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A Polish AR-Space with no Nontrivial Isotopy

by

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Summary. The Polish space Y constructed in [vM1] admits no nontrivial isotopy. Yet, there exists a Polish group that acts transitively on Y.

1. Introduction. We consider separable metric spaces only.

THEOREM 1.1. *The countable dense homogeneous Polish AR-space Y constructed in* [vM1] *has the following properties*:

(1) *Y admits no nontrivial isotopy with a continuum as the parameter set*;
(2) *Y admits a transitive action of a Polish group and*, *hence*, *Y is a coset space*;
(3) *Y has the homeomorphism extension property for compacta* (*that is*, *Y is compactly homogeneous*);
(4) *for any bijection Φ of Y with* $\mathrm{int}(\mathrm{Fix}(\Phi)) = \emptyset$ (*in particular*, *by a result of van Mill* [vM1], *for any nonidentity homeomorphism of Y*), *Y is countable dense homogeneous with respect to conjugates of Φ.*

Recall that a space X is *countable dense homogeneous* (abbreviated CDH) if for any countable dense subsets A and B of X there exists a homeomorphism h of X such that $h(A) = B$; by a result of Bennett [B], such a connected X is necessarily homogeneous. In (4) of Theorem 1.1, we have in mind the following "conjugated" variant of the countable dense homogeneity: Let Φ be a bijection of a space X such that $\mathrm{int}(\mathrm{Fix}(\Phi)) = \emptyset$. We say that X is *countable dense homogeneous with respect to conjugates of Φ*

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(abbreviated Φ^*-CDH) if for any two countable dense sets A and B of X, there exists a homeomorphism h of X such that $h^{-1}(\Phi(h(A))) = B$.

As shown in [vM1], every homeomorphism of Y which is the identity on a nonempty open subset (more generally, on a non-Z-set in Y) must be the identity. It follows that Y is not strongly locally homogeneous. (Recall that X is *strongly locally homogeneous* if every $x \in X$ has a neighborhood U so that for any $x, x' \in U$ there exists a homeomorphism h that moves x to x' and is the identity outside U.) By a result of van Mill [vM2], every strongly locally homogeneous Polish space X admits a transitive action of a Polish group (and, hence, X is a coset space). So, Theorem 1.1 shows that, beyond the class of strongly locally homogeneous spaces, there are homogeneous coset spaces with a nice local structure. On the other hand, in [vM2], van Mill has constructed a homogeneous Polish space Z which is not a coset space. The space Z, however, has a very bad local structure and, in particular, is far from being an AR. Possibly, as a rule, a homogeneous Polish space X with a nice local structure must be a coset space. (The referee has kindly informed us that, recently, van Mill has constructed a counterpart of the space Z which can be identified with a convex set in ℓ_2. This shows that, in our vague statement above, the AR-property is not strong enough to guarantee that a homogeneous X is a coset space.)

As noted in [vM1], the space Y admits a topological copy S, which is a convex subset of the infinite-dimensional Hilbert space H; moreover, $S \times S$ is homeomorphic to H.

2. The space Y

Definition 2.1. Let P be a compactum. A countable collection $\mathcal{P}$ in the Hilbert cube Q is *Z-embedding-dense* for P if $\mathcal{P}$ consists of pairwise disjoint topological copies of P which are Z-sets and such that every map $\alpha : P \to Q$ can be approximated by an embedding $e : P \to Q$ with $e(P) \in \mathcal{P}$.

Employing the fact that the space of mappings of P into the Hilbert cube Q is separable and the basic facts on Z-sets (see, e.g., [To]) one can easily construct a Z-embedding-dense collection $\mathcal{P}$ for an arbitrary compactum P (see [vM1, Lemma 3.1]).

Letting P be the Hilbert cube itself, choose any Z-embedding-dense collection $\mathcal{P} = \{P_1, P_2, \ldots\}$ and let

$$Y = Q \setminus \bigcup_{k=1}^{\infty} P_k.$$

It is easily seen that Y is Polish and, as a complement of a countable union of Z-sets, is an AR (see, e.g., [To]).

3. No nontrivial isotopy on Y. Let $(T, *)$ be a pointed nontrivial continuum, where $*$ is a fixed point of T. Write $P'_k = P_k \times T$ and consider the collection $\mathcal{P}' = \{P'_1, P'_2, \ldots\}$ in $Q' = Q \times T$. Let

$$Y' = Q' \setminus \bigcup_{k=1}^{\infty} P'_k \subset Q'.$$

DEFINITION 3.1. A map $h : Y' \to Y'$ is (n,m)-*continuous* if the natural extension

$$\hat{h} : (Y' \cup P'_n)/\{P'_n\} \to (Y' \cup P'_m)/\{P'_m\}$$

is continuous.

