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*MEASURE AND INTEGRATION*

# The Young Measure Representation for Weak Cluster Points of Sequences in $M$-spaces of Measurable Functions

by

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*Presented by Czesław OLECH*

**Summary.** Let $\langle X, Y\rangle$ be a duality pair of $M$-spaces $X, Y$ of measurable functions from $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^d$. The paper deals with $Y$-weak cluster points $\overline{\phi}$ of the sequence $\phi(\cdot, z_j(\cdot))$ in $X$, where $z_j : \Omega \to \mathbb{R}^m$ is measurable for $j \in \mathbb{N}$ and $\phi\colon \Omega \times \mathbb{R}^m \to \mathbb{R}^d$ is a Carathéodory function. We obtain general sufficient conditions, under which, for some negligible set $A_\phi$, the integral $I(\phi, \nu_x) := \int_{\mathbb{R}^m} \phi(x, \lambda)\, d\nu_x(\lambda)$ exists for $x \in \Omega \setminus A_\phi$ and $\overline{\phi}(x) = I(\phi, \nu_x)$ on $\Omega \setminus A_\phi$, where $\nu = \{\nu_x\}_{x\in\Omega}$ is a measurable-dependent family of Radon probability measures on $\mathbb{R}^m$.

**1. Notations and some basic facts on Young measures.** Let $\mu$ denote a complete separable $\sigma$-finite $\sigma$-additive positive measure on a $\sigma$-algebra $\mathfrak{A}$ of subsets of a set $\Omega$. Measurability will always mean $\mathfrak{A}$-measurability. Let $E$ be a separable Banach space. We will denote by $L^\infty(\Omega, E; \mu)$, or briefly $L^\infty(E)$, the Banach space (of all equivalence classes) of essential $E$-norm-bounded measurable functions $u\colon \Omega \to E$ with norm $\|u\|_{L^\infty} := \operatorname{ess\,sup}_{x\in\Omega} \|u(x)\|_E$. Let $L^1(\Omega, E; \mu)$, or briefly $L^1(E)$, denote the Bochner–Lebesgue space (of all equivalence classes) of $\mu$-integrable strongly measurable functions from $\Omega$ into $E$.

Let $\mathcal{M}(\mathbb{R}^m)$ be the Banach space of bounded signed Radon measures on $\mathbb{R}^m$ and $C_0(\mathbb{R}^m)$ be the Banach space of all continuous functions $f\colon \mathbb{R}^m \to \mathbb{R}$ with $\lim_{|\lambda|\to\infty} f(\lambda) = 0$ equipped with the sup-norm, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^m$. It is known that $(C_0(\mathbb{R}^m))^* \cong \mathcal{M}(\mathbb{R}^m)$. Let $L^\infty_\omega(\mathcal{M}(\mathbb{R}^m))$ denote the Banach space (of all equivalence classes) of $C_0(\mathbb{R}^m)$-

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weakly measurable functions $\nu : \Omega \to \mathcal{M}(\mathbb{R}^m)$ with norm $\|\nu\|_\infty := \|x \mapsto |\nu_x|(\mathbb{R}^m)\|_{L^\infty} < \infty$, where $|\nu_x|(\mathbb{R}^m)$ is the total variation of $\nu_x$ on $\mathbb{R}^m$ and, for abbreviation, we write $L^\infty$ (resp. $\nu_x$) instead of $L^\infty(\mathbb{R})$ (resp. $\nu(x)$). It is known that $L^\infty_\omega(\mathcal{M}(\mathbb{R}^m))$ can be interpreted as dual space $(L^1(C_0(\mathbb{R}^m)))^*$ via the injection $\nu \mapsto \langle \cdot, \nu\rangle_\mu$, where $\langle h, \nu\rangle_\mu := \int_\Omega \langle v(x), h(x)\rangle\, d\mu(x)$ for all $h \in L^1(C_0(\mathbb{R}^m))$. Given a measurable function $z\colon \Omega \to \mathbb{R}^m$, define the *parametrized Dirac measure* $\delta_z \in L^\infty_\omega(\mathcal{M}(\mathbb{R}^m))$ by

$$x \in \Omega \mapsto \delta_z(x) := \delta_{z(x)} \quad \text{(the Dirac measure supported at } z(x)).$$

An element $\nu \in L^\infty_\omega(\mathcal{M}(\mathbb{R}^m))$ is called a *Young parametrized measure* if $\nu_x(\mathbb{R}^m) = 1$ $\mu$-a.e. Define $(\phi \circ z)(x) := \phi(x, z(x))$. A function $f\colon \Omega\times\mathbb{R}^m \to E$ is said to be *Carathéodory* if $f(\cdot, u)$ is measurable for every $u \in \mathbb{R}^m$ and $f(x, \cdot)$ is continuous for almost all $x \in \Omega$.

The formulations and proofs of the main results of the present paper are based on the following fundamental theorem [2, 3] about the Young measure representation in case of the pair $\langle X, Y\rangle = \langle L^1(\mathbb{R}), L^\infty(\mathbb{R})\rangle$ (see Theorem 1.1; cf. [20, p. 98–100], [8, Section 8.1, pp. 518–525], [5, 21]).

Theorem 1.1 (The Young measure representation; Ball [3], Balder [2]). *Suppose that a sequence of measurable functions $z_j\colon \Omega \to \mathbb{R}^m$ satisfies the global tightness condition with respect to $\mu$:*

$$\text{(GB)} \qquad \lim_{L\to\infty} \sup_{j\in\mathbb{N}} \mu\{x \in \Omega : |z_j(x)| \ge L\} = 0.$$

*Then there exist a subsequence $z_{j_k}$ and a Young measure $\nu = \{\nu_x\}_{x\in\Omega}$ such that $\delta_{z_{j_k}}$ is $L^1(C_0(\mathbb{R}^m))$-weakly convergent to $\nu$ in $L^\infty_\omega(\mathcal{M}(\mathbb{R}^m))$. Moreover, given a Carathéodory function $\psi\colon \Omega \times \mathbb{R}^m \to \mathbb{R}$, the following statements hold.*

