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CONVEX AND DISCRETE GEOMETRY

Measure and Helly's Intersection Theorem for Convex Sets

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Summary. Let $\mathcal{F} = \{F_\alpha\}$ be a uniformly bounded collection of compact convex sets in $\mathbb{R}^n$. Katchalski extended Helly's theorem by proving for finite $\mathcal{F}$ that $\dim(\bigcap \mathcal{F}) \geq d$, $0 \leq d \leq n$, if and only if the intersection of any $f(n,d)$ elements has dimension at least d where $f(n,0) = n+1 = f(n,n)$ and $f(n,d) = \max\{n+1, 2n-2d+2\}$ for $1 \leq d \leq n-1$. An equivalent statement of Katchalski's result for finite $\mathcal{F}$ is that there exists $\delta > 0$ such that the intersection of any $f(n,d)$ elements of $\mathcal{F}$ contains a d-dimensional ball of measure δ where $f(n,0) = n+1 = f(n,n)$ and $f(n,d) = \max\{n+1, 2n-2d+2\}$ for $1 \leq d \leq n-1$. It is proven that this result holds if the word finite is omitted and extends a result of Breen in which $f(n,0) = n+1 = f(n,n)$ and $f(n,d) = 2n$ for $1 \leq d \leq n-1$. This is applied to give necessary and sufficient conditions for the concepts of "visibility" and "clear visibility" to coincide for continua in $\mathbb{R}^n$ without any local connectivity conditions.

1. Introduction. Katchalski [6] significantly generalized Helly's intersection theorem on convex sets by proving the theorem stated in the abstract. Let $\mathcal{F} = \{F_\alpha\}$ be a uniformly bounded collection of compact convex sets in $\mathbb{R}^n$. Suppose $0 \leq d \leq n$, j is a positive integer and $\delta > 0$. The collection $\mathcal{F}$ is said to have *property* (j,d,δ) if any j elements of $\mathcal{F}$ contain a common closed d-dimensional ball of radius δ. Breen [2] proved that if $\mathcal{F} \subseteq \mathbb{R}^n$ is a uniformly bounded collection of compact convex sets then $\dim(\bigcap \mathcal{F}) \geq d$ if and only if for some $\delta > 0$, $\mathcal{F}$ has property $(i(n,d), d, \delta)$ where $i(n,d) = 2n$, $1 \leq d \leq n-1$, and $i(n,0) = i(n,n) = n+1$. Two of the main tools she employed were Katchalski's theorem [6] and an intersection result of Falconer [5].

Our proof was in part motivated by an alternative proof for finite $\mathcal{F}$ outlined by Katchalski in [6] using the Bonnice–Klee theorem [1]. If $\mathcal{F} =$

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$\{F_\alpha\}$ is a uniformly bounded collection of compact convex sets in $\mathbb{R}^n$, then $\mathcal{F}$ is said to be *H-closed* provided $\mathcal{F}$ is closed in the sense of the Hausdorff metric. If $F \in \mathcal{F}$ the cone generated by F is defined as the set $\{\lambda x \mid \lambda \geq 0$ and $x \in F\}$ and will be denoted by $\operatorname{cone} F$; note that $\operatorname{cone} F$ is not necessarily closed. The symbol $C(\mathcal{F})$ denotes $\{\operatorname{cone} F \mid F \in \mathcal{F}\}$. If $r > 0$ the symbol $B(x,r)$ denotes the closed ball of radius r with center x. Let $\mathcal{C} = \{C_\alpha\}$ be a collection of closed convex cones with apex 0_v (the origin) in $\mathbb{R}^n$. Suppose $1 \leq d \leq n$, j is a positive integer, $\delta > 0$, and $r > 0$. The symbol $\mathcal{B}_r(\mathcal{C})$ denotes $\{B_\alpha \mid B_\alpha = C_\alpha \cap B(0_v, r),\ C_\alpha \in \mathcal{C}\}$. Also, $\mathcal{C}$ is said to have *property* (j,d,δ,r) if $\mathcal{B}_r(\mathcal{C})$ has property (j,d,δ). If j is a positive integer and $\mathcal{F}$ is a family of sets then $\mathcal{F}^j$ is defined as $\{\bigcap \mathcal{A} \mid \mathcal{A} \subset \mathcal{F},\ |\mathcal{A}| = j\}$. Also, if F is a compact convex set, $\operatorname{rad}_j(F)$ denotes the nonnegative number with the property that F contains a closed j-dimensional convex ball of radius $\operatorname{rad}_j(F)$ and for any $\kappa > 0$ the set F does not contain a j-dimensional ball of radius $\operatorname{rad}_j(F) + \kappa$.