It was shown in [vM1] that, for a homeomorphism $h : Y \to Y$ and n, there exists m such that the obvious counterpart of $\hat{h}$, that is, the map $(Y \cup P_n)/\{P_n\} \to (Y \cup P_m)/\{P_m\}$, is continuous. Moreover, the assignment $n \mapsto m$ is a permutation. A similar fact holds for the space Y'.

PROPOSITION 3.2. *For every isotopy* $(h_t) : Y \to Y$, $t \in T$, *with* $h_* = \mathrm{id}$, *there exists a permutation* $p : \mathbb{N} \to \mathbb{N}$ *such that* $h : Y' \to Y'$ *given by* $h(y,t) = (h_t(y), t)$, $(y,t) \in Y'$, *is* $(n, p(n))$-*continuous.*

Proof. We follow the proof of [vM1, Proposition 3.4].

Let M be the closure of the graph of h in the product $Q' \times Q'$ and let π_1, π_2 be the restrictions to M of the respective projections of $Q' \times Q' \to Q'$. Then M is a continuum, both π_1 and π_2 are surjections, and $\pi_1^{-1}(\bigcup \mathcal{P}') = \pi_2^{-1}(\bigcup \mathcal{P}')$. Moreover, modifying the argument of [ACvM, Lemma 3.6], one sees that both π_1 and π_2 are monotone. To see that π_1 is monotone fix $(x,t) \in Q'$. Suppose $\pi_1^{-1}(x,t) \subset U \cup V$ for some nonempty open and disjoint subsets of M. Since π_1 is closed, there exists an open connected set $W \subset Q$ with $x \in W$ and $\pi_1^{-1}(W \times \{t\}) \subset U \cup V$. It follows that $(W \setminus Y) \times \{t\} = [(W \times \{t\}) \cap \pi_1(U \cap M)] \cup [(W \times \{t\}) \cap \pi_1(V \cap M)]$, which yields a separation of a connected set $W \setminus Y$, a contradiction.

Now, using the monotonicity of π_1 and π_2 and the Sierpiński theorem, one finds m such that $\pi_1^{-1}(P'_n) = \pi_2^{-1}(P'_m)$. Let $p(n) = m$; clearly, p is a permutation.

Suppose $\{y_k\}$ is a sequence in Y' such that $\lim_{k\to\infty} d(y_k, P'_n) = 0$. It follows that $\lim_{k\to\infty} d((y_k, h(y_k)), \pi_1^{-1}(P'_n)) = 0$. Since $\pi_1^{-1}(P'_n) = \pi_2^{-1}(P'_m)$, we have $\lim_{k\to\infty} d((y_k, h(y_k)), \pi_2^{-1}(P'_m)) = 0$. This implies

$$\lim_{k\to\infty} d(\pi_2(y_k, h(y_k)), P'_m) = 0.$$

Thus $\{h(y_k)\}$ converges to P'_m in $(Y' \cup P'_m)/\{P'_m\}$. ■

THEOREM 3.3. *Let* $(h_t) : Y \to Y$, $t \in T$, *be an isotopy with* $h_* = \mathrm{id}$. *Then* $h_t = \mathrm{id}$ *for all* $t \in T$.

Proof. Suppose $h_{t_0}(y_0) \neq y_0$ for some $t_0 \neq *$. Write $h(y,t) = (h_t(y), t)$ for $(y,t) \in Y'$. Pick $\alpha : Q \to Q$ with

$$y_0 \in \alpha(Q) \quad \text{and} \quad h_{t_0}(y_0) \notin \alpha(Q).$$

Enlarge y_0 to an open neighborhood $\widetilde{W}$ in Q such that, for $W = \widetilde{W} \cap Y$,

$$\overline{h_{t_0}(W)} \cap \alpha(Q) = \emptyset.$$

Since $y_0 \in \alpha(Q) \cap \widetilde{W}$ and $\alpha(Q) \cap \overline{h_{t_0}(W)} = \emptyset$, there exists an embedding $e_n : Q \to P_n$ so close to α that

$$P_n \cap \widetilde{W} \neq \emptyset \quad \text{and} \quad P_n \cap \overline{h_{t_0}(W)} = \emptyset.$$

