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GENERAL TOPOLOGY

Multiplication is Discontinuous in the Hawaiian Earring Group (with the Quotient Topology)

by

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Summary. The natural quotient map q from the space of based loops in the Hawaiian earring onto the fundamental group provides a naturally occuring example of a quotient map such that $q \times q$ fails to be a quotient map. With the quotient topology, this example shows $\pi_1(X,p)$ can fail to be a topological group if X is locally path connected.

1. Introduction. The Hawaiian earring HE is the union of a null sequence of circles joined at a common point. If the fundamental group $\pi_1(HE,p)$ is endowed with a certain natural topology, we prove $\pi_1(HE,p)$ fails to be a topological group with the standard operations, and in the bargain obtain a naturally occurring example of a map $q : Y \to Z$ such that $q \times q : Y \times Y \to Z \times Z$ fails to be a quotient map.

Following the definitions in [2], there is a natural quotient topology one can impart on the familiar based fundamental group $\pi_1(X,p)$ of a topological space X.

If $L(X,p)$ denotes the space of p based loops in X with the compact open topology, and if $q : L(X,p) \to \pi_1(X,p)$ is the natural surjection, then we endow $\pi_1(X,p)$ with the quotient topology such that $A \subset \pi_1(X,p)$ is closed in $\pi_1(X,p)$ if and only of $q^{-1}(A)$ is closed in $L(X,p)$.

For spaces X sufficiently simple on the small scale, $\pi_1(X,p)$ has the discrete topology and is certainly a topological group [7], [4], [3].

More generally Proposition 3.1 of [2] asserts that $\pi_1(X,p)$ is always a topological group with the familiar operations. However Tyler Lawson no-

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ticed in 2006 that the proof of Proposition 3.1 depends on the questionable assumption that the product of the quotient maps $q\times q : L(X,p)\times L(X,p) \to \pi_1(X,p) \times \pi_1(X,p)$ is again a quotient map (since for general topological spaces, if $q : Y \to Z$ is a quotient map then $q \times q : Y \times Y \to Z \times Z$ can fail to be a quotient map [13]).

Jeremy Brazas has recently found various examples of spaces X such that $\pi_1(X,p)$ fails to have continuous multiplication [3]. Particular attention in [3] is given to spaces X constructed in the following manner. Let A be a totally disconnected subset of the positive real line, and consider X as the union of planar circles of radius $1 + a$, centered on the positive real line, and joined at the common point $(0,0)$. Such spaces X are locally simply connected, but fail to be locally path connected. For example Brazas proves $\pi_1(X,p)$ fails to have continuous multiplication if A is the positive rationals.

In contrast to the examples in [3], the Hawaiian earring HE is locally path connected but not locally simply connected.

Various papers have referenced (or generalized) the false Proposition 3.1 of [2], including some of the author, and we will comment later in this paper on what can be discarded, safely ignored or possibly repaired.

As a consequence of Theorem 1 we know $\pi_1(HE,p)$ is a Hausdorff space but *not* a topological group with the familiar operations and this begs the question "Is $\pi_1(HE,p)$ regular?"

2. Main result and implications. The *Hawaiian earring HE* is the union of a null sequence of circles joined at a common point p.

Formally HE is the following subspace of the plane $\mathbb{R}^2$. For an integer $n \geq 1$ let X_n denote the circle of radius $1/n$ centered at $(1/n,0)$ and define $HE = \bigcup_{n=1}^{\infty} X_n$.

Let $p = (0,0)$ and let $Y_N = \bigcup_{n=1}^{N} X_n$.

Let $R_N : HE \to Y_N$ denote the natural retraction collapsing $\bigcup_{n=N+1}^{\infty} X_n$ to p. The natural restriction between Y_{N+1} and Y_N determines an inverse limit space $\lim_{\leftarrow} Y_N$ such that $\lim_{\leftarrow} Y_N$ is canonically homeomorphic to HE via $h : HE \to \lim_{\leftarrow} Y_n$ of the format $h(x) = (p,\dots,p,x,x,\dots)$. The map h induces a continuous homomorphism $\phi : \pi_1(HE,p) \to \lim_{\leftarrow} \pi_1(Y_N,p)$, since in general maps between spaces induce continuous homomorphisms [2]. Of critical importance is the nontrivial fact that ϕ is one-to-one [12], [5], [6].

Let $L(HE,p)$ denote the space of maps $f : [0,1] \to HE$ such that $f(0) = f(1) = p$, and endow $L(HE,p)$ with the compact open topology. Since HE is a compact metric space, this is equivalent to the topology of uniform convergence.