We shall make explicit use of the following result of Falconer [5].

Proposition 1. *Let $\mathcal{F} = \{F_\alpha\}$ be a uniformly bounded H-closed collection of compact convex sets in $\mathbb{R}^n$. If $\dim(\bigcap \mathcal{F}) < n$ then there exist $F_{\alpha_1}, \ldots, F_{\alpha_k}$ such that $\dim(\bigcap_{j=1}^k F_{\alpha_j}) = q < n$ where $k \leq 2(n-q)$.*

Two linear flats I and J each of dimension 1 or more will be called *skew* if $I \cap J = \emptyset$ and whenever $I_1 \subset I$ and $J_1 \subset J$ are flats of dimension 1 or more then no translate I_1 is contained in I_2 and vice versa. Two convex sets S and L each of dimension 1 or more will be called *skew* if there exist two skew linear flats I and J with $S \subset I$ and $L \subset J$. We shall need the following proposition.

Proposition 2. *Let $L \subset \mathbb{R}^n$, $n \geq 4$, be an $n-3$-dimensional subspace and let $F \subset L$ be a convex set with $0_v \in F$ and $1 \leq \dim F \leq n-3$. Let S be a convex set S of dimension 2 which is skew to L. Let $G = \operatorname{conv}(S \cup F)$ and suppose that $1 \leq m \leq n-3$. If $\dim F = m$ then $\dim G = m+3$.*

Proof. Suppose $n = 4$. Then $m = 1$. Since S is skew to L, S does not intersect L nor is S parallel to any nontrivial flat of L, and since $\dim S = 2$ we have $\dim G \geq 3$. We claim $\dim G \geq 4$. Suppose not. Then $\dim G = 3$ and if I is the linear flat generated by S then $\dim I = 2$ and so either I must intersect L or be parallel to a flat in L, each of which is a contradiction. Thus $\dim G = 4$. Thus the assertion is true for $n = 4$. We now suppose that it is true for n and prove it for $n+1$. If $m = 1$ then G is contained in a copy of $\mathbb{R}^4$ and the same argument as the one just given yields the assertion. Without loss of generality we may suppose that $2 \leq m \leq n-2$. Thus if $2 \leq m \leq n-3$ then G is contained in a copy of $\mathbb{R}^n$ and the induction hypothesis gives the assertion. Thus we may suppose $m = n-2$. The same argument as given in the first sentence gives $\dim G \geq n$. If $\dim G = n$ then if J is the $n-2$-dimensional subspace space generated by F and if I is the 2-dimensional linear flat generated by S then in $\mathbb{R}^n$ either I must intersect J or be parallel to a flat in J, each of which is a contradiction. Thus $\dim G = n+1 = (n-2)+3 = m+3$ and the assertion follows. ∎

2. The intersection of cones and convex sets

THEOREM 3. *Let $\mathcal{F} = \{F_\alpha\}$ be a uniformly bounded collection of H-closed compact convex sets in $\mathbb{R}^n, n \geq 2$, with $0_v \in \bigcap \mathcal{F}$. Then* $\dim(\bigcap \mathcal{F}) \geq d \geq 2$ *if and only if* $\dim(\bigcap C(\mathcal{F})) \geq d \geq 2$.