For $e'_n(x,t) = (e_n(x), t)$, $(x,t) \in Q'$, we have $e'_n(Q') \cap (Q \times \{*\}) \neq \emptyset$, that is,

$$P'_n \cap (Q \times \{*\}) \neq \emptyset.$$

Since $h = \mathrm{id}$ on $(Q \times \{*\}) \cap Y'$, h is (n,n)-continuous (that is, $p(n) = n$), which contradicts the fact that

$$P'_n \cap \overline{W \times \{t_0\}} \neq \emptyset \quad \text{and} \quad \overline{h(W \times \{t_0\})} \cap P'_{p(n)} = \overline{h(W \times \{t_0\})} \cap P'_n = \emptyset. \ \blacksquare$$

COROLLARY 3.4. *The space Y admits no nontrivial flow. More generally, if a group G acts on Y then, for every $g \in G$ that can be joined to the unit $e \in G$ by a continuum, we have $gy = y$ for every $y \in Y$.*

4. A transitive action of a Polish group on Y. Let $H(Q)$ be the group of homeomorphisms of the Hilbert cube Q. Consider

$$H(Q|Y) = \{h \in H(Q) \mid (\forall n \in \mathbb{N})\ h(P_n) = P_n\} = \{h \in H(Q) \mid h(Y) = Y\},$$

a subgroup of $H(Q)$. It is easily seen that the group $H(Q|Y)$ acts transitively on Y. However, $H(Q|Y)$ with the topology inherited from $H(Q)$ is not completely metrizable (actually, $H(Q|Y)$ is a genuine $F_{\sigma\delta}$-subset of $H(Q)$). It is clear that if a group G acts on a space X, then G equipped with a stronger compatible topology (that is, giving rise to a topological group) will act on X as well. If such a stronger Polish topology exists on G then G is referred to as *Polishable.* Below we show that this is the case for $G = H(Q|Y)$; this fact also follows from a general condition for Polishability established in [vM2].

THEOREM 4.1. *The group $H(Q|Y)$ is Polishable.*

Proof. Let $\mathrm{Aut}(\mathbb{Z})$ be the group of permutations of the integers with the pointwise convergence topology; $\mathrm{Aut}(\mathbb{Z})$ is a Polish topological group. Consider the group homomorphism $\varphi : H(Q|Y) \to \mathrm{Aut}(\mathbb{Z})$ given by $\varphi(h) = p(h) \in \mathrm{Aut}(\mathbb{Z})$, $h \in H(Q|Y)$, where the value $p(n) = m$ is determined by $h(P_n) = P_m$. Then the graph $\Gamma(\varphi) = \Gamma$ is a subgroup of $H(Q) \times \mathrm{Aut}(\mathbb{Z})$. Since $(h, \varphi(h)) \mapsto h$ is continuous from Γ onto $H(Q|Y)$, it is enough to show that Γ is closed in $H(Q) \times \mathrm{Aut}(\mathbb{Z})$. To see this consider a sequence

$\{h_k\}_{k=1}^{\infty} \subset H(Q|Y)$ that converges in $H(Q)$ such that $\{\varphi(h_k)\}$ converges in $\mathrm{Aut}(\mathbb{Z})$. It follows that, for every n, the sequence $\{\varphi(h_k)(n)\}_{k=1}^{\infty}$ stabilizes, that is, $h_k(P_n) = P_m$ for some m and all but finitely many k. Thus, letting $h = \lim_{k\to\infty} h_k$, we have $h(P_n) = P_m$. Now, it is easily seen that $h \in H(Q|Y)$ and $\varphi(h) = \lim_{k\to\infty} \varphi(h_k)$; hence, $(h, \varphi(h)) \in \varGamma$. ∎

Recall that, by the Effros theorem [E], if a Polish topological group G acts transitively on a Polish space X then G/G_x is homeomorphic to X, where $G_x = \{g \in G \mid \ gx = x\}$ is the stabilizer of x (x may be chosen arbitrarily in X). Hence, in such a case, X is a coset space. The above theorem yields:

Corollary 4.2. *The space Y admits a transitive action of a Polish group, and hence is a coset space.*

Remark 1. According to Corollary 3.4, the group $H(Q|Y)$ neither with its original topology nor with the above Polish topology contains a nontrivial continuum.

