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Morita Equivalences of Functor Categories and Decompositions of Functors Defined on a Category Associated to Algebras with One-Side Units

by

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Summary. Conditions which imply Morita equivalences of functor categories are described. As an application a Dold–Kan type theorem for functors defined on a category associated to associative algebras with one-side units is proved.

1. Introduction. Let $\mathcal{F}$ denote the category whose objects are finite sets $[n] = \{0, \dots, n\}$ and whose morphisms are maps $f : [n] \to [m]$. The category $\varDelta$ of finite totally ordered sets is a subcategory of $\mathcal{F}$ with the same object set. The morphisms of $\varDelta$ are all arrows of $\mathcal{F}$ which preserve the natural order $\{0 \le 1 \le \dots \le n\}$. Let S be the subcategory of $\varDelta$ consisting of all order preserving surjections. Its morphisms are compositions of elementary order preserving surjections $s_i : [n] \to [n-1]$ such that $s_i(i) = s_i(i+1)$. Let D be the subcategory of $\varDelta$ consisting of all order preserving injections. Its morphisms are compositions of elementary order preserving injections $d_i : [n-1] \to [n]$ such that i does not belong to the image of d_i.

We will also consider subcategories $\mathcal{F}_\bullet$ and $\mathcal{F}^\bullet$ which have the same objects as $\mathcal{F}$. The morphisms of $\mathcal{F}_\bullet$ (resp. $\mathcal{F}^\bullet$) are based maps $f : [n] \to [m]$ such that $f(0) = 0$, (resp. $f(n) = m$). Let $\varDelta^\bullet = \varDelta \cap \mathcal{F}^\bullet$, $\varDelta_\bullet = \varDelta \cap \mathcal{F}_\bullet$ and $D^\bullet = D \cap \mathcal{F}^\bullet$. The maps $d_i : [n-1] \to [n]$, for $i = 0, \dots, n-1$, belong to $D^\bullet$. The categories S^{op} and $D^\bullet$ are isomorphic.

We will prove the following result.

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1.1. Theorem. *Let $\mathcal{C}'$ be a category with the same object set as $\mathcal{F}$ and with two subcategories, with the same object sets, S' and D', isomorphic to S and $D^\bullet$ respectively. If the relations*

$$s'_j d'_i = d'_{i-1} s'_j \quad \text{for } j < i-1, \qquad s'_j d'_i = d'_i s'_{j-1} \quad \text{for } i < j, \qquad s'_i d'_i = \mathrm{id}$$

hold in $\mathcal{C}'$, then, for every functor $M : (\mathcal{C}')^{\mathrm{op}} \to \mathrm{Ab}$ and $n \geq 1$, there exists a decomposition

$$M[n] = \coprod_{1\le k\le n} \coprod_{s\in S'([n],[k])} \bigcap_{0\le i\le k-1} \mathrm{Ker}(M(d'_i) : M[k] \to M[k-1]).$$

If $\mathcal{C}' = \Delta^\bullet$, then such a decomposition can be obtained as a consequence of the Dold–Kan Theorem [1–3] which concerns simplicial abelian groups, i.e. contravariant functors from the simplicial category Δ to the category Ab of abelian groups.

In [5–6] a similar fact is proved for functors defined on the category Γ of finite based sets. It can be obtained from 1.1 for the category $\mathcal{C}' = QD^\bullet$ described below. It follows from the definition that the categories $\mathcal{F}_\bullet$ and $\mathcal{F}^\bullet$ are isomorphic. Usually $\Gamma = \mathcal{F}_\bullet$ but we will consider the category $\mathcal{F}^\bullet$. Let $(D^\bullet)^*$ be the subcategory of $\mathcal{F}^\bullet$, with the same objects, whose morphisms are compositions of the surjections $d^*_i : [n] \to [n-1]$, for $i = 0, \ldots, n-1$, such that

$$d^*_i(j) = j \quad \text{if } j < i, \qquad d^*_i(j) = j-1 \quad \text{if } i < j, \qquad d^*_i(i) = n-1.$$

Then $QD^\bullet$ is the subcategory of $\mathcal{F}^\bullet$ generated by $D^\bullet$ and $(D^\bullet)^*$.

