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MEASURE AND INTEGRATION

Construction of an Uncountable Difference between $\Phi(B)$ and $\Phi_f(B)$

by

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Summary. We construct a set B and homeomorphism f where f and f^{-1} have property N such that the symmetric difference between the sets of density points and of f-density points of B is uncountable.

1. Introduction. The notion of f-density is a fairly new one. The *density* of a point, x, of a measurable set, A, is defined as

$$\lim_{h\to 0}\frac{m(A\cap(x-h,x+h))}{2h}.$$

If this limit exists and is equal to 1, then x is called a *density point* of A, and we denote the set of these points $\Phi(A)$. In contrast, the *f-density* of a set A at a point x, where f is a homeomorphism and both it and f^{-1} have property N, is defined as

$$\lim_{h\to 0}\frac{m(f(A)\cap(f(x)-h,f(x)+h))}{2h}.$$

Similarly, x is said to be an *f-density point* of A if this limit exists and is equal to 1, and we call the set of these points $\Phi_f(A)$. A function having property N is characterized by mapping null sets to null sets, and measurable sets (if f is continuous) to measurable sets. The property is important to f-density because it allows us to prove an analogue of the Lebesgue density theorem for f-density points. We remember that the Lebesgue density theorem says

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that a set and its density points only differ by a null set. Thus, the analogue for f-density says that a set and its f-density points only differ by a null set.

The symmetric difference between two sets A and B, denoted $A \triangle B$, is defined as $(A \cap B^c) \cup (A^c \cap B)$. It is fairly simple to construct a set A and f with the required conditions such that $\Phi(A) \triangle \Phi_f(A) \neq \emptyset$. It is more difficult, however, to construct a symmetric difference between $\Phi(A)$ and $\Phi_f(A)$ that is uncountable. One reason for this difficulty is the following theorem:

THEOREM 1. *If A is a set, and f is a homeomorphism such that f and f^{-1} have property N, then $\Phi(A) \triangle \Phi_f(A)$ is a null set.*

The theorem follows immediately from the Lebesgue density theorem ($A \triangle \Phi(A)$ is null) and its analogue for f-density ($A \triangle \Phi_f(A)$ is null) [1]. Thus, the symmetric difference must be both uncountable and null. A well-known set with these properties is the Cantor set, and so it is natural to build our construction of an uncountable difference off this set.

Lastly, we define the "overall density" of a set A on an interval I as

$$\frac{m(A \cap I)}{m(I)}.$$

This fraction is sometimes called "average density". It is a notion that becomes important in proofs below.

2. Description of the set B and function f. Denote by I_1 the first interval removed in the construction of the Cantor set, and let a_1 and b_1 be the left and right endpoints, respectively, of I_1. In I_1, construct the set A_1 such that it is a two-sided interval set with density points at a_1 and b_1. Specifically, let the left-hand half of I_1 consist of the set

$$\bigcup_{n=1}^{\infty} \left(\frac{\frac{1}{2^n} + \frac{1}{3^n}}{6} + \frac{1}{3}, \frac{\frac{1}{2^{n-1}}}{6} + \frac{1}{3} \right).$$

This is merely the standard interval set which has been scaled and translated to fit inside the left-hand half of the interval I_1. Similarly, let the right-hand half of I_1 consist of the mirror-image of the left-hand set. Specifically, this is

$$\bigcup_{n=1}^{\infty} \left(\frac{\frac{-1}{2^{n-1}}}{6} + \frac{2}{3}, \frac{\frac{-1}{2^n} - \frac{1}{3^n}}{6} + \frac{2}{3} \right).$$

We define the set A_1 as the union of these two interval sets. It has been proved in [2] that 0 is a right-hand density point of the standard interval set, and so it is obvious that a_1 and b_1 are left-hand and right-hand density points, respectively, of A_1.