5. Different kinds of homogeneity of Y. The fact that Y is CDH was verified in [vM1] by an application of the well-known back-and-forth technique. (Actually, it is shown that, for any countable dense sets $A, B \subset Y$, there exists $h \in H(Q|Y)$ with $h(A) = B$.) This same technique yields the compact homogeneity of Y. Let K and L be compacta in Y and h a homeomorphism of K onto L. Observe that K and L are Z-sets in the Hilbert cube Q. So, h can be extended to a homeomorphism h_0 of Q. Employing the fact that elements of $\mathcal{P}$ are Z-sets in Q (and are homeomorphic to each other), we can modify h_0 step by step to a homeomorphism h_n of Q that agrees with h_{n-1} on $K \cup P_1 \cup \cdots \cup P_n$ and sends it into $L \cup \bigcup \mathcal{P}$, and whose inverse h_n^{-1} agrees with h_{n-1}^{-1} on $L \cup P_1 \cup \cdots \cup P_n$ and sends it into $K \cup \bigcup \mathcal{P}$. This can be achieved so that $\lim h_n = \overline{h}$ is a homeomorphism of Q. Then $\overline{h}(Y) = Y$ (hence, $\overline{h} \in H(Q|Y)$) and $\overline{h}|K = h$. This shows (3) of Theorem 1.1.

Remark 2. The homeomorphism extension property fails for local compacta of Y. Recently van Mill [vM3] showed that the Hilbert cube Q contains a countable compact set $\varDelta$ so that every homeomorphism of Y which restricts to the identity on $\varDelta \cap Y$ is necessarily the identity on Y. Moreover, $\varDelta \setminus Y$ is a convergent sequence space and $D = \varDelta \cap Y$ is (countable) discrete in Y (hence, D is necessarily a Z-set in Y). Pick $y, y' \in Y \setminus \varDelta$, $y \neq y'$. Then the homeomorphism h of $D \cup \{y\}$ onto $D \cup \{y'\}$ which is the identity on D and sends y to y' cannot be extended to a homeomorphism of Y.

Before we give the proof of (4) of Theorem 1.1 below, let us comment on the $\varPhi^*$-CDH property. The requirement that $\mathrm{int}(\mathrm{Fix}(\varPhi)) = \emptyset$ is natural if one expects the existence of h for every choice of A and B. Formally, the CDH and Φ^*-CDH properties are incomparable. Obviously, for a homeomorphism Φ of X with $\mathrm{int}(\mathrm{Fix}(\Phi)) = \emptyset$, the Φ^*-CDH property implies the CDH property. To see this in a more general setting let a group G act on X so that, for some $g_0 \in G$, $g_0 x = \Phi(x)$ is as required. By the definition of Φ^*-CDH, we can find a homeomorphism h of X so that $h^{-1} \circ g_0 \circ h(A) = B$. Then, for the conjugated action $g * x = h^{-1}(g(h(x)))$ of G on X, we have $g_0 * A = B$.

PROPOSITION 5.1. *Fix any bijection Φ of Y with* $\mathrm{int}(\mathrm{Fix}(\Phi)) = \emptyset$ *(in particular, any nonidentity homeomorphism of Y). Then, for any two countable dense sets A and B in Y, there exists a homeomorphism h of Q such that $\Phi(h(A)) = h(B)$ and $h(\bigcup \mathcal{P}) = \bigcup \mathcal{P}$. In particular, Y is Φ^**-CDH.

Proof. Define $\Phi(x) = x$ for $x \in Q \setminus Y$. Enumerate $A = \{a_i\}_{i=1}^{\infty}$ and $B = \{b_i\}_{i=1}^{\infty}$. We will inductively construct, for every $n \geq 1$, towers of finite subsets $\{A_n\}$ and $\{B_n\}$ of A and B, respectively, a finite subfamily $\mathcal{P}_n$ of $\mathcal{P}$, and a homeomorphism $h_n \in H(Q)$ such that

(1) $\{a_1, \ldots, a_n\} \subset A_n$ and $\{b_1, \ldots, b_n\} \subset B_n$;
(2) $\Phi(h_n(\{a_1, \ldots, a_n\})) \subset h_n(B_n)$ and $\Phi^{-1}(h_n(\{b_1, \ldots, b_n\})) \subset h_n(A_n)$; furthermore, $\Phi(h_n(A_n)) = h_n(B_n)$;
(3) $\{P_1, \ldots, P_n\} \subset \mathcal{P}_n$;
(4) $\{h_n(P_1), \ldots, h_n(P_n)\} \subset \mathcal{P}_n$ and $\{P_1, \ldots, P_n\} \subset h_n(\mathcal{P}_n)$;
(5) $h_n | A_{n-1} \cup B_{n-1} \cup \bigcup \mathcal{P}_{n-1} = h_{n-1} | A_{n-1} \cup B_{n-1} \cup \bigcup \mathcal{P}_{n-1}$;
(6) $d(h_{n-1}, h_n) < 2^{-1-n}$.

