompact $k$-space. Let $H$ be a subset of $\widetilde{M}_+(T,\mathcal{K}(T),\mathrm{cc}(E,F))$ endowed with the wn-topology. Then the following conditions* (1) *and* (2) *are equivalent*:

(1) *$H$ is relatively compact.*
(2) (a) *The set $\{M(T);\ M\in H\}$ is relatively compact in* $\mathrm{cc}(E,F)$,
    (b) *$H$ is uniformly tight.*

Since $\widetilde{M}_+(T,\mathcal{K}(T),\mathrm{cc}(E,F))$ is a completely regular space, condition (1) of the corollary is equivalent to that of the theorem. The result is evident if $T$ is a locally compact space or a complete metric space. If $T$ is a hemicompact $k$-space the proof is similar to that in [9, Theorem 5.2, p. 884].

Finally, note that $\widetilde{M}_+(T,\mathcal{K},\mathrm{cc}(E,F))$ with the $wn$-topology is a Hausdorff space when axioms (i)–(v) are satisfied. Since two weak set-valued measures $M$ and $M'$ are equal if and only if $\delta^*(y|M(\cdot))=\delta^*(y|M'(\cdot))$ for all $y\in F$, the proof follows from that of Topsøe ([16, p. 204]).

**3. The space $\widetilde{M}_+(T,\mathcal{K}(T),\mathrm{ck}(E))$.** In this section we prove that the $wn$-topology, the $sn$-topology and the $s$-topology coincide in $\widetilde{M}_+(T,\mathcal{K}(T),\mathrm{ck}(E))$. Now $E$ is a Banach space and $F=E'$ is its topological dual. The norms on $E$ and $E'$ are denoted by $|\cdot|$. Let $B'(0,1)$ be the closed unit ball of $E'$, endowed with the relative topology $\sigma(B'(0,1),E)$ generated by the weak topology $\sigma(E',E)$ in $E'$. We denote by $\mathrm{ck}(E)$ the space of all convex compact non-empty subsets of $E$, and by $\widetilde{M}_+(T,\mathcal{K}(T),\mathrm{ck}(E))$ the subspace of $\widetilde{M}_+(T,\mathcal{K}(T),\mathrm{cc}(E,E'))$ consisting of all elements with values in $\mathrm{ck}(E)$. Note that a weak set-valued $M$ with values in $\mathrm{cc}(E,E')$ is a set-valued measure, that is, for any sequence $(A_n)$ of pairwise disjoint sets in $\mathcal{B}(T)$ with union $A$, we have $M(A)=\lim_{n\to\infty}\sum_{k=0}^{n}M(A_k)$ where the limit is taken with respect to the Hausdorff topology [15]. The Hausdorff topology derives from the distance $\delta$ defined by $\delta(C,C')=\sup\{|\delta^*(y|C)-\delta^*(y|C')|;\ y\in E',\ |y|\le 1\}$ for all $C$ and $C'$ in $\mathrm{cc}(E,E')$. The space $(\mathrm{ck}(E),\delta)$ is a complete metric space [2]. We start with the following

Lemma 3.1. *Let $(C_i)_{i\in I}$ be a net on* $\mathrm{ck}(E)$, *and let $(z_i)_{i\in I}$ be a net on $B'(0,1)$. If $(C_i)$ converges to $C_0$ in* $\mathrm{ck}(E)$, *and $(z_i)$ converges to $z_0$ in $B'(0,1)$, then $(\delta^*(z_i|C_i))$ converges to $\delta^*(z_0|C_0)$.*

*Proof.* We have

$$\begin{aligned}|\delta^*(z_i|C_i)-\delta^*(z_0|C_0)| &\le |\delta^*(z_i|C_i)-\delta^*(z_i|C_0)|+|\delta^*(z_i|C_0)-\delta^*(z_0|C_0)|\\ &\le \sup_{|y|\le 1}|\delta^*(y|C_i)-\delta^*(y|C_0)|+|\delta^*(z_i|C_0)-\delta^*(z_0|C_0)|.\end{aligned}$$

Since $C_0\in\mathrm{ck}(E)$ and the map $\delta^*(\cdot|C_0):B'(0,1)\to\mathbb{R}$, $y\mapsto\delta^*(y|C_0)$, is continuous, the net $(\delta^*(z_i|C_0))_i$ converges to $\delta^*(z_0|C_0)$. Moreover

$$\lim_i(\sup_{|y|\leq 1}|\delta^*(y|C_i)-\delta^*(y|C_0)|)=0$$

because $(C_i)$ converges to $C_0$. The lemma is therefore proved.