(Y1) *If $\psi \circ z_{j_k}$ is $L^\infty(\mathbb{R})$-weakly convergent to $\overline{\psi}$ in $L^1(\mathbb{R})$, then, for some $\mu$-negligible set $A_\psi \in \mathfrak{A}$, the integral $\int_{\mathbb{R}^m} \psi(x, \lambda)\, d\nu_x(\lambda) \in \mathbb{R}$ exists for $x \in \Omega \setminus A_\psi$ and*

$$\overline{\psi}(x) = \int_{\mathbb{R}^m} \psi(x, \lambda)\, d\nu_x(\lambda) \quad \text{on } \Omega \setminus A_\psi. \tag{1.1}$$

(Y2) *If $\psi \circ z_{j_k}$ is sequentially $L^\infty(\mathbb{R})$-weakly pre-compact in $L^1(\mathbb{R})$, then, for some $\mu$-negligible set $A_\psi \in \mathfrak{A}$, the integral $I(\psi, \nu_x) := \int_{\mathbb{R}^m} \psi(x, \lambda)\, d\nu_x(\lambda) \in \mathbb{R}$ exists for all $x \in \Omega \setminus A_\psi$ and $\psi \circ z_{j_k}$ is $L^\infty(\mathbb{R})$-weakly convergent to $\widetilde{\psi} \in L^1(\mathbb{R})$, where $\widetilde{\psi}(x) := I(\psi, \nu_x)$ for $x \in \Omega \setminus A_\psi$ and $\widetilde{\psi}(x) := 0$ otherwise.*

The generalization of Theorem 1.1 for the $L^{\Psi^*}(\mathbb{R})$-weak limit of $\tau \circ z_{j_k}$ in the Orlicz space $L^\Psi(\mathbb{R})$ is proved by P. Málek et al. [11, Th. 4.2.1, pp. 171–176] in the case when $\Psi$ and $\Psi^*$ are complementary non-power Orlicz functions, $\Psi$ satisfies the $\triangle_2$-condition [12], and $\tau\colon \mathbb{R}^m \to \mathbb{R}$ is continuous.

**2. Formulation of results.** A linear space $Z \subset L^0(\mathbb{R}^m)$ is called an *$M$-space* if the inclusions $z \in Z$ and $\alpha \in L^\infty(\mathbb{R})$ imply that $\alpha z \in Z$ [16, 18]. If $m = 1$ then it is easy to check that $M$-spaces $Z$ are just vector lattices. The *Köthe associate space $Z'$ with respect to $\mu$* of an $M$-space $Z$ is defined e.g. in [7, 9] for $m = 1$, and in [15, 14, 18] for $m \geq 2$. By [16, Theorem 3.1], equivalently in case $m \geq 2$, $Z'$ is defined by

$$Z' = \{z' \in L^0(\mathbb{R}^m) : z'(x) \in \operatorname{vsupp} Z(x)\ \mu\text{-a.e.}, \langle z, z'\rangle_\mu \in \mathbb{R}, \forall z \in Z\}.$$

Here $\langle z, z'\rangle_\mu := \int_\Omega (z(x), z'(x))\, d\mu(x)$, where $(\cdot,\cdot)$ denotes the Euclidean scalar product on $\mathbb{R}^m$, and the so-called *vector support* $\operatorname{vsupp} Z$ can be equivalently defined by

$$\operatorname{vsupp} Z(x) := \overline{\{z_1(x), z_2(x), \ldots\}} \quad \mu\text{-a.e.}$$

for some sequence $z_n \in Z$ such that $z \in Z \Rightarrow z(x) \in \overline{\{z_1(x), z_2(x), \ldots\}}$ $\mu$-a.e. If $Z, Y \subset L^0(\mathbb{R}^m)$ are $M$-spaces and $Y \subset Z'$, then $\langle Z, Y\rangle$ is a duality pair with respect to $\langle z, z'\rangle_\mu$ ($z \in Z, z' \in Y$), and we write $\langle Z, Y\rangle_\mu$.

Let $\langle Z, Y\rangle$ be a duality pair of vector spaces. A set $\mathcal{N} \subset Z$ is called *sequentially $Y$-weakly pre-compact in $Z$* (or *conditionally sequentially $Y$-weakly compact in $Z$*) if each sequence $z_j \in \mathcal{N}$ has some $Y$-weak Cauchy subsequence $z_{j(k)}$. The space $Z$ is called *sequentially $Y$-weakly complete* if each $Y$-weak Cauchy sequence is $Y$-weakly convergent in $Z$.

Theorem 2.1. *Let $X, Y \subset L^0(\mathbb{R}^d)$ be $M$-spaces*, $\operatorname{supp} X = \Omega$, $\operatorname{vsupp} X(x) = \operatorname{vsupp} Y(x)$ *$\mu$-a.e., and $Y \subset X'$, where $X'$ is the Köthe associate space of $X$ with respect to $\mu$. Suppose that a sequence $z_j \in L^0(\mathbb{R}^m)$ satisfies* (GB) *with respect to $\mu$, and a Carathéodory function $\phi\colon \Omega \times \mathbb{R}^m \to \mathbb{R}^d$ satisfies $\phi(x, \mathbb{R}^m) \subset \operatorname{vsupp} X(x)$ $\mu$-a.e. Moreover, let $z_{j_k}$ and $\nu$ be as in Theorem* 1.1. *Then the following statements hold.*

(Y3) *If $\phi \circ z_{j_k}$ is $Y$-weakly convergent to $\overline{\phi}$ in $X$, then, for some $\mu$-negligible set $A_\phi \in \mathfrak{A}$, the integral $\int_{\mathbb{R}^m} \phi(x, \lambda)\, d\nu_x(\lambda)$ exists in* $\operatorname{vsupp} X(x)$ *for $x \in \Omega \setminus A_\phi$ and*

$$\overline{\phi}(x) = \int\limits_{\mathbb{R}^m} \phi(x, \lambda)\, d\nu_x(\lambda) \quad \text{on } \Omega \setminus A_\phi. \tag{2.1}$$

(Y4) *If $X = Y'$ and $\phi \circ z_{j_k}$ is sequentially $Y$-weakly pre-compact in $X$, then, for some $\mu$-negligible set $A_\phi \in \mathfrak{A}$, the integral $I(\phi, \nu_x) := \int_{\mathbb{R}^m} \phi(x, \lambda)\, d\nu_x(\lambda)$ exists in* $\operatorname{vsupp} X(x)$ *for all $x \in \Omega \setminus A_\phi$ and $\phi \circ z_{j_k}$ is $Y$-weakly convergent to $\widetilde{\phi}$ in $X$, where $\widetilde{\phi}(x) := I(\phi, \nu_x)$ for $x \in \Omega \setminus A_\phi$ and $\widetilde{\phi}(x) := 0$.*

Condition 2.2 (Local tightness condition, [11, p. 171], [20]). *A sequence $z_j \in L^0(\mathbb{R}^m)$ satisfies*

(LB) $$\lim_{L\to\infty}\sup_{j\in\mathbb{N}}\mu\{x\in C_q : |z_j(x)|\ge L\} = 0 \quad (\forall q\in\mathbb{N})$$

*for a nondecreasing sequence* $C_q \in \mathfrak{A}$ *with* $\mu(C_q)<\infty$ *and* $\bigcup_{q\in\mathbb{N}} C_q = \Omega$.