Clearly, $h = \lim h_n$ is then as required.

Inductive construction. Define $h_0 = \mathrm{id}$ and let $A_0 = B_0 = \mathcal{P}_0 = \emptyset$. Suppose that A_{n-1}, B_{n-1}, $\mathcal{P}_{n-1}$, and h_{n-1} have been constructed for $n \geq 1$ so that (1)–(6) are satisfied.

Assume $a_n \in A \setminus (A_{n-1} \cup B_{n-1})$ and let $c = \Phi(h_{n-1}(a_n))$. It follows that $c \notin h_{n-1}(B_{n-1})$. If $c \notin h_{n-1}(A_{n-1})$ and $c \neq h_{n-1}(a_n)$ (the latter condition, in particular, implies $h_{n-1}(a_n) \in Y$), we set $g^{(1)} = h_{n-1}$. If, however, $c \in h_{n-1}(A_{n-1})$ or $c = h_{n-1}(a_n)$, then there exists a homeomorphism $g^{(1)}$ such that $h_{n-1} | A_{n-1} \cup B_{n-1} \cup \bigcup \mathcal{P}_{n-1} = g^{(1)} | A_{n-1} \cup B_{n-1} \cup \bigcup \mathcal{P}_{n-1}$, $\Phi(g^{(1)}(a_n)) \in Y \setminus h_{n-1}(A_{n-1} \cup B_{n-1})$, and $\Phi(g^{(1)}(a_n)) \neq g^{(1)}(a_n)$; to obtain the latter condition use the fact that $\mathrm{int}(\mathrm{Fix}(\Phi)) = \emptyset$. Moreover, $g^{(1)}$ can be made as close to h_{n-1} as we wish. Let $A' = A_{n-1} \cup \{a_n\}$. Note that, in both cases, we have $\Phi(g^{(1)}(a_n)) \in Y \setminus g^{(1)}(A' \cup B_{n-1})$. Now, there exists a homeomorphism $g^{(2)}$ (as close to $g^{(1)}$ as we wish) such that $g^{(1)} | A' \cup B_{n-1} \cup \bigcup \mathcal{P}_{n-1} = g^{(2)} | A' \cup B_{n-1} \cup \bigcup \mathcal{P}_{n-1}$ and $g^{(2)}(b') = \Phi(g^{(1)}(a_n))$ for a certain $b' \in B$; it follows that $b' \notin A' \cup B_{n-1}$.

Let $B' = B_{n-1} \cup \{b'\}$. Clearly, we have $\Phi(g^{(2)}(A')) = g^{(2)}(B')$. Assume $b_n \in B \setminus (A' \cup B')$ and let $d = \Phi^{-1}(g^{(2)}(b_n))$. It follows that $d \notin g^{(2)}(A')$. If $d \notin g^{(2)}(B')$ and $d \neq g^{(2)}(b_n)$ (the latter condition, in particular, implies $g^{(2)}(b_n) \in Y$), we set $g^{(3)} = g^{(2)}$. If $d \in g^{(2)}(B')$ or $d = g^{(2)}(b_n)$, then there exists a homeomorphism $g^{(3)}$ such that we have $g^{(3)}|A' \cup B' \cup \bigcup \mathcal{P}_{n-1} = g^{(2)}|A' \cup B' \cup \bigcup \mathcal{P}_{n-1}$, $\Phi^{-1}(g^{(3)}(b_n)) \in Y \setminus g^{(3)}(A' \cup B')$, and $\Phi^{-1}(g^{(3)}(b_n)) \neq g^{(3)}(b_n)$. Moreover, $g^{(3)}$ can be made as close to $g^{(2)}$ as we wish. In both cases, we have $\Phi^{-1}(g^{(3)}(b_n)) \in Y \setminus g^{(3)}(A' \cup B' \cup \{b_n\})$. Now, there exists a homeomorphism $g^{(4)}$ (as close to $g^{(3)}$ as we wish) such that $g^{(3)}|A' \cup B' \cup \{b_n\} \cup \bigcup \mathcal{P}_{n-1} = g^{(4)}|A' \cup B' \cup \{b_n\} \cup \bigcup \mathcal{P}_{n-1}$ with $g^{(4)}(a') = \Phi^{-1}(g^{(3)}(b_n))$ for a certain $a' \in A$; it follows that $a' \notin A' \cup B' \cup \{b_n\}$. We let $A_n = A' \cup \{a'\}$ and $B_n = B' \cup \{b_n\}$.