THEOREM 3.2. *Let $T$ be a completely regular Hausdorff space, and $E$ be a Banach space. Let $(M_i)_{i\in I}$ be a net on $\widetilde{M}_+(T,\mathcal{K}(T),\mathrm{ck}(E))$ and $M_0\in\widetilde{M}_+(T,\mathcal{K}(T),\mathrm{ck}(E))$. Then $(M_i)$ converges to $M_0$ in the wn-topology if and only if $(M_i)$ converges to $M_0$ in the s-topology.*

*Proof.* By Section 1.6 it is evident that the $s$-topology is finer than the $wn$-topology, so we need only prove that convergence in the $wn$-topology implies convergence in the $s$-topology. Assume that $(M_i)$ converges to $M_0$ in the $wn$-topology. To show that $(M_i)$ converges to $M_0$ in the $s$-topology it suffices to prove that for every $f\in C_+(T),(\int fM_i)$ is a Cauchy net. If this were not so, there would exist $g\in C_+(T)$ and $\varepsilon>0$ such that for every $i\in I$ we would find $k_i,j_i\in I$ with $k_i,j_i\geq i$ and $y_i\in B'(0,1)$ such that $|\int g\delta^*(y_i|M_{j_i}(\cdot))-\int g\delta^*(y_i|M_{k_i}(\cdot))|\geq\varepsilon$. We may assume without loss of generality that $g\leq 1$. Since $B'(0,1)$ is a compact space for the topology $\sigma(B'(0,1),E)$, the net $(y_i)_{i\in I}$ has a convergent subnet. Assume for simplicity that $(y_i)$ itself converges to $z\in B'(0,1)$. Consider the net $(\int g\delta^*(y_i|M_{k_i}(\cdot)))_{i\in I}$. We have

$$\int g\delta^*(y_i|M_{k_i}(\cdot))-\int g\delta^*(z|M_{k_i}(\cdot))\leq\int g\delta^*(y_i-z|M_{k_i}(\cdot))$$

because for every $y$ and $y'$ in $E'$ one has $\int g\delta^*(y+y'|M_{k_i}(\cdot))\leq\int g\delta^*(y|M_{k_i}(\cdot))+\int g\delta^*(y'|M_{k_i}(\cdot))$. Since $g\leq 1$ and $\delta^*(y|M_{k_i}(\cdot))$ is a non-negative measure one has $\int g\delta^*(y|M_{k_i}(\cdot))\leq\delta^*(y|M_{k_i}(T))$ for every $y\in E'$. We then have

$$\int g\delta^*(y_i|M_{k_i}(\cdot))-\int g\delta^*(z|M_{k_i}(\cdot))\leq 2\delta^*\left(\frac{1}{2}(y_i-z)\middle|M_{k_i}(T)\right).$$

It follows that

$$\left|\int g\delta^*(y_i|M_{k_i}(\cdot))-\int g\delta^*(z|M_{k_i}(\cdot))\right|\\ \leq 2\sup\left(\delta^*\left(\frac{1}{2}(y_i-z)\middle|M_{k_i}(T)\right),\delta^*\left(\frac{1}{2}(z-y_i)\middle|M_{k_i}(T)\right)\right).$$