THEOREM 2.3. *Let* $\mu(\Omega)=\infty$ *and let* $X$, $Y$ *and* $\phi$ *be as in Theorem* 2.1. *If a sequence* $z_j\in L^0(\mathbb{R}^m)$ *satisfies* (LB) *with respect to* $\mu$, *then the statements* (Y3)–(Y4) *of Theorem* 2.1 *remain true.*

A normed space $Z\subset L^0(\mathbb{R}^m)$ with norm $\|\cdot\|_Z$ is called a *normed $M$-space* if the inclusions $z\in Z$ and $\alpha\in L^\infty(\mathbb{R})$ imply that $\alpha z\in Z$ and $\|\alpha z\|_Z \le \|\alpha\|_{L^\infty}\|z\|_Z$ [16, 18]. The *regular part* $Z^\circ$ of a normed $M$-space $Z$ is defined to be the normed $M$-subspace of all elements $z\in Z$ satisfying $\lim_{\mu_*(D)\to 0}\|\chi_D z\|_Z = 0$, where $\mu_* := \mu$ if $\mu(\Omega)<\infty$ and $\mu_*$ is a fixed finite positive measure equivalent to $\mu$ if $\mu(\Omega)=\infty$, and $\chi_D$ denotes the characteristic function of $D\in\mathfrak{A}$.

PROPOSITION 2.4. *Let* $X, Y\subset L^0(\mathbb{R}^d)$ *be $M$-spaces with* $X\subset Y'$, *where* $Y'$ *is the Köthe associate space of* $Y$ *with respect to* $\mu$. *Suppose that a sequence* $z_j\in L^0(\mathbb{R}^m)$ *and a Carathéodory function* $\phi\colon \Omega\times\mathbb{R}^m\to\mathbb{R}^d$ *satisfy one of the following conditions*:

(SC1) *There exist nondecreasing continuous functions* $g,\gamma\colon[0,\infty)\to[0,\infty)$ *such that*

- (a) $\lim_{t\to\infty} g(t)=\infty$ *and* $\lim_{t\to\infty}\gamma(t)/g(t)=0$;
- (b) $\{(g\circ|z_j|)u_0\}_{j\in\mathbb{N}}$ *is $Y$-weakly bounded in* $X$, *where* $u_0\colon\Omega\to(0,\infty)$ *is measurable*, $u_0 Y\subset L^1(\mathbb{R}^d)$, *and* $\operatorname{vsupp} X(x) = \operatorname{vsupp} Y(x)$ *$\mu$-a.e.*;
- (c) $|\phi(x,\lambda)|\le\gamma(|\lambda|)u_0(x)$ *for $\mu$-almost all* $x\in\Omega$ *and all* $\lambda\in\mathbb{R}^m$;

(SC2) *There exists a Banach $M$-space* $\Gamma$ *with* $Y\subset\Gamma^\circ$, $(\Gamma^\circ)'\subset X$ *and*

$$\sup_{j\in\mathbb{N}}\|\phi\circ z_j\|_{(\Gamma^\circ)'}<\infty.$$

*Then the sequence* $\phi\circ z_j$ *is sequentially $Y$-weakly pre-compact in* $X$.

REMARK 2.5. Proposition 2.4/(SC1) is a generalization of [20, Proposition 6.5] (where $Y=L^1(\mathbb{R})$ with $\mu(\Omega)<\infty$).

In the case of $\phi\colon\Omega\times\mathbb{R}^m\to E$ with $\dim E=\infty$, results analogous to Theorems 2.1 and 2.3 can be proved but only for a pair $\langle X,Y\rangle$ of Köthe–Bochner spaces $X, Y$ of $E$-/$E^*$-valued functions (see Theorem 2.6). Given a separable Banach space $E$ and a vector lattice $K\subset L^0(\mathbb{R})$, the *Köthe–Bochner space* $K(E)$ is defined as the space (of equivalence classes) of strongly measurable $E$-valued functions $z$ such that $\|z(\cdot)\|_E\in K$.

THEOREM 2.6. *Let* $K,\widetilde{K}\subset L^0(\mathbb{R})$ *be vector lattices*, $E$ *be a Banach space and* $E^*$ *be its dual. Assume that*:

(a) $\operatorname{supp} K = \operatorname{supp} \widetilde{K} = \Omega$ *and* $\widetilde{K} \subset K'$, *where* $K'$ *is the Köthe associate space of* $K$ *with respect to* $\mu$;

(b) $E$ *is separable and reflexive with* $\dim E = \infty$.

*If* $\phi\colon \Omega \times \mathbb{R}^m \to E$ *is a Carathéodory function and a sequence* $z_j \in L^0(\mathbb{R}^m)$ *satisfies either* (GB) *or* (LB), *then the statements* (Y3)–(Y4) *of Theorem* 2.1 *remain true for the Köthe–Bochner spaces* $X = K(E)$ *and* $Y = \widetilde{K}(E^*)$ *provided* (2.1) (*resp.* $\widetilde{\phi}$) *is substituted by*

$$\overline{\phi}(x) = (P)\text{-}\int_{\mathbb{R}^m} \phi(x,\lambda)\, d\nu_x(\lambda) \qquad on\ \Omega \setminus A_\phi \tag{2.2}$$

(*resp.* $\widetilde{\phi}(x) = (P)\text{-}\int_{\mathbb{R}^m} \phi(x,\lambda)\, d\nu_x(\lambda)$ *for* $x \in \Omega \setminus A_\phi$), *where, for* $x \in \Omega \setminus A_\phi$, *the above integral exists as the Pettis integral of the function* $\phi(x,\cdot) : \mathbb{R}^m \to E$ *with respect to the measure* $\nu_x$.