Let $q : L(HE,p) \to \pi_1(HE,p)$ denote the canonical quotient map such that $q(f) = q(g)$ if and only if f and g are path homotopic in HE.

Let $p_n = (2/n, 0)$. Define the *oscillation number* $O_n : L(HE, p) \to \{0, 1, 2, 3, 4, \ldots\}$ to be the maximum number m such that there exists a set $T = \{0, t_1, \ldots, t_{2m}\} \subset [0, 1]$ such that $0 < t_1 < \cdots < t_{2m} = 1$ with $f(t_{2i}) = p$ and $f(t_{2i+1}) = p_n$.

REMARK 1. Fixing n and m and allowing k to vary, suppose $f_k \to f$ uniformly in $L(HE, p)$ and $O_n(f_k) \geq m$ as shown by the sets $T_k \subset [0, 1]$ such that $|T_k| = 2m + 1$. Then if $T \subset [0, 1]$ is a subsequential limit of $\{T_k\}$ in the Hausdorff metric, then T shows $O_n(f) \geq m$.

Given $f \in L(HE, p)$, and natural numbers m and n, we obtain a lower bound on $O_n(f)$ as follows. Suppose $f \in L(HE, p)$ and recall $R_m : HE \to Y_m$ is the natural retraction collapsing X_k to p for all $k > m$. If $n > m$ then $p_n \notin Y_m$ (and hence $O_n(R_m(f)) = 0$) and if $n \leq m$ then $O_n(R_m(f)) = O_n(f)$. Thus if $f_1 = R_m(f)$ then $O_n(f_1) \leq O_n(f)$. Let U be a contractible open subspace of Y_m such that $p \in U$. Consider the open set $J = f_1^{-1}(Y_m \backslash \{p\})$ with open interval components $J_1, J_2, \ldots$. Note $J \subset (0, 1)$ and hence $\mathrm{diam}(J_n) \to 0$. Uniform continuity of f ensures $\mathrm{diam}(f_1(J_n)) \to 0$. Thus, with finitely many exceptions, $f_1(\overline{J_i}) \subset U$. If $f_1(\overline{J_i}) \subset U$, replace $f|_{\overline{J_n}}$ by the constant loop at p, to create a function $f_2 : [0, 1] \to Y_m$. If $J^{\hat{}}$ denotes the union of intervals J_k in J such that $f_1(\overline{J_i}) \subset U$, the standard pasting lemma ensures $f_2 = p|_{\overline{J^{\hat{}}}} \cup f_1|_{([0,1] \backslash J^{\hat{}})}$ is continuous, and applying a contraction of U to $f_2|_{\overline{J^{\hat{}}}}$, we see f_2 is path homotopic to f_1 in Y_m. By construction $O_n(f_2) = O(f_1)$.

Recall $\pi_1(Y_m, p)$ is canonically isomorphic to F_m, the free group on m generators $\{x_1, \ldots, x_m\}$ with letters x_i corresponding to one counterclockwise orbit around the circle X_i. Let $w \in F_m$ denote the reduced finite word in F_m corresponding to $[f_2] \in \pi_1(Y_m, p)$. Let $g : [0, 1] \to Y_m$ be the unique path of constant Euclidean speed determined by the word w. Notice $O_n(g)$ is the total number of occurrences of x_n and x_n^{-1} in the word w. Thus if $v \in F_m$ denotes the (unreduced) finite word determined by f_2, then standard word reduction in F_m from v to w (by successive deletion of consecutive inverse pairs $x_i x_i^{-1}$ or $x_j^{-1} x_j$) shows $|v| \geq |w|$. Note $R_m(f)$, f_2 and g are path homotopic in Y_m and hence in HE. Thus in the particular case that $R_m(f)$ is path homotopic to f in HE, the previous discussion is summarized in the following remark.

REMARK 2. Suppose f and g are in the same path component of $L(HE, p)$ and suppose $g : [0, 1] \to Y_m$ is a path of constant speed corresponding to a maximally reduced finite word w in the free group F_m on m letters. Then $O_n(f) \geq O_n(g)$.

REMARK 3. Since $\phi : \pi_1(HE, p) \to \lim_{\leftarrow} \pi_1(Y_n, p)$ is continuous and one-to-one, and since $\lim_{\leftarrow} \pi_1(Y_n, p)$ is a T_2 space, the space $\pi_1(HE, p)$ is T_2. In particular $\pi_1(HE, p)$ is T_1 and hence the path components of $L(HE, p)$ are closed subspaces of $L(HE, p)$.