1.2. Proposition. *The categories $\Delta^\bullet$ and $QD^\bullet$ satisfy the assumptions of Theorem* 1.1. ∎

A general result which implies decompositions of functors defined on $\Delta^\bullet$ and $QD^\bullet$ is proved in [7]. In this note we will prove that there exists a category $\mathcal{U}$ satisfying the assumptions of 1.1 and such that, for every category $\mathcal{C}'$ satisfying the assumptions of 1.1, there exists an appropriate functor $\mathcal{U} \to \mathcal{C}'$. Hence decompositions for $\Delta^\bullet$ and $QD^\bullet$ can be obtained as special cases of the decomposition for $\mathcal{U}$. The category $\mathcal{U}$ will be defined in Section 2 as a subcategory of a monoidal category associated to associative algebras with one-side units (2.4(i)). (The category Δ can be considered (2.2(i)) as a subcategory of a monoidal category (PRO) associated to unital associative algebras.) The category $\mathcal{U}$ does not satisfy the assumptions of the theorems proved in [7]. In Section 3 we will generalize certain results of that paper which are consequences of Morita equivalences of appropriate functor categories. Theorem 1.1 will be proved in Section 4, using the results of Section 3.

2. Categories associated to algebras with one-side units. Let R be a commutative ring. If A is an associative R-algebra, then it induces a monoidal functor $\mathbf{A} : S \to R\text{-Mod}$, where R-Mod is the category of R-modules ([4]). For every $[n] \in S$,

$$\mathbf{A}[n] = A \otimes_R \cdots \otimes_R A = A^{\otimes n+1},$$
$$s_i(a_0 \otimes \cdots \otimes a_n) = a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n.$$

A coassociative R-coalgebra (C, ρ) induces a functor $\mathbf{C} : S^{\mathrm{op}} \to R\text{-Mod}$ in a similar way:

$$s_i^{\mathrm{op}}(c_0 \otimes \cdots \otimes c_m) = c_0 \otimes \cdots \otimes c_{i-1} \otimes \rho c_i \otimes c_{i+1} \cdots \otimes c_m.$$

This implies the following fact.

2.1. Proposition. *Let $S^* = S^{\mathrm{op}}$ and let P be the category generated by S, S^* and the relations*

$$s_j s_i^* = s_{i-1}^* s_j \quad \text{for } j < i-1, \qquad s_j s_i^* = s_i^* s_{j-1} \quad \text{for } i < j-1.$$

If B has the structure of an associative R-algebra and an associative R-coalgebra, then it induces a functor $\mathbf{B}$ on P such that $\mathbf{B}[n] = B^{\otimes n+1}$. ∎

The following examples and Proposition 2.4 imply Proposition 1.2.

2.2. Examples. (i) Let A be an R-algebra with unit e. Then the functor $\mathbf{A} : S \to R\text{-Mod}$ can be extended to a functor $\Delta \to R\text{-Mod}$ such that

$$d_i(a_0 \otimes \cdots \otimes a_{n-1}) = a_0 \otimes \cdots \otimes a_{i-1} \otimes e \otimes a_i \cdots \otimes a_n.$$

The unit e give us two R-coalgebra structures $\rho_1, \rho_2 : A \to A \otimes_R A$ such that

$$\rho_1(a) = a \otimes e, \qquad \rho_2(a) = e \otimes a.$$

There are two functors associated with these coalgebra structures, $p_1' : P \to \Delta_\bullet \subset \Delta$ and $p_2' : P \to \Delta^\bullet \subset \Delta$, such that

$$p_1'(s_i^*) = d_{i+1}, \qquad p_2'(s_i^*) = d_i, \qquad p_1'(s_i) = p_2'(s_i) = s_i.$$

(ii) $QD^\bullet$ is the category with morphisms generated by $d_j \in D^\bullet$ and d_i^* together with the relations

$$d_j^* d_i = d_{i-1} d_j^* \quad \text{if } j < i, \qquad d_i^* d_i = \mathrm{id}, \qquad d_j^* d_i = d_i d_{j-1}^* \quad \text{if } j > i.$$

There exists a natural surjection $\mu : P \to QD^\bullet$ such that $\mu(s_i) = d_i^*$ and $\mu(s_i^*) = d_i$.

Let V be an R-module with a given element $e \in V$ and an R-homomorphism $f : V \to R$ such that $f(e) = 1$. Let $\rho(v) = e \otimes v$ and $v_1.v_2 = f(v_1)v_2$. Then one can define a functor $\mathbf{V}_0 : QD^\bullet \to R\text{-Mod}$ such that $\mathbf{V} = \mathbf{V}_0 \mu$.