Proof. The necessity is immediate; we consider the sufficiency. Let $k = \dim(\bigcap C(\mathcal{F})) \geq 2$ and define $j = n - \dim(\bigcap C(\mathcal{F})) = n-k$. We first establish the assertion in the case of $j = 0$, i.e. $\dim(\bigcap C(\mathcal{F})) = k = n$. Suppose that $\dim(\bigcap \mathcal{F}) < n$. Then by Proposition 1 of Falconer there exist $F_{\alpha_1}, \dots, F_{\alpha_k}$ such that $\dim(\bigcap_{j=1}^k F_{\alpha_j}) = q < n$. Note that $\bigcap C(\mathcal{F}) \subseteq \bigcap_{j=1}^k C(F_{\alpha_j}) = C(\bigcap_{j=1}^k F_{\alpha_j})$, which implies that $\dim(\bigcap C(\mathcal{F})) < \dim(\bigcap_{j=1}^k C(F_{\alpha_j})) = \dim(\bigcap_{j=1}^k F_{\alpha_j}) < n$, a contradiction. Thus the assertion is true for $d = n$, $n = 2$, and we may suppose that $k < n$ and $n \geq 3$.

Let $P(j)$ be the conclusion of the theorem for j; the last paragraph shows that $P(0)$ is true. We now suppose that $P(j)$ is true and prove that $P(j+1)$ is true. Since $\dim(\bigcap C(\mathcal{F})) \geq 2$ we may choose a hyperplane L with $0_v \in L$ such that L does not support $\bigcap C(\mathcal{F})$. In particular, L cannot support any element of $\mathcal{F}$, which implies by a routine argument that $L \cap \mathcal{F}$ is H-closed and that if $\mathcal{F}^L = L \cap \mathcal{F}$ then $\dim(\bigcap C(\mathcal{F}^L)) = k-1 \geq 1$. Let V denote the subspace generated by $\bigcap C(\mathcal{F}^L)$. Regarding $\mathbb{R}^n$ as a subset of $\mathbb{R}^{n+1}$, since $1 \leq k-1 \leq (n+1)-3$ we may choose a compact convex set $S \subset \mathbb{R}^{n+1}$ of dimension 2 which is skew to V. Define $\mathcal{G} = \{\operatorname{conv}(\{S\} \cup F_\alpha^L) \mid F_\alpha^L \in \mathcal{F}^L\}$. Note that if $M = \operatorname{conv}(S \cup \bigcap C(\mathcal{F}^L))$ then $M \subset \bigcap C(\mathcal{G})$. Since S is skew to $V \subset L$ and $1 \leq k-1 \leq (n+1)-3$ we see by applying Proposition 2 in $\mathbb{R}^{n+1}$ that $\dim M = ((k-1)+3) = k+2$ and so $\dim(\bigcap C(\mathcal{G})) \geq \dim M \geq k+2$. Since

$$(n+1) - \dim\left(\bigcap C(\mathcal{G})\right) = (n+1)-(k+2) = n-k-1 \leq n-k = j,$$

the induction hypothesis on j applied in $\mathbb{R}^{n+1}$ gives $\dim(\bigcap \mathcal{G}) \geq k+2$.

We next assert that $\bigcap \mathcal{G} = \operatorname{conv}(S \cup \bigcap \mathcal{F}^L)$. This follows if we show that $\bigcap \mathcal{G} \subset \operatorname{conv}(S \cup \bigcap \mathcal{F}^L)$. Let $x \in \bigcap \mathcal{G}$. If $x \in S \cup \bigcap \mathcal{F}^L$ we are done; if not then for each F_α^L, F_β^L there exist positive scalars $\lambda_\alpha, \lambda_\beta$ less than 1 and points s_α, s_β in S, $f_\alpha \in F_\alpha^L$ and $f_\beta \in F_\beta^L$ with

$$x = \lambda_\alpha s_\alpha + (1-\lambda_\alpha) f_\alpha = \lambda_\beta s_\beta + (1-\lambda_\beta) f_\beta.$$