Denote by I_2^1 and I_2^2 the next two intervals removed from $[0, 1]$ in the construction of the Cantor set, and denote by a_2^1 and b_2^1 the left-hand and right-hand endpoints of I_2^1, and a_2^2 by and b_2^2 the left-hand and right-hand endpoints of I_2^2. Scale and translate the set

$$E = \bigcup_{n=1}^{\infty} \left(\frac{1}{2^n} + \frac{1}{4^n}, \frac{1}{2^{n-1}} \right)$$

so that it fits in the left-hand half of the interval I_2^1, which means the transformation of E with the use of the function $g(x) = a_2^1 + \frac{b_2^1 - a_2^1}{2} \cdot x$. Specifically, this is

$$\bigcup_{n=1}^{\infty} \left(\frac{\frac{1}{2^n} + \frac{1}{4^n}}{18} + \frac{1}{9}, \frac{\frac{1}{2^{n-1}}}{18} + \frac{1}{9} \right).$$

Put the mirror-image set inside the right-hand side of I_2^1. Place the same set that is in I_2^1 inside I_2^2. Denote by A_2^1 the set that is inside I_2^1, and by A_2^2 the set that is inside I_2^2.

In a similar way, define all A_i^j for $i \in \mathbb{N}$ and $j \in \{1, 2, \ldots, 2^{i-1}\}$, where the I_i^j's removed in construction of the Cantor set consist of two-sided interval sets of the form $\left(\frac{1}{2^n} + \frac{1}{(i+2)^n}, \frac{1}{2^n}\right)$ but scaled and translated to fit inside their respective intervals. Define the set

$$B = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{2^{i-1}} A_i^j.$$

For the two-sided interval set A_i^j within each I_i^j we introduce the following notation. The centermost interval of A_i^j is a union of both the right-hand interval set and the left-hand interval set. Call the right-hand endpoint of this interval $z_0^{i,j}$, and note that $z_0^{i,j}$ is slightly to the right of the midpoint of I_i^j. Then denote all successive endpoints of the interval set to the right of $z_0^{i,j}$ by $z_n^{i,j}$ where $n > 0$, and those to the left of $z_0^{i,j}$ by $z_n^{i,j}$ where $n < 0$. So we note that

$$A_i^j = \bigcup_{n=-\infty}^{\infty} (z_{2n-1}^{i,j}, z_{2n}^{i,j}).$$

Now define

$$f_i^j(z_n^{i,j}) = z_{n+1}^{i,j},$$

and let $f_i^j(x)$ be linear for $x \in (z_n^{i,j}, z_{n+1}^{i,j})$.

Define

$$f(x) = \begin{cases} f_i^j(x) & \text{for } x \in I_i^j \text{ where } i \in \mathbb{N},\ j = 1, \ldots, 2^{i-1}, \\ x & \text{for } x \in C. \end{cases}$$

We note that

$$[0,1] = C \cup \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{2^{i-1}} I_i^j,$$

so f is defined on all of $[0,1]$.

3. f is a homeomorphism, and f and f^{-1} have property N

Theorem 2. *f is 1-to-1.*

Proof. We first show that on all I_i^j, $f_i^j(x)$ is 1-to-1. By definition of the set A_i^j and the function f_i^j, $z_n^{i,j} < z_{n+1}^{i,j}$ and $f_i^j(z_n^{i,j}) < f_i^j(z_{n+1}^{i,j})$ for $n \in \mathbb{Z}$. Since these points are connected by linear line segments, the function is strictly monotone, and therefore 1-to-1.

If $x \notin I_i^j$ for all i and j, then $x \in C$, and since $f(x) = x$ for the Cantor set, $f(x)$ is 1-to-1. ∎

Lemma 1. *For every interval I_i^j, $f(I_i^j) = I_i^j$.*

Proof. Let $x \in I_i^j$. Assume without loss of generality that x is in the right-hand half of I_i^j (i.e., $x > z_0^{i,j}$). So there exists an n such that $z_n^{i,j} \leq x \leq z_{n+1}^{i,j}$. By definition of f, $f([z_n^{i,j}, z_{n+1}^{i,j}]) = [z_{n+1}^{i,j}, z_{n+2}^{i,j}] \subset I_i^j$. So $f(I_i^j) \subset I_i^j$. Now we show reverse containment. Let $x \in I_i^j$. Once again, assume without loss of generality that x is in the right-hand half of I_i^j (i.e., $x > z_0^{i,j}$), and note that there exists an n such that $z_n^{i,j} \leq x \leq z_{n+1}^{i,j}$. By definition of f^{-1}, $f^{-1}([z_n^{i,j}, z_{n+1}^{i,j}]) = [z_{n-1}^{i,j}, z_n^{i,j}] \subset I_i^j$. So $f^{-1}(I_i^j) \subset I_i^j$, and hence $f(I_i^j) = I_i^j$. ∎