2.3. Definition. (i) $\tilde{P}$ is the category generated by S, D and the relations

$$s_j d_i = d_{i-1} s_j \quad \text{for } j < i-1, \qquad s_j d_i = d_i s_{j-1} \quad \text{for } i < j.$$

(ii) $\tilde{P}_r$ (resp. $\tilde{P}_l$) is the factor category of $\tilde{P}$ associated to the relations

$$s_{i-1}d_i = \mathrm{id} \quad (\text{resp. } s_i d_i = \mathrm{id}).$$

(iii) $\mathcal{U} = (\tilde{P}_l)^\bullet$ is the subcategory of $\tilde{P}_l$ generated by S and $D^\bullet$.

The following results are easy to prove.

2.4. Proposition.

(i) *If A is an algebra with a right (resp. left) unit then the functor* $\mathbf{A}$ *can be extended to a functor defined on $\tilde{P}_r$ (resp. $\tilde{P}_l$).*

(ii) *There exist natural projections $\tilde{P} \to \Delta$, $\tilde{P}_r \to \Delta$, $\tilde{P}_l \to \Delta$ and Δ can be considered as the factor category of $\tilde{P}$ associated to the relations* $s_{i-1}d_i = s_i d_i = \mathrm{id}$.

(iii) *There exists a natural projection $p : P \to \mathcal{U}$. The functor $p_2' : P \to \Delta^\bullet$ defined in* 2.2(i) *factorizes through a functor $\mathcal{U} \to \Delta^\bullet$. The factor category of $\mathcal{U}$ associated to the relations* $s_{i-1}d_i = \mathrm{id}$ *is equal to the $\Delta^\bullet$. The natural surjection $\mu : P \to QD^\bullet$ factorizes through a surjection $\mathcal{U} \to QD^\bullet$. $QD^\bullet$ is isomorphic to the factor category of $\mathcal{U}$ associated to the relations $s_{i-1}d_i = d_{i-1}s_{i-1}$.*

3. Decompositions of categories and Morita equivalences in functor categories. Let $\mathcal{C}$ be a small category. The category whose morphisms are all identity morphisms (resp. endomorphisms) of $\mathcal{C}$ will be denoted by $\mathrm{Id}_{\mathcal{C}}$ (resp. $E_{\mathcal{C}}$). The morphism sets of $\mathcal{C}$ will be denoted by $\mathcal{C}(c, c')$ and the morphism set functor by $\mathcal{C} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Set}$. The category of all functors from $\mathcal{C}$ to $\mathcal{D}$ will be denoted by $(\mathcal{C}, \mathcal{D})$.

We will use R-categories whose morphism sets are R-modules and whose compositions are R-module homomorphisms. If $\mathcal{A}$ is a small category, then $R[\mathcal{A}]$ is an R-category with the same objects as $\mathcal{A}$. The morphisms of $R[\mathcal{A}]$ form free R-modules generated by the morphisms of $\mathcal{A}$. The functor category $(\mathcal{A}, R\text{-Mod})$ is isomorphic to the category of R-functors from $R[\mathcal{A}]$ to R-Mod and will be denoted by $R[\mathcal{A}]$-Mod.

If $M : \mathcal{C} \to R\text{-Mod}$ and $M' : \mathcal{C}^{\mathrm{op}} \to R\text{-Mod}$ then $M' \otimes_{R[\mathcal{C}]} M$ is the coend of the bifunctor $M' \otimes_R M$. Every R-bifunctor $U : R[\mathcal{C}^{\mathrm{op}} \times \mathcal{C}'] \to R\text{-Mod}$ gives us functors

$$- \otimes_{R[\mathcal{C}']} U : (\mathcal{C}'^{\mathrm{op}}, R\text{-Mod}) \to (\mathcal{C}^{\mathrm{op}}, R\text{-Mod}),$$
$$\mathrm{Hom}_{R[\mathcal{C}]}(U, -) : (\mathcal{C}^{\mathrm{op}}, R\text{-Mod}) \to (\mathcal{C}'^{\mathrm{op}}, R\text{-Mod}).$$

In particular, given an R-functor $F : R[\mathcal{C}'] \to R[\mathcal{C}]$, we can take $U_F(c, c') = R[\mathcal{C}](c, F(c'))$ and $U_F'(c', c) = R[\mathcal{C}](F(c'), c)$. In this case we will use the following notation:

$$T_{\mathcal{C}'}\mathcal{C} = -\otimes_{R[\mathcal{C}']} U_F = -\otimes_{R[\mathcal{C}']} R[\mathcal{C}] : R[\mathcal{C}']^{\mathrm{op}}\text{-Mod} \to R[\mathcal{C}]^{\mathrm{op}}\text{-Mod},$$
$$H_{\mathcal{C}'}\mathcal{C} = \mathrm{Hom}_{R[\mathcal{C}']}(R[\mathcal{C}], -) = \mathrm{Hom}_{R[\mathcal{C}']}(U'_F, -) : R[\mathcal{C}']^{\mathrm{op}}\text{-Mod} \to R[\mathcal{C}]^{\mathrm{op}}\text{-Mod}.$$