Thus $\lambda_\alpha s_\alpha - \lambda_\beta s_\beta = (1-\lambda_\beta) f_\beta - (1-\lambda_\alpha) f_\alpha$. Since S is skew to V, both $\lambda_\alpha s_\alpha - \lambda_\beta s_\beta$ and $(1-\lambda_\beta) f_\beta - (1-\lambda_\alpha) f_\alpha$ equal 0_v. Note that $f_\alpha \neq 0_v$ for all α: if one $f_\beta = 0_v$ then $f_\alpha = 0_v$ for all α and then $x = 0_v$ and so $x \in \bigcap \mathcal{F}^L$, a contradiction as $x \notin S \cup \bigcap \mathcal{F}^L$. Thus f_α and f_β are positive scalar multiples of each other for any α and β, and as $\mathcal{F}^L$ is H-closed we may produce a set $F_\theta^L \in \mathcal{F}^L$ where $\|f_\theta\| = \inf \|f_\alpha\| > 0$ over all α and $x = \lambda_\theta s_\theta + (1-\lambda_\theta) f_\theta$ with $f_\theta \in (\bigcap \mathcal{F}^L)$ and the assertion follows.

Let $s = \dim(\bigcap \mathcal{F}^L)$. Note that $s \geq 1$, for if $s = 0$ then since $\bigcap \mathcal{G} = \operatorname{conv}(S \cup \bigcap \mathcal{F}^L)$ and $\dim S = 2$ we see that $\dim(\bigcap \mathcal{G}) = 3$, which is a contradiction as $\dim(\bigcap \mathcal{G}) \geq k+2 \geq 4$. Further, as $1 \leq s = \dim(\bigcap \mathcal{F}^L) \leq \dim(\bigcap C(\mathcal{F}^L)) = k-1$ and $1 \leq k-1 \leq (n+1)-3$, we see by applying Proposition 2 in $\mathbb{R}^{n+1}$ that $k+2 \leq \dim(\bigcap \mathcal{G}) = \dim(\bigcap \mathcal{F}^L) + 3 = s+3$ and so $s \geq d-1$ and $s = \dim(\bigcap \mathcal{F}^L) \geq k-1$. Thus we may choose a nontrivial closed line segment $h = [0_v, x] \subset \bigcap \mathcal{F}^L \subset \bigcap C(\mathcal{F})$, and as $\dim(\bigcap C(\mathcal{F})) \geq 2$ we may choose a hyperplane $L_1 \neq L$ with $0_v \in L_1, h \cap L_1 = 0_v$ and such that L_1 does not support $\bigcap C(\mathcal{F})$. Repeating for L_1 the same construction done for L gives $\dim(\bigcap \mathcal{F}^{L_1}) \geq k-1$. The latter together with the facts that $h \subset \bigcap C(\mathcal{F})$ and $h \cap L_1 = 0_v$ implies that $\dim(\bigcap \mathcal{F}) \geq k$, which establishes the theorem. ■

3. The intersection of convex sets

THEOREM 4. *Let $\mathcal{F} = \{F_\alpha\}$ be a uniformly bounded collection of compact convex sets in $\mathbb{R}^n$. Then $\dim(\bigcap \mathcal{F}) \geq d$, $0 \leq d \leq n$, if and only if for some $\delta > 0$, $\mathcal{F}$ has property $(f(n,d), d, \delta)$ where $f(n,0) = n+1$ and $f(n,d) = \max\{n+2, 2n-2d+2\}$ for $1 \leq d \leq n$.*