Proposition 2.7. *Let* $z_j \in L^0(\mathbb{R}^m)$ ($j \in \mathbb{N}$). *Then* (LB) *with respect to* $\mu$ *follows from the condition*:

(LK) *For* $q \in \mathbb{N}$ *there exist a normed lattice with monotone norm* $K(q) \subset L^0(\mathbb{R})$ *and a continuous nondecreasing function* $g_q\colon [0,\infty) \to [0,\infty)$ *such that* $\lim_{t\to\infty} g_q(t) = \infty$ *and* $\sup_{j\in\mathbb{N}} \|\chi_{C_q} g_q(|z_j(\cdot)|)\|_{K(q)} < \infty$ *for a nondecreasing sequence* $C_q \in \mathfrak{A}$ *with* $\mu(C_q) < \infty$ *and* $\bigcup_{q\in\mathbb{N}} C_q = \Omega$.

Remark 2.8. Proposition 2.7 is an extension of the statement in [3, Remark 1, p. 209] (where $K(q) = L^1(\mathbb{R})$).

Remark 2.9. If $Z \subset L^0(\mathbb{R}^m)$ is a normed $M$-space and $\sup_{j\in\mathbb{N}} \|z_j\|_Z < \infty$, then (LB) holds. Indeed, by [9, Corollary of Theorem IV.3.1], [23] ($m = 1$) and [16, Theorem 2.1/(3)] ($m \geq 2$), the sequence $z_j$ is bounded in $L^0(\mathbb{R}^m)$ equipped with the quasi-norm $\|z\|_{L^0(\mathbb{R}^m)} := \int_\Omega \frac{|z(x)|}{1+|z(x)|}\, d\mu_*(x)$. Hence, by [9, Section III.1.3–III.1.4], this sequence is bounded in $\mu$ on any $C_q$, and so (LB) follows. In particular, $Z$ can be assumed to be either a Banach lattice of scalar-valued functions (a solid space) or a non-solid generalized Orlicz space (see, e.g., [1, 12, 17]) of $\mathbb{R}^m$-valued functions with $m \geq 2$.

## 3. Proofs of results of Section 2

*Proof of Theorem 2.1.* We divide this proof into Steps 3.1–3.2.

Step 3.1 (Proof of (Y3)). Given $y \in Y$, define $\phi_y\colon \Omega \times \mathbb{R}^m \to \mathbb{R}$ by $\phi_y(x,\lambda) := (y(x), \phi(x,\lambda))$. As $Y$ is an $M$-space we have $\alpha y \in Y$ for every $\alpha \in L^\infty(\mathbb{R})$, and from $Y \subset X'$ we infer that

$$\langle \phi \circ z_{j_k}, \alpha y\rangle_\mu = \langle \phi_y \circ z_{j_k}, \alpha\rangle_\mu \in \mathbb{R}.$$

By Theorem 1.1/(Y2) for $\phi_y$ together with the assumption for $\phi \circ z_{j_k}$, we deduce that

$$\langle \phi \circ z_{j_k}, \alpha y\rangle_\mu = \langle \phi_y \circ z_{j_k}, \alpha\rangle_\mu \to \langle \overline{\phi}, \alpha y\rangle_\mu = \langle \widetilde{\phi}_y, \alpha\rangle_\mu \in \mathbb{R}$$

for all $\alpha \in L^\infty(\mathbb{R})$, where, for some $\widetilde{D}_{\phi y} \in \mathfrak{A}$ with $\mu(\Omega \setminus \widetilde{D}_{\phi y}) = 0$, the integral $\int_{\mathbb{R}^m} \phi_y(x,\lambda)\, d\nu_x(\lambda) \in \mathbb{R}$ exists for $x \in \widetilde{D}_{\phi y}$, and $\widetilde{\phi}_y(x) := \int_{\mathbb{R}^m} \phi_y(x,\lambda)\, d\nu_x(\lambda)$ for $x \in \widetilde{D}_{\phi y}$ and $\widetilde{\phi}_y(x) := 0$ otherwise. Hence,

$$\begin{aligned}\langle \overline{\phi}, \chi_D y\rangle_\mu &= \langle \widetilde{\phi}_y, \chi_D\rangle_\mu \\ &= \int_D \Big[\int_{\mathbb{R}^m} (y(x), \phi(x,\lambda))\, d\nu_x(\lambda)\Big]\, d\mu(x) \in \mathbb{R} \quad (D \in \mathfrak{A}, D \subset \widetilde{D}_{\phi y}).\end{aligned}$$

On the other hand, $\langle \overline{\phi}, \chi_D y\rangle_\mu = \int_D (y(x), \overline{\phi}(x))\, d\mu(x)$ for any $D \in \mathfrak{A}$ with $D \subseteq \widetilde{D}_{\phi y}$. By the Radon–Nikodym theorem, we deduce that for $y \in Y$ there exists $D_{\phi y} \in \mathfrak{A}$ such that $D_{\phi y} \subset \widetilde{D}_{\phi y}$, $\mu(\widetilde{D}_{\phi y} \setminus D_{\phi y}) = 0$, and

$$(y(x), \overline{\phi}(x)) = \int_{\mathbb{R}^m} (y(x), \phi(x,\lambda))\, d\nu_x(\lambda) \in \mathbb{R} \quad (\forall x \in D_{\phi y}). \tag{3.1}$$

Now, we consider $X \subset L^0(\Omega, \mathbb{R}^d)$ and $Y \subset X'$ for $d > 1$ (the case $d = 1$ can be handled analogously upon using [9, Corollary IV.3.2] for $\operatorname{supp} Y = \operatorname{supp} X = \Omega$). By [16, Theorem 3.1], there exists a sequence of representative families $G_q = \{u_{1q}, \dots, u_{dq}\}$ of the $M$-space $Y$ such that the sets $\operatorname{supp} G_q \in \mathfrak{A}$ are mutually disjoint, and

(1) $\mu(\operatorname{supp} Y \setminus \bigcup_{q=1}^\infty \operatorname{supp} G_q) = 0$;
(2) $|u_{1q}(x)| = \dots = |u_{d(q)q}(x)| = 1$ and $|u_{iq}(x)| = 0$ ($i \notin \{1, \dots, d(q)\}$) for $x \in \operatorname{supp} G_q$ and $d(q) = \dim \operatorname{vsupp} Y(x)$ on $\operatorname{supp} G_q$.