3.1. Definition. Let $\mathcal{N}$ be the inclusion subcategory of D consisting of all order preserving injections which are compositions of the injections $d_n : [n-1] \to [n]$. We will say that $\mathcal{A}$ is an $\mathcal{N}$-*category* if the morphism sets of $\mathcal{A}$ are finite, and if there exists a functor $\pi : \mathcal{A} \to \mathcal{N}$ which is an inclusion on object sets.

The following facts are immediate consequences of the definitions.

3.2. Proposition. *Let $\mathcal{A}$ or $\mathcal{A}^{\mathrm{op}}$ be an $\mathcal{N}$-category. Let $\mathcal{C} = E_{\mathcal{A}}$ be the endomorphism category of $\mathcal{A}$.*

(i) *There exists a canonical projection $p_{\mathcal{A}} : R[\mathcal{A}] \to R[\mathcal{C}]$ of R-categories.*
(ii) *Assume that A_0 consists of all morphisms of $\mathcal{A}$ which are not endomorphisms. For every functor $M : \mathcal{A}^{\mathrm{op}} \to R\text{-Mod}$ and every functor $N : \mathcal{C}^{\mathrm{op}} \to R\text{-Mod}$ we have*

$$T_{\mathcal{A}}\mathcal{C}(M)(x) = M(x) / \sum_{a\in A_0,\, a:x\to y} \mathrm{Im}\, M(a),$$
$$H_{\mathcal{A}}\mathcal{C}(M)(x) = \bigcap_{a\in A_0,\, a:y\to x} \mathrm{Ker}\, M(a),$$
$$T_{\mathcal{C}}\mathcal{A}(N)(x) = \bigoplus_{y\in \mathrm{Ob}\,\mathcal{A}} N(y) \otimes_{R[\mathcal{C}](y,y)} R[\mathcal{A}](x,y),$$
$$H_{\mathcal{C}}\mathcal{A}(N)(x) = \prod_{y\in \mathrm{Ob}\,\mathcal{A}} \mathrm{Hom}_{R[\mathcal{C}](y,y)}(R[\mathcal{A}](y,x), N(y)).$$

3.3. Definition. Let $\mathcal{C}$, $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}'$ be small categories with the same object sets such that $\mathcal{C}_1$, $\mathcal{C}_2$ are subcategories of $\mathcal{C}'$ and $\mathcal{C} = \mathcal{C}_1 \cap \mathcal{C}_2$.

(i) $\mathcal{C}' = \mathcal{C}_1 \cdot_{\mathcal{C}} \mathcal{C}_2$ if the morphisms of $\mathcal{C}'$ can be represented as compositions $f_1 f_2$ of morphisms f_i from $\mathcal{C}_i$, uniquely up to morphisms from $\mathcal{C}$. If $x, y \in \mathrm{Ob}\,\mathcal{C}'$, then
$$\mathcal{C}'(x,y) = \Big(\coprod_{z\in \mathrm{Ob}\,\mathcal{C}} \mathcal{C}_1(z,y) \times \mathcal{C}_2(x,z) \Big)_{\sim}$$
where $(f_1 f, f_2) \sim (f_1, f f_2)$ for morphisms f_i of $\mathcal{C}_i$ and f from $\mathcal{C}$.
(ii) $\mathcal{C}_1 \cdot_{\mathcal{C}} \mathcal{C}_2 = \mathcal{C}_1 \mathcal{C}_2$ if $\mathcal{C} = \mathrm{Id}_{\mathcal{C}_1} = \mathrm{Id}_{\mathcal{C}_2}$.

3.4. Examples. $\Delta = DS$, $\Delta^\bullet = D^\bullet S$, $QD^\bullet = D^\bullet (D^\bullet)^*$.

3.5. Proposition. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ be small categories with the same object sets such that $\mathcal{A}$ and $\mathcal{B}^{\mathrm{op}}$ are $\mathcal{N}$-categories, $\mathcal{C} = \mathcal{A} \cap \mathcal{B} = E_{\mathcal{A}} = E_{\mathcal{B}}$*

and $\mathcal{D} = \mathcal{A} \cdot_{\mathcal{C}} \mathcal{B}$.