Proof. The necessity is immediate; we consider the sufficiency. We proceed by induction on n. If $n \leq 2$, $d = 1$, or $d = n$ the conclusion follows from the results of Breen [2] and Falconer [5] respectively. Thus we may suppose that $n \geq 3$ and $d \geq 2$. Without loss of generality by Helly's theorem [11] we may assume that $0_v \in \bigcap \mathcal{F}$. For each $F_\alpha \in \mathcal{F}$ let $\mathcal{H}_\alpha$ be the set of all closed half-spaces H^+ containing F_α. It is well known that $F_\alpha = \bigcap \mathcal{H}_\alpha$ [8] and therefore if $\mathcal{H} = \{H^+ \mid H^+ \in \mathcal{H}_\alpha,\ F_\alpha \in \mathcal{F}\}$ then $\bigcap \mathcal{F} = \bigcap \mathcal{H}$. As $\mathcal{F}$ has property $(f(n,d), d, \delta)$ so does $\mathcal{H}$. As $\mathcal{F}$ is uniformly bounded we may enclose the closure of $\bigcup \mathcal{F}$ in the interior of a cube I. Then the family $\mathcal{P}$ of polytopes which is the closure of the family $\{H^+ \cap I \mid H^+ \in \mathcal{H}\}$ in the Hausdorff metric, has property $(f(n,d), d, \delta), \bigcap \mathcal{F} = \bigcap \mathcal{H}$, $\mathcal{P}$ is H-closed, and each element of $C(\mathcal{P})$ is closed since it is a polytope [7]. Since $\dim(\bigcap \mathcal{F}) \geq \dim(\bigcap \mathcal{P})$ and $\bigcap \mathcal{P} \subset \bigcap \mathcal{F}$, to prove the theorem it suffices to prove $\dim(\bigcap \mathcal{P}) \geq d$. Therefore, without loss of generality, we suppose that $\mathcal{F}$ is an H-closed family of polytopes, and each element of $\mathcal{C} = C(\mathcal{F})$ is closed. By a corollary of the Bonnice–Klee theorem [1, p. 11], $\dim(\bigcap \mathcal{C}) \geq 1$.

Recall that $\mathcal{B} = \mathcal{B}_r(\mathcal{C}) = \{B_\alpha \mid B_\alpha = C_\alpha \cap B(0_v, r),\ C_\alpha \in \mathcal{C}\}$ and that if $r = 2 \cdot \operatorname{diam}(\bigcup \mathcal{F})$ then $\mathcal{B}$ has property $(f(n,d), d, \delta)$ as does its closure $\mathcal{K}$. Since $\dim(\bigcap \mathcal{C}) = \dim(\bigcap \mathcal{K})$ we see that $\dim(\bigcap \mathcal{K}) \geq 1$. Thus we may choose a point $u \in \operatorname{relint}(\bigcap \mathcal{K})$. We may then choose a hyperplane H with $u \in H$, $\bigcap \mathcal{K} \nsubseteq H$, with $\bigcap \mathcal{K}$ intersecting both open half-spaces of H and H does not support $\bigcap \mathcal{K}$. Note that $\bigcap \mathcal{K} \subset E_\beta$ for any $E_\beta \in \mathcal{K}^{f(n,d)}$. Since $\bigcap \mathcal{K} \not\subset H$, each $E_\beta \in \mathcal{K}^{f(n,d)}$ must intersect at least one of the open half-spaces of H since if not then $\bigcap \mathcal{K} \subset E_\beta \subset H$, a contradiction. But then E_β must intersect both the open half-spaces of H since if not then H supports $\bigcap \mathcal{K}$, a contradiction. This together with the hypothesis that $\mathcal{K}$ has property $(f(n,d), d, \delta)$ implies that $\dim(H \cap E_\beta) \geq d - 1 \geq 1$.