By the definition [16] of the representative family $G_q$, we have $u_{iq} \in Y$ and the linear hull of $\{u_{1q}(x), \dots, u_{dq}(x)\}$ coincides with $\operatorname{vsupp} Y(x)$ for $x \in \operatorname{supp} G_q$. Hence, by (3.1), for $\chi_{\operatorname{supp} G_q} u_{pq} \in Y$ ($1 \le p \le d(q)$) there exists $D_{pq} \in \mathfrak{A}$ such that $D_{pq} \subset \operatorname{supp} G_q$, $\mu(\operatorname{supp} G_q \setminus D_{pq}) = 0$, and

$$(\chi_{\operatorname{supp} G_q}(x) u_{pq}(x), \overline{\phi}(x)) = \int_{\mathbb{R}^m} (\chi_{\operatorname{supp} G_q}(x) u_{pq}(x), \phi(x,\lambda))\, d\nu_x(\lambda) \in \mathbb{R}$$

for $x \in D_{pq}$. By the assumption, there exists $D_0 \in \mathfrak{A}$ with $\mu(\Omega \setminus D_0) = 0$ such that $\overline{\phi}(x), \phi(x,\lambda) \in \operatorname{vsupp} X(x) = \operatorname{vsupp} Y(x)$ for all $x \in D_0$ and for all $\lambda \in \mathbb{R}^m$. Hence, for $x \in D_0 \cap \bigcap_{p=1}^{d(q)} D_{pq}$ and $1 \le p \le d(q)$, the integral $\int_{\mathbb{R}^m} \phi(x,\lambda)\, d\nu_x(\lambda)$ exists in the finite-dimensional Euclidean space $\operatorname{vsupp} Y(x) = \operatorname{vsupp} X(x)$ and

$$(u_{pq}(x), \overline{\phi}(x)) = \Big(u_{pq}(x), \int_{\mathbb{R}^m} \phi(x,\lambda)\, d\nu_x(\lambda)\Big) \in \mathbb{R}.$$

Therefore,

$$\overline{\phi}(x) = \int\limits_{\mathbb{R}^m} \phi(x,\lambda)\, d\nu_x(\lambda) \in \operatorname{vsupp} X(x)$$

for $x \in D_\phi := \bigcup_{q=1}^{\infty}[D_0 \cap \bigcap_{p=1}^{d(q)} D_{pq}]$, and $\mu(\Omega \setminus D_\phi) = 0$. Hence the statement (Y3) follows for $A_\phi := \Omega \setminus D_\phi$.

Step 3.2 (Proof of (Y4)). Observe that, as $X = Y'$, there exist a subsequence $j(k)$ of $j_k$ and $\overline{\phi} \in X$ such that $\phi \circ z_{j(k)}$ is $Y$-weakly convergent to $\overline{\phi}$ in $X$, due to the $Y$-weak completeness theorem of J. Dieudonné [7] (if $X$ is a normed lattice with $Y = X'$); W. Luxemburg and A. Zaanen [10], P. P. Zabrejko [23, Theorem 32] (if $X$ is a normed lattice); H. Nakano [13] ($d = 1$ with $Y = X'$); O. Burkinshaw and P. Dodds [4, Corollary 4.2 of Theorem 4.1] ($d = 1$) and [15, Theorem 2.8/(1)], [18] ($d \geq 2$).

By Theorem 2.1/(Y3) applied to $\phi \circ z_{j(k)}$, we can find $A_\phi \in \mathfrak{A}$ such that $\mu(A_\phi) = 0$ and the integral $\int_{\mathbb{R}^m} \phi(x,\lambda)\, d\nu_x(\lambda)$ exists in $\operatorname{vsupp} X(x)$ for $x \in \Omega \setminus A_\phi$.

We proceed to show that $\phi \circ z_{j_k}$ is $Y$-weakly convergent to $\widetilde{\phi}$ in $X$, where $\widetilde{\phi}(x) := \int_{\mathbb{R}^m} \phi(x,\lambda)\, d\nu_x(\lambda)$ for $x \in \Omega \setminus A_\phi$ and $\widetilde{\phi}(x) := 0$ otherwise.

On the contrary, suppose that $\phi \circ z_{j_k}$ is not $Y$-weakly convergent to $\widetilde{\phi}$ in $X$. Then there exist $\varepsilon > 0$, $h_0 \in Y$ and a subsequence $q_k$ of $j_k$ such that $|\langle \phi \circ z_{q_k}, h_0\rangle_\mu - \langle \widetilde{\phi}, h_0\rangle_\mu| > \varepsilon > 0$. By the above $Y$-weak completeness theorem together with Theorem 2.1/(Y3), for the sequence $\phi \circ z_{q_k}$ we can find a subsequence $i_k$ of $q_k$, $\widehat{\phi} \in X$ and $A_{\widehat{\phi}} \in \mathfrak{A}$ such that $\langle \phi \circ z_{i_k}, h\rangle_\mu \to \langle \widehat{\phi}, h\rangle_\mu$ $(\forall h \in Y)$, $\mu(A_{\widehat{\phi}}) = 0$, the integral $\int_{\mathbb{R}^m} \phi(x,\lambda)\, d\nu_x(\lambda)$ exists in $\operatorname{vsupp} X(x)$ for $x \in \Omega \setminus A_{\widehat{\phi}}$, and $\widehat{\phi}(x) = \int_{\mathbb{R}^m} \phi(x,\lambda)\, d\nu_x(\lambda)$ on $\Omega \setminus A_{\widehat{\phi}}$. Therefore, $\widehat{\phi}$ and $\widetilde{\phi}$ define the same element (equivalence class) in $X$, and $\langle \widehat{\phi}, h_0\rangle_\mu = \langle \widetilde{\phi}, h_0\rangle_\mu$. Hence, we get a contradiction. ∎

Proposition 3.1 ([15, Lemma 4.2.2]). *Let $\mu(\Omega) = \infty$. Then, for a sequence $z_j \in L^0(\mathbb{R}^m)$, the condition* (LB) *holds with respect to $\mu$ if and only if the condition* (GB) *holds with respect to $\mu_*$.*

*Proof of Theorem 2.3.* By Proposition 3.1, (LB) for $\mu$ and $z_j$ implies (GB) for $\mu_*$ and $z_j$. So, we may apply Theorem 2.1 for $z_j$ with respect to $\mu_*$. Recall that if $\mu(\Omega) = \infty$ then the measure $\mu$ is called *separable* (see [9, 23]) provided $\mu_*$ is separable, which is equivalent to separability of $L^0(\mathbb{R}^m)$. We divide the proof into Steps 3.3–3.4.