Theorem 3. *f is onto.*

Proof. If $x \notin I_i^j$ for any i and j, then $x \in C$, and so $f(x) = x$. By this fact, Lemma 1 above, and $[0,1] = C \cup \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{2^{i-1}} I_i^j$, f is onto. ∎

Theorem 4. *f is continuous.*

Proof. Since $f([0,1]) = [0,1]$ and f is monotone, f is continuous. ∎

Theorem 5. *f has property N.*

Proof. Let S be a null set. Since

$$S = (S \cap C) \cup (S \setminus C)$$

we have

$$f(S) = f((S \cap C) \cup (S \setminus C)) = f(S \cap C) \cup f(S \setminus C).$$

Since $f(x) = x$ for $x \in C$, we see that $f(S \cap C)$ is null set. We must prove that $f(S \setminus C)$ is a null set. Since this part of S is not in C, it is in some set of I_i^j's. Therefore, it is defined using a collection of f_i^j's. Since these f_i^j's consist of line segments connected at endpoints, $f(S \setminus C)$ is a null set. Since the union of two nullsets is a nullset, $f(S)$ is a null set, and f has property N. It is analogous to show that f^{-1} has property N. ■

4. $C \subset \Phi(B)$. We first introduce lemmas which will help us in proving $C \subset \Phi(B)$.

LEMMA 2. *If*

$$A_i^j = \bigcup_{n=-\infty}^{\infty} (z_{2n-1}^{i,j}, z_{2n}^{i,j}),$$

$f_i^j(z_n^{i,j}) = z_{n+1}^{i,j}$, $I \subset I_i^j$ *is an interval, and*

$$\frac{m(A_i^j \cap I)}{m(I)} > 1 - \frac{1}{n}$$

for some $n \in \mathbb{N} \setminus \{1\}$, *then*

$$\frac{m(f(A_i^j) \cap I)}{m(I)} \leq \frac{1}{n}.$$

Proof. Since $A_i^j = \bigcup_{n=-\infty}^{\infty}(z_{2n-1}^{i,j}, z_{2n}^{i,j})$ and $f_i^j(z_n^{i,j}) = z_{n+1}^{i,j}$ (the intervals between points are connected by line segments), then $\bigcup_{n=-\infty}^{\infty}(z_{2n}^{i,j}, z_{2n+1}^{i,j}) = f(A_i^j) \sim (A_i^j)^c = \bigcup_{n=-\infty}^{\infty}[z_{2n}^{i,j}, z_{2n+1}^{i,j}]$ because $f(A_i^j)$ and $(A_i^j)^c$ only differ at the endpoints of the intervals which are a nullset. Thus, if

$$\frac{m(A_i^j \cap I)}{m(I)} > 1 - \frac{1}{n},$$

then

$$\frac{m((A_i^j)^c \cap I)}{m(I)} = \frac{m(f(A_i^j) \cap I)}{m(I)} \leq 1 - \left(1 - \frac{1}{n}\right) = \frac{1}{n}. \ ■$$

LEMMA 3.

$$m((A_i^j)^c) = \frac{\frac{1}{i+2}}{1 - \frac{1}{i+2}} \cdot \frac{1}{3^i} = \frac{1}{3^i(i+1)}.$$

Proof. We note that A_i^j is the two-sided interval set

$$\bigcup_{n=-\infty}^{\infty} \left(\frac{1}{2^n} + \frac{1}{(i+2)^n}, \frac{1}{2^{n-1}}\right)$$

scaled to fit inside I_i^j, and the measure of I_i^j is $\frac{1}{3^i}$. For measurement purposes, it will be equivalent to just consider a single interval set inside I_i^j. So we see that

$$(A_i^j)^c = \bigcup_{n=1}^{\infty} \left[\frac{1}{2^n}, \frac{1}{2^n} + \frac{1}{(i+2)^n}\right],$$

and so, before we scale the set, we have

$$m((A_i^j)^c) = \sum_{n=1}^{\infty} \frac{1}{(i+2)^n} = \frac{\frac{1}{i+2}}{1-\frac{1}{i+2}} = \frac{i+2}{(i+1)(i+2)} = \frac{1}{i+1}.$$