Suppose that $\theta_\beta = \operatorname{rad}_{d-1}(H \cap E_\beta)$ and let $\theta = \inf\{\theta_\beta \mid E_\beta \in \mathcal{K}^{f(n,d)}\}$. We next assert that $\theta > 0$. Suppose that $\theta = 0$. Then there exists a sequence $\{E_{\beta_i}\}$ in $\mathcal{K}^{f(n,d)}$ such that $\theta_{\beta_i} \to 0$ as $i \to \infty$ and for each i, $E_{\beta_i} = K_{\alpha(1,\beta_i)} \cap K_{\alpha(2,\beta_i)} \cap \cdots \cap K_{\alpha(f(n,d),\beta_i)}$. Without loss of generality (avoiding subsequences) we may suppose that $E_{\beta_i} \to Q$ for some compact convex set Q. Since for each i, $\bigcap \mathcal{K} \subset E_{\beta_i}$, we have $\bigcap \mathcal{K} \subset Q$ and so Q must intersect both open half-spaces of H since $\bigcap \mathcal{K}$ does. Further, since $\mathcal{K}$ has property $(f(n,d), d, \delta)$, each E_{β_i} contains some closed d-dimensional ball of radius δ; a standard argument in the Hausdorff metric then shows that Q must contain a closed d-dimensional ball of radius δ. Thus $\dim Q \geq d$. Since Q must intersect both open half-spaces of H we have $\dim(Q \cap H) \geq d - 1 \geq 1$ and by a routine argument $E_{\beta_i} \cap H \to Q \cap H$. Since $\theta = 0$ and $\theta_{\beta_i} = \operatorname{rad}_{d-1}(H \cap E_{\beta_i})$, and $E_{\beta_i} \cap H \to Q$, we must have $\dim(Q \cap H) \leq d - 2$, a contradiction. Thus $\theta = \inf_i \theta_i > 0$.

Now $\theta > 0$ implies that if $\mathcal{K}_1 = \{H \cap K_\alpha \mid K_\alpha \in \mathcal{K}\}$ then $\mathcal{K}_1$ has property $(f(n,d), d-1, \theta)$. Since $f(n-1, d-1) \leq f(n,d)$, $\mathcal{K}_1$ has property $(f(n-1, d-1), d-1, \theta)$. The induction hypothesis applied in the hyperplane H yields $\dim(\bigcap \mathcal{K}_1) \geq d - 1 \geq 1$. Since $\bigcap \mathcal{K}_1 = H \cap \bigcap \mathcal{K}$ and $\bigcap \mathcal{K}$ intersects both open half-spaces of H, this implies that $\dim(\bigcap \mathcal{K}) \geq d \geq 2$. Then since $\dim(\bigcap \mathcal{C}) = \dim(\bigcap \mathcal{B}) \geq \dim(\bigcap \mathcal{K})$ we see that $\dim(\bigcap \mathcal{C}) \geq d \geq 2$ and an application of Theorem 3 establishes the theorem.

4. The equivalence of visibility and clear visibility. If $S \subset \mathbb{R}^n$ is a nonempty set, the symbols $S(x)$, $\operatorname{conv} S$, and $\operatorname{Ker} S$ denote, respectively, $\{y \mid [x,y] \subset S\}$, the convex hull of S, and $\{x \mid [x,y] \subset S \ \ \forall y \in S\}$. If $A \subset S$ and $\varepsilon > 0$ let A_ε denote all points in S whose distance from A is less than ε. If K is a nonempty subset of S and $x \in S$, and $\operatorname{conv}(\{x\} \cup K) \subset S$, we say that x is *visible* via S from K. Suppose $0 \leq d \leq n$, j is a positive integer and $\delta > 0$. A is said to be (j, d, δ) *visible* if given a set K of j elements of A, each point of K is visible via S from a common d-dimensional ball B_K of radius δ

contained in S. Moreover, A is said to be (j,d) *clearly visible* if given a set K of j elements of A there exists a relatively open subset O_K of S containing K such that each point of O_K is visible via S from a common d-dimensional ball B_K contained in S. Finally, A is said to be (j,d,δ) *clearly visible* if given a set K of j elements of A there exists a relatively open subset O_K of S containing K such that each point of O_K is visible via S from a common d-dimensional ball B_K of radius δ contained in S.