Step 3.3 (Proof of (Y3)). Denote by $\alpha_* \in L^1((0,\infty))$ the Radon–Nikodym derivative $d\mu_*/d\mu$. Define

$$\widetilde{Y} := \{\tilde{z}' : \alpha_* \tilde{z}' \in Y\},$$

$$\widetilde{Y}'_{\mu_*} := \{z \in L^0(\mathbb{R}^m) : z(x) \in \operatorname{vsupp} \widetilde{Y}(x)\ \mu_*\text{-a.e.}, \langle z, \tilde{z}'\rangle_{\mu_*} \in \mathbb{R},\ \forall \tilde{z}' \in \widetilde{Y}\},$$

where $\langle z, \tilde{z}'\rangle_{\mu_*} := \int_\Omega (z(x), \tilde{z}'(x))\, d\mu_*(x)$. Then $\widetilde{Y}'_{\mu_*}$ is in fact the Köthe associate space of $\widetilde{Y}$ with respect to $\mu_*$. Observe that, for $\alpha_*\tilde{z}' = z' \in Y$,

$$\langle z, \tilde{z}'\rangle_{\mu_*} = \int_\Omega (z(x), z'(x)/\alpha_*(x))\alpha_*(x)\, d\mu(x) = \langle z, z'\rangle_\mu.$$

As $\zeta \in L^1(\Omega, C_0(\mathbb{R}^m); \mu)$ if and only if $\widetilde{\zeta} := \zeta/\alpha_* \in L^1(\Omega, C_0(\mathbb{R}^m); \mu_*)$, we have

$$\langle \nu, \widetilde{\zeta}\rangle_{\mu_*} := \int_\Omega \Big[ \int_{\mathbb{R}^m} \widetilde{\zeta}(x,\lambda)\, d\nu_x(\lambda)\Big]\, d\mu_*(x) = \langle \nu, \zeta\rangle_\mu.$$

Hence, $\delta_{z_{j_k}}$ is $L^1(\Omega, C_0(\mathbb{R}^m); \mu_*)$-weakly convergent to $\nu$ in $L^\infty_\omega(\Omega, \mathcal{M}(\mathbb{R}^m); \mu_*)$ and $\phi \circ z_{j_k}$ is $\widetilde{Y}$-weakly convergent to $\overline{\phi}$ in $X$ with respect to the duality pair $\langle X, \widetilde{Y}\rangle_{\mu_*}$. By Theorem 2.1/(Y3), there exists $A_\phi \in \mathfrak{A}$ such that $\mu_*(A_\phi) = 0$, the integral $\int_{\mathbb{R}^m} \phi(x,\lambda)\, d\nu_x(\lambda)$ exists in $\operatorname{vsupp} X(x)$ for $x \in \Omega \setminus A_\phi$, and (2.1) holds for all $x \in \Omega \setminus A_\phi$. As $\mu$ is equivalent to $\mu_*$, we see that $\mu(A_\phi) = 0$.

Step 3.4 (Proof of (Y4)). Observe that $X = Y'$ implies $X = \widetilde{Y}'_{\mu_*}$. Since the sequence $\phi \circ z_{j_k}$ is sequentially $Y$-weakly pre-compact in $X$, we conclude that $\phi \circ z_{j_k}$ is sequentially $\widetilde{Y}$-weakly pre-compact in $X$ with respect to the duality pair $\langle X, \widetilde{Y}\rangle_{\mu_*}$. By Theorem 2.1/(Y4), there exists $A_\phi \in \mathfrak{A}$ such that $\mu_*(A_\phi) = 0$, the integral $\int_{\mathbb{R}^m} \phi(x,\lambda)\, d\nu_x(\lambda)$ exists in $\operatorname{vsupp} X(x)$ for all $x \in \Omega \setminus A_\phi$, and $\phi \circ z_{j_k}$ is $\widetilde{Y}$-weakly convergent to $\widetilde{\phi}$ in $X$ with respect to $\langle \cdot, \cdot\rangle_{\mu_*}$, where $\widetilde{\phi}(x) := \int_{\mathbb{R}^m} \phi(x,\lambda)\, d\nu_x(\lambda)$ for $x \in \Omega \setminus A_\phi$ and $\widetilde{\phi}(x) := 0$ otherwise. Since $\mu$ is equivalent to $\mu_*$, we conclude that $\mu(A_\phi) = 0$ and $\phi \circ z_{j_k}$ is $Y$-weakly convergent to $\widetilde{\phi}$ in $X$. ∎

*Proof of Proposition 2.4.* We divide this proof into Steps 3.5–3.6.

Step 3.5. Assume that (SC1) holds. We claim that the sequence $\phi \circ z_j$ is $Y$-absolutely bounded in $X$, i.e.

$$(3.2)\quad y \in Y \ \Rightarrow \ \lim_{\mu_*(D)\to 0} \sup_{j\in\mathbb{N}} \int_D |(y(x), (\phi \circ z_j)(x))|\, d\mu(x) = 0,$$
$$\sup_{j\in\mathbb{N}} \int_\Omega |(y(x), (\phi \circ z_j)(x))|\, d\mu(x) < \infty.$$

Indeed, we deduce that

$$\begin{aligned}\int_D |(y(x), (\phi \circ z_j)(x))|\, d\mu(x) &\le \int_D |y(x)|\gamma(|z_j(x)|)u_0(x)\, d\mu(x)\\ &= \Big( \int_{D\cap\{\gamma(|z_j(\cdot)|)\le l\}} + \int_{D\cap\{\gamma(|z_j(\cdot)|)\ge l\}} \Big)|y(x)|\gamma(|z_j(x)|)u_0(x)\, d\mu(x)\\ &\le l \int_D |y(x)|u_0(x)\, d\mu(x) + \int_{\{\gamma(|z_j(\cdot)|)\ge l\}} |y(x)|\gamma(|z_j(x)|)u_0(x)\, d\mu(x).\end{aligned}$$