(i) *The projection* $p_{\mathcal{A}} : R[\mathcal{A}] \to R[\mathcal{C}]$ *induces a projection* $R[\mathcal{D}] \to R[\mathcal{B}]$ *which gives an* $R[\mathcal{D}^{\mathrm{op}}]$*-module structure on* $R[\mathcal{B}]$.

(ii) *The projection* $p_{\mathcal{B}} : R[\mathcal{B}] \to R[\mathcal{C}]$ *gives an* $R[\mathcal{D}]$*-module structure on* $R[\mathcal{A}]$.

(iii) *For every functor* $M : \mathcal{D}^{\mathrm{op}} \to R\text{-Mod}$ *and every functor* $N : \mathcal{C}^{\mathrm{op}} \to R\text{-Mod}$, *there are natural isomorphisms*

$$T_{\mathcal{B}}\mathcal{C}(M) = M \otimes_{R[\mathcal{D}]} R[\mathcal{A}], \quad H_{\mathcal{A}}\mathcal{C}(M) = \mathrm{Hom}_{R[\mathcal{D}]}(R[\mathcal{B}], M),$$
$$T_{\mathcal{C}}\mathcal{B}(N) = N \otimes_{R[\mathcal{A}]} R[\mathcal{D}], \quad H_{\mathcal{C}}\mathcal{A}(N) = \mathrm{Hom}_{R[\mathcal{B}]}(R[\mathcal{D}], N).$$

Proof. The result is a consequence of 3.1–3.3. ■

Let

$$u(x,y) : R[\mathcal{B}](y,x) \otimes_{R[\mathcal{C}](y,y)} R[\mathcal{A}](x,y) \to R[\mathcal{C}](x,x)$$

be defined by using composition in $\mathcal{D}$ and the projection $p_{\mathcal{D}} = p_{\mathcal{A}} \otimes p_{\mathcal{B}} : R[\mathcal{D}] \to R[\mathcal{C}]$. We will consider the following homomorphisms induced by u:

$$j(x,y) : R[\mathcal{B}](y,x) \to \mathrm{Hom}_{R[\mathcal{C}](x,x)}(R[\mathcal{A}](x,y), R[\mathcal{C}](x,x)),$$
$$j'(x,y) : R[\mathcal{A}](x,y) \to \mathrm{Hom}_{R[\mathcal{C}](x,x)}(R[\mathcal{B}](y,x), R[\mathcal{C}](x,x)).$$

3.6. Theorem. *Suppose that the assumptions of* 3.5 *are satisfied and that, for every pair* $x, y \in \mathcal{D}$, $R[\mathcal{A}](x,y)$ *is a free* $R[\mathcal{C}](x,x)$*-module and* $j(x,y)$ *and* $j'(x,y)$ *are isomorphisms. Then the pairs of adjoint functors* $T_{\mathcal{C}}\mathcal{B}, H_{\mathcal{A}}\mathcal{C}$ *and* $H_{\mathcal{C}}\mathcal{A}, T_{\mathcal{B}}\mathcal{C}$ *define Morita equivalences of categories*

$$(\mathcal{D}^{\mathrm{op}}, R\text{-Mod}) \quad \textit{and} \quad (\mathcal{C}^{\mathrm{op}}, R\text{-Mod}).$$

Proof. The result is an immediate consequence of the following facts which can be proved by induction on the cardinality of the object set of $\mathcal{D}$ using the same arguments as in the proof of 1.6 in [7].

(i) The natural transformations $\mathrm{Id} \to H_{\mathcal{A}}\mathcal{C}T_{\mathcal{C}}\mathcal{B}$ and $T_{\mathcal{B}}\mathcal{C}H_{\mathcal{C}}\mathcal{A} \to \mathrm{Id}$ are equivalences of endofunctors defined on $(\mathcal{C}^{\mathrm{op}}, R\text{-Mod})$.

(ii) The natural transformations $T_{\mathcal{C}}\mathcal{B}H_{\mathcal{A}}\mathcal{C} \to \mathrm{Id}$ and $\mathrm{Id} \to H_{\mathcal{C}}\mathcal{A}T_{\mathcal{B}}\mathcal{C}$ are equivalences of endofunctors defined on $(\mathcal{D}^{\mathrm{op}}, R\text{-Mod})$.