Finally, assuming $P_n \notin \mathcal{P}_{n-1}$, we can find a homeomorphism g_1 as close to $g^{(4)}$ as we wish and such that $g_1|A_n \cup B_n \cup \bigcup \mathcal{P}_{n-1} = g^{(4)}|A_n \cup B_n \cup \bigcup \mathcal{P}_{n-1}$ and $g_1(P_n) \in \mathcal{P}$. Similarly, if $g_1^{-1}(P_n) \notin \mathcal{P}_{n-1}$, we can find a homeomorphism g_2 as close to g_1 as we wish and such that $g_2|A_n \cup B_n \cup \bigcup \mathcal{P}_{n-1} \cup P_n = g_1|A_n \cup B_n \cup \bigcup \mathcal{P}_{n-1} \cup P_n$ and $g_2(P) = P_n$ for some $P \in \mathcal{P}$. Let $\mathcal{P}_n = \mathcal{P}_{n-1} \cup \{g_1(P_n)\} \cup \{P\}$. The inductive construction is completed by letting $h_n = g_2$. ∎

Proposition 5.1, together with the comments preceding its statement, yields

COROLLARY 5.2. *For a nontrivial action of a group G on the space Y and countable dense subsets A and B of Y there exists $g_0 \in G$ so that, for a certain homeomorphism h of X, the conjugated action $g * x = h^{-1}(g(h(x)))$ sends A onto B when $g = g_0$.*

REMARK 3. In Proposition 5.1, the homeomorphism h can be chosen as close to the identity in $H(Q)$ as we wish. As a consequence, for countable dense subsets A and B of Y, any homeomorphism g of Y can be approximated by conjugations $h^{-1} \circ g \circ h$ that send A onto B. However, this approximation is not in the limitation topology on the group of homeomorphisms $H(Y)$ of Y because $h \in H(Q|A)$ is not necessarily close to the identity in the limitation topology. Actually, it can be shown that Y is not homogeneous "via small homeomorphisms". More precisely, there exists a continuous function $\varepsilon : Y \to (0, \infty)$ such any homeomorphism h of Y which satisfies $d(h(x), x) < \varepsilon(x)$ for every $x \in Y$ must be the identity on Y (that is, if h is in the ε-neighborhood of the identity in the limitation topology, then h must be the identity itself).

6. Other counterparts of Y. The most elementary example that can be obtained via the procedure described in Section 2 is the space $Q \setminus A$, where A is a countable dense subset of Q; simply, apply the procedure to a one-point space P. The resulting space, however, is strongly locally homogeneous. On the other hand, choosing a countable Z-embedding-dense collection $\mathcal{P}$ in the Hilbert cube Q for $P = [0,1]$, we obtain the space $Q \setminus \bigcup \mathcal{P}$ which is a counterpart of the space Y. This space (which can be checked to be topologically different from Y) shares all the properties of Y from Theorem 1.1.

In case P is a compactum with $\dim(P) \le k$, Z_k-embedding-dense collections can be constructed in the interior of the $(2k+1)$-dimensional cube I^{2k+1}, which replaces the Hilbert cube Q (for the definition of a Z_k-set see [To]). In particular, there exists a Z_1-embedding-dense collection $\mathcal{I} = \{I_n\}_{n=1}^{\infty}$ in $\dot{I}^m$, the interior of I^m for $m \ge 4$. Actually, we can assume that each I_n is a finite union of line segments. Then the resulting space $Y_I = I^m \setminus \bigcup_{n=1}^{\infty} I_n$ seems to share all the properties of Y listed in Theorem 1.1 with the exception of (3). Obviously, Y_I is not an AR-space; yet, it must be locally connected, connected, and l-connected for some l. The following counterpart of property (3) holds: Y_I has the homeomorphism extension property for compacta in $\dot{I}^m$ which are Z_1-sets in I^m. The tricky case of $m = 3$ will be discussed in the forthcoming paper by S. Spież and the author.

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