Since $\gamma$ is nondecreasing, we can choose $m_l \to \infty$ such that $\{t \geq 0 : \gamma(t) \geq l\} \subset \{t \geq 0 : t \geq m_l\}$. Then

$$\begin{aligned}\int\limits_{\{\gamma(|z_j(\cdot)|)\geq l\}} &|y(x)|\gamma(|z_j(x)|)u_0(x)\,d\mu(x) \\ &\leq \int\limits_{\{|z_j(\cdot)|\geq m_l\}} |y(x)|\gamma(|z_j(x)|)u_0(x)\,d\mu(x) \\ &\leq \frac{1}{M_l}\int\limits_{\{|z_j(\cdot)|\geq m_l\}} |y(x)|g(|z_j(x)|)u_0(x)\,d\mu(x) \\ &\leq \frac{1}{M_l}\int\limits_{\Omega} |y(x)|g(|z_j(x)|)u_0(x)\,d\mu(x) \leq \frac{C}{M_l} \to 0\end{aligned}$$

as $l \to \infty$ uniformly in $j$, where $C \in (0,\infty)$, $g(t) \geq M_l\gamma(t)$ for $t \geq m_l$, and $M_l \to \infty$ as $l \to \infty$. Hence, for any $\varepsilon > 0$ there exists $l_0$ such that

$$\int\limits_{\{\gamma(|z_j(\cdot)|)\geq l_0\}} |y(x)|\gamma(|z_j(x)|)u_0(x)\,d\mu(x) \leq \varepsilon \quad \forall j \in \mathbb{N}.$$

As $y \in Y$ and $u_0 Y \subset L^1(\mathbb{R}^d)$ we have $\lim_{\mu_*(D)\to 0}\int_D |y(x)|u_0(x)\,d\mu(x) = 0$. Therefore, there exists $\delta > 0$ such that $\mu_*(D) < \delta$ implies

$$\int\limits_D |y(x)u_0(x)|\,d\mu(x) \leq \frac{\varepsilon}{l_0}.$$

Hence, we infer that

$$\mu_*(D) < \delta \;\Rightarrow\; \int\limits_D |(y(x),(\phi\circ z_j)(x))|\,d\mu(x) \leq l_0\,\frac{\varepsilon}{l_0} + \varepsilon = 2\varepsilon.$$

So, the first part of (3.2) follows. The second part of (3.2) follows by the same arguments.

Since $\operatorname{vsupp} X(x) = \operatorname{vsupp} Y(x)$ $\mu$-a.e. and $X \subset Y'$, (3.2) implies that the sequence $\phi\circ z_{j_k}$ is sequentially $Y$-weakly pre-compact in $X$, due to the $Y$-weak pre-compactness theorem of J. Dieudonné [7] (if $X$ is a normed lattice with $X = X''$, $Y = X'$); W. Luxemburg and A. Zaanen [10], P. P. Zabrejko [23, Theorem 33] (if $X$ is a normed lattice); H. Nakano [13] ($m = 1$ with $X = X'', Y = X'$); O. Burkinshaw and P. Dodds [4, Theorem 3.4, Proposition 2.4] ($m = 1$), and [15, Theorem 2.8/(2)], [18] ($m \geq 2$).

Step 3.6. Assume that (SC2) holds. It is known that $(\Gamma^\circ)'$ can be interpreted as the dual space $(\Gamma^\circ)^*$ by the injection $z' \mapsto \langle\cdot, z'\rangle_\mu$ (see, e.g., [1, 23], [9, Theorems VI.1.4 and IV.3.6] ($d = 1$), [15, Corollary 2.2, Proposition 2.2], [18] ($d \geq 2$)). By [9, Theorem IV.3.3] ($m = 1$) and [16, Theorem 2.5], [15, 18] ($m \geq 2$), the separability of $\mu$ implies the separability of $\Gamma^\circ$. Hence, by the Alaoglu–Bourbaki theorem together with [9, Theorem V.7.6], the $\Gamma^\circ$-weak topology on any closed ball of $(\Gamma^\circ)^*$ is compact and metrizable. Therefore, for any sequence $a_i$ in the $(\Gamma^\circ)'$-norm-bounded set $\{\phi\circ z_{j_k}\}_{k\in\mathbb{N}}$ there exist a subsequence $p(i)$ of the sequence $i$ and $a\in(\Gamma^\circ)'$ such that $a_{p(i)}$ is $\Gamma^\circ$-weakly convergent to $a$ in $(\Gamma^\circ)'$. Since $Y\subset\Gamma^\circ$ and $(\Gamma^\circ)'\subset X$, $a_{p(i)}$ is $Y$-weakly convergent to $a$ in $X$. Hence, $a_{p(i)}$ is a $Y$-weak Cauchy sequence in $X$. Thus, the statement of Proposition 2.4/(SC2) follows. ∎

*Proof of Theorem 2.6.* It suffices to modify Step 3.1 of the proof of Theorem 2.1/(Y3). Since $\operatorname{supp}\widetilde{K}=\Omega$, by [9, Corollary IV.3.2] there exists a sequence of disjoint sets $\Omega_q\in\mathfrak{A}$ such that $\chi_{\Omega_q}\in\widetilde{K}$ and $\mu(\Omega\setminus\bigcup_{q=1}^\infty\Omega_q)=0$. Since $E$ is a separable reflexive space, so is $E^*$. Hence, there exists $\{\widetilde{u}_p\}_{p\in\mathbb{N}}$ dense in $E^*$. By (3.1) for $\chi_{\Omega_q}\widetilde{u}_p\in Y$, for some $\widetilde{D}_{pq}\in\mathfrak{A}$, $\widetilde{D}_{pq}\subset\Omega_q$ and $\mu(\Omega_q\setminus\widetilde{D}_{pq})=0$ and $\langle\chi_{\Omega_q}(x)\widetilde{u}_p,\overline{\phi}(x)\rangle=\int_{\mathbb{R}^m}\langle\chi_{\Omega_q}(x)\widetilde{u}_p,\phi(x,\lambda)\rangle\,d\nu_x(\lambda)\in\mathbb{R}$ for $x\in\widetilde{D}_{pq}$. Therefore, for $x\in\bigcap_{p\in\mathbb{N}}\widetilde{D}_{pq}$,

$$\langle\widetilde{u}_p,\overline{\phi}(x)\rangle=\int_{\mathbb{R}^m}\langle\widetilde{u}_p,\phi(x,\lambda)\rangle\,d\nu_x(\lambda)\in\mathbb{R}.$$