By Lemma 3.1 the nets $(\delta^*(\frac{1}{2}(y_i-z)|M_{k_i}(T)))_i$ and $(\delta^*(\frac{1}{2}(z-y_i)|M_{k_i}(T)))_i$ converge to 0. Hence $\lim_i|\int g\delta^*(y_i|M_{k_i}(\cdot))-\int g\delta^*(z|M_{k_i}(\cdot))|=0$. Taking account of the hypothesis one has $\lim_i\int g\delta^*(z|M_{k_i}(\cdot))=\int g\delta^*(z|M_0(\cdot))$. Then we may conclude that $\lim_i\int g\delta^*(y_i|M_{k_i}(\cdot))=\int g\delta^*(z|M_0(\cdot))$. Analogously, $\lim_i\int g\delta^*(y_i|M_{j_i}(\cdot))=\int g\delta^*(z|M_0(\cdot))$. It follows from the equality of those limits that $\lim_i|\int g\delta^*(y_i|M_{k_i}(\cdot))-\int g\delta^*(y_i|M_{j_i}(\cdot))|=0$. That is a contradiction.

We denote by $t_{sn}$, $t_{wn}$ and $t_s$ the strong-narrow, weak-narrow and simple topology, respectively. If $t$ and $t'$ are two topologies on the same set, we write $t \preceq t'$ if $t$ is coarser than $t'$.

Let $G \in \mathcal{G}(T)$ and $K \in \mathcal{K}(T)$ and assume that $K \subset G$ and $T$ is a completely regular Hausdorff space. Put $\mathcal{F} = \{f \in C_+(T);\, f < 1_G\}$ where $1_G$ is the indicator function of $G$. Since $T$ is a completely regular Hausdorff space, there exists $f \in \mathcal{F}$ such that $f(x) = 1$ for all $x \in K$. The family $\mathcal{F}$ is filtering to the right and $1_G = \sup\{f;\, f \in \mathcal{F}\}$. Now let $M \in \widetilde{M}_+(T, \mathcal{K}(T), \mathrm{ck}(E))$. Since $M$ is $\mathcal{K}(T)$-inner regular and positive, we have $M(G) = \overline{\mathrm{co}} \bigcup\{\int fM;\, f \in \mathcal{F}\} = \mathrm{cl} \bigcup\{\int fM;\, f \in \mathcal{F}\}$. The second equality follows from the fact that $\bigcup\{\int fM;\, f \in \mathcal{F}\}$ is a convex set in $E$. Indeed, if $x \in \int fM$ and $y \in \int gM$ with $f \in \mathcal{F}$ and $g \in \mathcal{F}$, then $h = \sup(f, g) \in \mathcal{F}$ and $M(h) \supseteq M(f) \cup M(g)$ because $M$ is positive. Therefore $rx+(1-r)y \in M(h)$ where $0 \le r \le 1$.

Theorem 3.3. *Let $T$ be a completely regular Hausdorff space and let $E$ be a Banach space. Then in $\widetilde{M}_+(T, \mathcal{K}(T), \mathrm{ck}(E))$ the wn-topology, sn-topology and s-topology are identical.*

*Proof.* Let $f \in C_+(T)$ and let

$$p_f : (\widetilde{M}_+(T; \mathcal{K}(T), \mathrm{ck}(E)), t_s) \to (\mathrm{ck}(E), \delta), \qquad M \mapsto p_f(M) = \int fM.$$

Then $p_f$ is continuous, and therefore lower semicontinuous. It follows that for every $G \in \mathcal{G}(T)$ the map

$$p_G : (\widetilde{M}_+(T, \mathcal{K}(T), \mathrm{ck}(E)), t_s) \to (\mathrm{ck}(E), \delta)$$
$$M \mapsto p_G(M) = M(G) = \mathrm{cl} \bigcup \Big\{ \int fM;\, f \in \mathcal{F} \Big\}$$

is lower semicontinuous. We deduce that $t_{sn} \preceq t_s$ because $t_{sn}$ is the weakest topology for which all maps $M \mapsto M(G)$ defined on $\widetilde{M}_+(T, \mathcal{K}(T), \mathrm{ck}(E))$ are lower semicontinuous. It follows from the proof of Theorem II.21 in [2, p. 52] that $t_{wn} \preceq t_{sn}$. The relations $t_{wn} \preceq t_{sn} \preceq t_s$ and Theorem 3.2 show that these three topologies are identical.

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