Remark 4. If Z is a metric space such that each path component of Z is a closed subspace of Z, then each path component of $Z \times Z$ is a closed subspace of $Z \times Z$. (If $(x_n, y_n) \to (x, y)$ and $\{(x_n, y_n)\}$ is in a path component of $Z \times Z$ then obtain paths α and β in Z connecting x to $\{x_n\}$ and y to $\{y_n\}$ and (α, β) is the desired path in $Z \times Z$.)

Theorem 1. *The product of quotient maps $q \times q : L(HE, p) \times L(HE, p) \to \pi_1(HE, p) \times \pi_1(HE, p)$ fails to be a quotient map, standard multiplication (by path class concatenation) $M : \pi_1(HE, p) \times \pi_1(HE, p) \to \pi_1(HE, p)$ is discontinuous, and the fundamental group $\pi_1(HE, p)$ fails to be a topological group with the standard group operations.*

Proof. Let $x_n \in L(HE, p)$ orbit X_n once counterclockwise.

Applying path concatenation, for integers $n \geq 2$ and $k \geq 2$ and $n \neq k$ let $a(n, k) \in L(HE, p)$ be a based loop corresponding to the finite word $(x_n x_k x_n^{-1} x_k^{-1})^{k+n}$ and let $w(n, k) \in L(HE, p)$ be a based loop corresponding to the finite word $(x_1 x_k x_1^{-1} x_k^{-1})^n$.

Let $F \subset \pi_1(HE, p) \times \pi_1(HE, p)$ denote the set of all doubly indexed ordered pairs $([a(n, k)], [w(n, k)])$.

Let $P \in L(HE, p)$ denote the constant map such that $f([0, 1]) = \{p\}$.

To prove $q \times q$ fails to be a quotient map it suffices to prove that F is not closed in $\pi_1(HE, p) \times \pi_1(HE, p)$ and $(q \times q)^{-1}(F)$ is closed in $L(HE, p) \times L(HE, p)$.

To prove F is not closed in $\pi_1(HE, p) \times \pi_1(HE, p)$ we will prove that $([P], [P]) \notin F$ but $([P], [P])$ is a limit point of F.

Recall $\phi : \pi_1(HE, p) \to \lim_{\leftarrow} \pi_1(Y_m, p)$ is one-to-one and $k \geq 2$. Thus $[P] \neq [w(n, k)]$ and $[P] \neq [a(n, k)]$. Thus $([P], [P]) \notin F$.

Suppose $[P] \in U$ and U is open in $\pi_1(HE, p)$. Let $V = q^{-1}(U)$. Then V is open in $L(HE, p)$ since, by definition, q is continuous.

Note $P \in V$. Thus there exist N and K such that if $n \geq N$ and $k \geq K$ then $a(n, k) \in V$. Note $(x_1 x_1^{-1})^N$ is path homotopic to P and hence $(x_1 x_1^{-1})^N \in V$. Observe that $w(N, k)$ (suitably parameterized over $[0, 1]$) converges to $(x_1 x_1^{-1})^N$ uniformly in $L(HE, p)$. Thus there exists $K_2 \geq K$ such that if $k \geq K_2$ then $w(N, k) \in V$. Hence $([w(N, K_2)], [a(N, K_2)]) \in U \times U$. This proves $([P], [P])$ is a limit point of F, and thus F is not closed in $\pi_1(HE, p) \times \pi_1(HE, p)$.

To prove $(q \times q)^{-1}(F)$ is closed in $L(HE, p) \times L(HE, p)$ suppose that $(f_m, g_m) \to (f, g)$ uniformly and $(f_m, g_m) \in (q \times q)^{-1}(F)$. Note $O_1(w(n, k)) = 2n$ and $O_N(a(N, k)) \geq 2(N + k)$.

Let $a(n_m, k_m)$ and $w(n_m, k_m)$ be path homotopic to respectively f_m and g_m.

By Remark 2, $O_1(g_m) \geq O_1(w(n_m, k_m)) = 2n_m$.

Thus if $\{n_m\}$ contains an unbounded subsequence then, by Remark 1, $O_1(g) \geq \limsup O_1(w(n_m, k_{n_m})) = \infty$ and we have a contradiction since $O_1(g) < \infty$. Thus $\{n_m\}$ is bounded and so takes on finitely many values.

In similar fashion, if $\{k_m\}$ is unbounded then there exists N and a subsequence $\{k_{m_l}\}$ such that $O_N(a(N, k_{m_l})) \to \infty$. It follows that $O_N(f) \geq \limsup O_N(a(N, k_{m_l})) = \infty$, contradicting the fact that $O_N(f) < \infty$.

Thus both $\{n_m\}$ and $\{k_m\}$ are bounded and hence (by the pigeon hole principle) there exists a path component $B \subset L(HE, p) \times L(HE, p)$ containing a subsequence (f_{m_l}, g_{m_l}).