Put $\psi(x,\lambda):=\|\phi(x,\lambda)\|_E$. Since $\operatorname{supp}K=\operatorname{supp}\widetilde{K}=\Omega$ and the sequence $\phi\circ z_{j_k}$ is $\widetilde{K}(E^*)$-weakly pre-compact in $K(E)$, by M. Talagrand [22, Corollary 9 of Theorem 6] and M. Nowak [19, Theorem 3.3] we deduce that the sequence $\psi\circ z_{j_k}$ is $\widetilde{K}$-weakly pre-compact in $\widetilde{K}'$. By Theorem 2.1/(Y4) for $\psi\circ z_{j_k}$, there exists $D_\psi\in\mathfrak{A}$ such that $\mu(\Omega\setminus D_\psi)=0$ and the integral $\int_{\mathbb{R}^m}\psi(x,\lambda)\,d\nu_x(\lambda)\in\mathbb{R}$ exists for all $x\in D_\psi$.

Fix $u^*\in E^*$. Then we can choose a sequence $\hat{u}_i:=\widetilde{u}_{p(i)}$ from the dense set $\{\widetilde{u}_p\}_{p\in\mathbb{N}}$ with $\|\hat{u}_i-u^*\|_{E^*}\to 0$ as $i\to\infty$. Hence, $x\in\bigcap_{p=1}^\infty\widetilde{D}_{pq}\cap D_\psi$ implies that $\langle\hat{u}_i,\phi(x,\lambda)\rangle\to\langle u^*,\phi(x,\lambda)\rangle$ for all $\lambda\in\mathbb{R}^m$, $\langle\hat{u}_i,\overline{\phi}(x)\rangle\to\langle u^*,\overline{\phi}(x)\rangle$, and $|\langle\hat{u}_i,\phi(x,\lambda)\rangle|\le\sup_{i\in\mathbb{N}}\|\hat{u}_i\|_{E^*}\psi(x,\lambda)<\infty$. Hence, by the Lebesgue dominated convergence theorem, we infer that

$$\int_{\mathbb{R}^m}\langle\hat{u}_i,\phi(x,\lambda)\rangle\,d\nu_x(\lambda)\to\int_{\mathbb{R}^m}\langle u^*,\phi(x,\lambda)\rangle\,d\nu_x(\lambda)\in\mathbb{R}$$

as $i\to\infty$ for $x\in\bigcap_{p=1}^\infty\widetilde{D}_{pq}\cap D_\psi$. Hence, $x\in\bigcap_{p=1}^\infty\widetilde{D}_{pq}\cap D_\psi$ implies that $\langle u^*,\overline{\phi}(x)\rangle=\int_{\mathbb{R}^m}\langle u^*,\phi(x,\lambda)\rangle\,d\nu_x(\lambda)\in\mathbb{R}$ for all $u^*\in E^*$. Therefore, for $x\in\bigcap_{p=1}^\infty\widetilde{D}_{pq}\cap D_\psi$, the Pettis integral $(P)$-$\int_{\mathbb{R}^m}\phi(x,\lambda)d\nu_x(\lambda)\in E$ exists and coincides with $\overline{\phi}(x)$ [6, p. 53]. So, we obtain

$$\Big(x\in D_\phi:=\bigcup_{q=1}^\infty\bigcap_{p=1}^\infty\widetilde{D}_{pq}\cap D_\psi\Big)\ \Rightarrow\ \overline{\phi}(x)=(P)\text{-}\int_{\mathbb{R}^m}\phi(x,\lambda)\,d\nu_x(\lambda)\in E,$$

and $\mu(\Omega\setminus D_\phi)=0$. Hence, the statement (Y3) of Theorem 2.6 follows for $A_\phi:=\Omega\setminus D_\phi$. ∎

LEMMA 3.2 ([15, Lemma 4.2.3]). *Let $K \subset L^0(\mathbb{R})$ be a normed lattice with monotone norm. Then for $\varepsilon \in (0,\infty)$ there exists $r(\varepsilon) \in (0,\infty)$ such that $\|z\|_K \le r(\varepsilon) \Rightarrow \|z\|_{L^0(\mathbb{R})} \le \varepsilon$.*

*Proof of Proposition 2.7.* By 2.7, $g_q(|z_j(x_0)|) \ge g_q(L)$ for $x_0 \in D_L^j := \{x \in \Omega : |z_j(x)| \ge L\}$. Since $K(q)$ is a normed lattice with monotone norm $\|\cdot\|_{K(q)}$, we infer that

$$\begin{aligned}\|\chi_{C_q} g_q(|z_j(\cdot)|)\|_{K(q)} &\ge \|\chi_{C_q\cap D_L^j} g_q(|z_j(\cdot)|)\|_{K(q)}\\ &\ge \|\chi_{C_q\cap D_L^j} g_q(L)\|_{K(q)} = g_q(L)\|\chi_{C_q\cap D_L^j}\|_{K(q)}.\end{aligned}$$

Hence, $\lim_{L\to\infty}\sup_{j\in\mathbb{N}}\|\chi_{C_q\cap D_L^j}\|_{K(q)} = 0$. By Lemma 3.2, for all $\varepsilon > 0$ there exists $r_q(\varepsilon) > 0$ such that, given $j \in \mathbb{N}$, if $\|\chi_{C_q\cap D_L^j}\|_{K(q)} \le r(\varepsilon)$ then $\|\chi_{C_q\cap D_L^j}\|_{L^0(\Omega,\mathbb{R})} = \frac{1}{2}\,\mu_*(C_q \cap D_L^j) \le \varepsilon$. Therefore, there exists $L_\varepsilon^q$ such that $L \ge L_\varepsilon^q$ implies that $\|\chi_{C_q\cap D_L^j}\|_{K(q)} \le r(\varepsilon)$ for all $j \in \mathbb{N}$. It follows that $\frac{1}{2}\,\mu_*(C_q \cap D_L^j) \le \varepsilon$ for all $j \in \mathbb{N}$ and all $L \ge L_\varepsilon^q$. This gives (GB) for $\mu_*$ and $z_j$ on $C_q \subset \Omega$. By Proposition 3.1, (LB) follows for $\mu$ and $z_j$. ∎

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