Theorem 5. *Let $S \subset \mathbb{R}^n$ be a nonconvex continuum. Then S being (j,d) clearly visible is equivalent to S being (j,d,δ) visible for some $\delta > 0$ if and only if $j \geq f(n,d)$ where $f(n,0) = n+1$ and $f(n,d) = \max\{n+2, 2n-2d+2\}$ for $1 \leq d \leq n$.*

Proof. We first demonstrate the sufficiency. To prove the equivalence of the two visibility conditions it suffices to show that S being (j,d,δ) visible for some $\delta > 0$ implies S being (j,d) clearly visible since (j,d) clear visibility of S always implies (j,d,δ) visibility of S for some $\delta > 0$ (Breen [3] or Stavrakas [9, Theorem 3]). Since $j \geq f(n,d)$, S is $(f(n,d),d,\delta)$ visible. This coupled with Theorem 4 shows that $\dim(\bigcap\{\operatorname{conv}(S(x)) \mid x \in S\} \geq d$ and Krasnosel'skiĭ's lemma [11] yields $\bigcap\{\operatorname{conv}(S(x)) \mid x \in S\} \subset \operatorname{Ker} S$, which implies that $\dim(\operatorname{Ker} S) = d$. This immediately implies the (j,d) clear visibility of S for any $j \geq 1$.

To prove the necessity, it suffices to construct a continuum S for which $j < f(n,d)$ such that S is (j,d,δ) visible for some $\delta > 0$ and S is not (j,d) clearly visible. To do this in $\mathbb{R}^2$ let $S = \bigcup_{n=1}^{\infty}\{[0_v, x_n] \mid x_1 = (1,0),$ $x_n = (1,1/n),\ n = 2,3,\ldots\}$. Note that $1 < f(2,1) = 4$, S is $(1,1,1/2)$ visible, but 0_v is not clearly visible from any one-dimensional subset of S and so in particular S is not $(1,1)$ clearly visible. ∎

Theorem 6. *Let $S \subset \mathbb{R}^n$ be a nonconvex continuum with points of local nonconvexity Q. Then the following are equivalent*:

(A) *S is $(f(n,d),d,\delta)$ visible for some $\delta > 0$.*
(B) $\dim(\operatorname{Ker} S) = d$.
(C) *S is $(f(n,d),d,\delta)$ clearly visible for some $\delta > 0$.*
(D) *Q_ε is $(f(n,d),d)$ clearly visible for some $\varepsilon > 0$.*
(E) *Q is $(f(n,d),d,\delta)$ clearly visible for some $\delta > 0$.*
(F) *Q_ε is $(f(n,d),d,\delta)$ visible for some $\varepsilon > 0$ and $\delta > 0$.*

Proof. The implications (B)⇒(C), (C)⇒(D) and (D)⇒(E) are immediate. The implication (E)⇒(F) is established in Stavrakas [9, Theorem 3]. The implication (A)⇒(B) was established in the first paragraph of the last proof. To prove (F)⇒(A) we note that in Stavrakas [10] it is proven that $\bigcap\{\operatorname{conv}(S(x)) \mid x \in Q_\varepsilon\} = \bigcap\{\operatorname{conv}(S(x) \mid x \in S\}$ and so the hypothesis (F) coupled with the same argument in the first paragraph of the proof of Theorem 5 yields the conclusion (B), which immediately implies (A). ■

We remark that examples given in Breen [3] illustrate for each $d \geq 1$ that the number $f(n,d)$ is best possible in the sense that the above theorem fails if $f(n,d)$ is replaced by a smaller integer. We remark that J. Cel [4] has used a generalized form of Krasnosel'skiĭ's lemma to represent starshaped sets in normed linear spaces.

In conclusion we acknowledge the many and profound accomplishments, in convexity and infinite-dimensional topology, of Victor Klee who passed away this year.

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