It follows from Remarks 3 and 4 that $(f, g) \in B$. Thus $(q \times q)^{-1}(F)$ is closed and hence $q \times q$ fails to be a quotient map.

In similar fashion we will prove that group multiplication $M : \pi_1(HE, p) \times \pi_1(HE, p) \to \pi_1(HE, p)$ is discontinuous, and hence $\pi_1(HE, p)$ will fail to be a topological group with the standard group operations. To achieve this we will exhibit a closed set $A \subset \pi_1(HE, p)$ such that $M^{-1}(A)$ is not closed in $\pi_1(HE, p) \times \pi_1(HE, p)$.

Consider the doubly indexed set $A = M(F) \subset \pi_1(HE, p)$ such that each element of A is of the form $[a(n, k)] * [w(n, k)]$ (with $*$ the denoting familiar path class concatenation).

On the one hand observe by definition (and since ϕ is one-to-one) $[a(n, k)] * [w(n, k)] \neq [P]$. Thus $[P] \notin A$ and $([P], [P]) \notin M^{-1}(A)$. Note $F \subset M^{-1}(A)$ and by the previous argument $([P], [P])$ is a limit point of F. Thus $M^{-1}(A)$ is not closed in $\pi_1(HE, p) \times \pi_1(HE, p)$.

On the other hand we will prove A is closed in $\pi_1(HE, p)$ by proving $q^{-1}(A)$ is closed in $L(HE, p)$. Suppose that $f_m \to f \in L(HE, p)$ and $f_m \in q^{-1}(A)$.

Obtain n_m and k_m such that $f_m \in [a(n_m, k_m)] * [w(n_m, k_m)]$.

In similar fashion to the previous proof, if $\{n_m\}$ is unbounded we obtain the contradiction $O_1(f) \geq \limsup O_1(f_m) = \infty$.

If $\{n_m\}$ is bounded and $\{k_m\}$ is unbounded we obtain N and a subsequence k_{m_l} and the contradiction $O_N(f) \geq \limsup O_N(f_{m_l}) = \infty$.

Thus both $\{n_m\}$ and $\{k_m\}$ are bounded. It follows by the pigeon hole principle that some path component $B \subset L(HE, p)$ contains a subsequence $\{f_{m_l}\}$ and Remark 3 implies that $f \in B$. Hence $q^{-1}(A)$ is closed in $L(HE, p)$ and thus A is closed in $\pi_1(HE, p)$. ∎

Theorem 1 contradicts some published claims to the contrary and we offer brief assessment of how this affects various published results.

It is falsely claimed in [2], [1] that group multiplication in $\pi_1(X, p)$ is continuous and that $\pi_1(X, p)$ is a topological group. However these mistakes do not appear to directly affect arguments elsewhere in the papers (some of which have also been challenged [10]).

The introduction in [10] mentions that $\pi_1(X,p)$ is a topological group, but apparently none of the results or proofs are affected by this remark.

Theorem 2 of [9] claims that $\pi_1(X,p)$ is regular iff $\pi_1(X,p)$ is a T_1 space. This claim is suspicious since the proof assumes (incorrectly) that $\pi_1(X,p)$ is a topological group, and Example 4.22 of [3] shows the T_1 property of $\pi_1(X,p)$ does not guarantee that $\pi_1(X,p)$ is completely regular.

The paper [8] develops a false generalization (Theorem 2.7) of Proposition 3.1 of [2] and the proof makes the common mistake of falsely assuming that the product of quotient maps is a quotient map. However the main application (the validity of Theorem 3.12 which constructs a completion $\overline{B_\infty}$ of the infinite braid group) in [8] is unlikely to be affected (since with the quotient topology, the pure braid subgroup of $\overline{B_\infty}$ is the topological inverse limit of the finite pure braid groups, and hence a topological group). Nevertheless it would be appropriate to provide a new and careful argument that multiplication in $\overline{B_\infty}$ is continuous.

In [14], Lemma 1.1 and its proof assert (falsely) that $\pi_1(X,p)$ is a topological group. In [11], Theorem 2.1 and its proof assert (falsely) that $\pi_n(X,p)$ is a topological group for all n, and the familiar mistake is to assume that the product of quotient maps is a quotient map.

3. Summary. The Hawaiian earring HE is a locally path connected compact metric space whose fundamental group $\pi_1(HE,p)$ is shown in this paper to have discontinuous group multiplication with a certain natural topology on $\pi_1(HE,p)$. The topology of $\pi_1(HE,p)$ is the quotient topology inherited under the natural map from the space of p-based loops in HE.

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