(iii) The composition of the natural transformations $T_{\mathcal{C}}\mathcal{B} \to \mathrm{Id} \to H_{\mathcal{C}}\mathcal{A}$ of functors from $(\mathcal{C}^{\mathrm{op}}, R\text{-Mod})$ to $(\mathcal{D}^{\mathrm{op}}, R\text{-Mod})$ is a natural equivalence.

(iv) The composition of natural transformations $H_{\mathcal{A}}\mathcal{C} \to \mathrm{Id} \to T_{\mathcal{B}}\mathcal{C}$ of functors from $(\mathcal{D}^{\mathrm{op}}, R\text{-Mod})$ to $(\mathcal{C}^{\mathrm{op}}, R\text{-Mod})$ is a natural equivalence. ■

4. Proof of Theorem 1.1. It follows from Section 2 that $\mathcal{U}$ is the category generated by S, $D^{\bullet}$ and the relations

$$s_j d_i = d_{i-1} s_j \quad \text{for } j < i-1, \qquad s_j d_i = d_i s_{j-1} \quad \text{for } i < j, \qquad s_i d_i = \mathrm{id}.$$

Let $\mathcal{C}'$ be a category satisfying the assumptions of 1.1. Then one can define a functor $\mathcal{U} \to \mathcal{C}'$, and Theorem 1.1 is a consequence of the following specialization.

4.1. Proposition. *For every functor* $M : \mathcal{U}^{\mathrm{op}} \to R$-Mod, *and for* $n \geq 1$, *there exists a decomposition*

$$M[n] = \coprod_{1\leq k\leq n} \coprod_{s\in S([n],[k])} \bigcap_{0\leq i\leq k-1} \operatorname{Ker}(M(d_i) : M[k] \to M[k-1]).$$

Proof. Let $t_i = s_{i-1}d_i : [n] \to [n]$ for $i = 1, \ldots, n+1$ be a morphism of $\tilde{P}_l$. Let T be the subcategory of $\tilde{P}_l$ with the same objects and with morphisms which are compositions of t_i. We have $t_i^2 = t_i$ and $t_i t_j = t_j t_i$. Let

$$T^\bullet = T \cap \mathcal{U} = T \cap \tilde{P}_l^\bullet, \quad \tilde{D}^\bullet = D^\bullet T^\bullet, \quad \tilde{S}^\bullet = T^\bullet S.$$

There exist decompositions

$$\mathcal{U} = D^\bullet T^\bullet S = \tilde{D}^\bullet \cdot_{T^\bullet} \tilde{S}^\bullet.$$

Let

$$M_0[k] = \bigcap_{0\leq i\leq k-1} \operatorname{Ker}(M(d_i) : M[k] \to M[k-1]).$$

It follows from the definitions that M_0 consists of elements annihilated by $D^\bullet$ and that it is a functor defined on $R[\tilde{D}^\bullet]$. Using Theorem 3.6 one can prove that

$$M = M_0 \otimes_{R[\tilde{D}^\bullet]} R[\mathcal{U}] = M_0 \otimes_{R[T^\bullet]} R[\tilde{S}^\bullet] = M_0 \otimes_{R[\mathrm{Id}_S]} R[S].$$

We have to check that the composition of the multiplication

$$R[\tilde{S}^\bullet] \otimes_{R[T^\bullet]} R[\tilde{D}^\bullet] \to R[\mathcal{U}]$$

with the natural projection $R[\mathcal{U}] \to R[T^\bullet]$ induces isomorphisms

$$R[\tilde{S}^\bullet](x,y) \to \operatorname{Hom}_{R[T^\bullet](y,y)}(R[\tilde{D}^\bullet](y,x), R[T^\bullet](y,y)),$$
$$R[\tilde{D}^\bullet](x,y) \to \operatorname{Hom}_{R[T^\bullet](x,x))}(R[\tilde{S}^\bullet](y,x), R[T^\bullet](x,x)).$$

Recall that $\tilde{D}^\bullet = D^\bullet T^\bullet$ and $\tilde{S}^\bullet = T^\bullet S$. Now the composition rules in $\tilde{D}^\bullet$ and $\tilde{S}^\bullet$ and the fact that we have isomorphisms

$$R[S](x,y) \to \operatorname{Hom}_R(R[D^\bullet](y,x), R), \quad R[D^\bullet](x,y) \to \operatorname{Hom}_R(R[S](y,x), R)$$

induced by composition of morphisms in Δ imply the result. ∎

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