Then, scaling this interval set, and therefore its measure to fit inside I_i^j, we find

$$m((A_i^j)^c) = \frac{1}{3^i} \cdot \frac{1}{i+1},$$

to obtain our lemma. ∎

LEMMA 4.

$$\frac{m(A_i^{j_1})}{m(I_i^{j_1})} < \frac{m(A_{i+1}^{j_2})}{m(I_{i+1}^{j_2})}$$

for each i *and each* $j_1 \in \{1, \ldots, 2^{i-1}\}$ *and* $j_2 \in \{1, \ldots, 2^i\}$.

Proof. From Lemma 3, we have

$$\frac{m(A_i^{j_1})}{m(I_i^{j_1})} = \frac{m(I_i^{j_1}) - m((A_i^{j_1})^c)}{m(I_i^{j_1})} = \frac{\frac{1}{3^i} - \frac{1}{3^i(i+1)}}{\frac{1}{3^i}}.$$

The inequality

$$\frac{\frac{1}{3^i} - \frac{1}{3^i(i+1)}}{\frac{1}{3^i}} < \frac{\frac{1}{3^{i+1}} - \frac{1}{3^{i+1}((i+1)+1)}}{\frac{1}{3^{i+1}}}$$

holds because

$$\begin{aligned} \frac{i}{i+1} &= \frac{\frac{i}{3^i(i+1)}}{\frac{1}{3^i}} = \frac{\frac{i+1}{3^i(i+1)} - \frac{1}{3^i(i+1)}}{\frac{1}{3^i}} < \frac{\frac{i+2}{3^{i+1}(i+2)} - \frac{1}{3^{i+1}(i+2)}}{\frac{1}{3^{i+1}}} \\ &= \frac{\frac{i+1}{3^{i+1}(i+2)}}{\frac{1}{3^{i+1}}} = \frac{i+1}{i+2} \end{aligned}$$

is true for $i \in \mathbb{N}$. ∎

LEMMA 5. *For any interval* I_i^j, *the interval* I^* *such that the ratio* $m(A_i^j \cap I^*)/m(I^*)$ *is at a minimum where* I^* *shares an endpoint with* I_i^j *is the interval which is* $3/4$ *the measure of* I_i^j *and shares either endpoint with* I_i^j. *Additionally, on this interval* I^*, *the overall density of* A_i^j *can be expressed as follows*:

$$\frac{m((A_i^j) \cap I^*))}{m(I^*)} = \frac{3i^2+5i}{3i^2+9i+6}.$$

Proof. Let I_i^j be an interval removed during construction of the Cantor set. For convenience, we examine the left-hand half of I_i^j. Contained in I_i^j is a two-sided interval set, A_i^j. We note that I^* will contain at least the left-hand half of I_i^j because the way the interval set is defined means

$$\frac{m(A_i^j \cap (z_{2n-1}^{i,j}, z_{2n}^{i,j}))}{m(z_{2n-1}^{i,j}, z_{2n}^{i,j})} < \frac{m(A_i^j \cap (z_{2n-3}^{i,j}, z_{2n-2}^{i,j}))}{m(z_{2n-3}^{i,j}, z_{2n-2}^{i,j})}.$$

Also, since A_i^j consists of mirror-image interval sets, we can deduce from Lemma 3 that for the left-hand half of I_i^j,

$$\frac{m(A_i^j \cap I_i^j \text{ left-hand})}{m(I_i^j \text{ left-hand})} = 1 - \frac{1}{i+1}.$$

So we must determine if we can make I^* bigger such that $m(A_i^j \cap I^*)/m(I^*)$ is even smaller. The "overall density" of the first $1/2$ of the interval set on the right-hand side is

$$\frac{\frac{1}{2} - \frac{1}{i+2}}{\frac{1}{2}}.$$

Since

$$\frac{i}{i+1} = 1 - \frac{1}{i+1} > \frac{\frac{1}{2} - \frac{1}{i+2}}{\frac{1}{2}} = \frac{i}{i+2}$$

for $i \geq 1$, we will always obtain a smaller $m(A_i^j \cap I^*)/m(I^*)$ when we include the first $1/2$ of the right-hand side. So we wonder if I^* can be made yet larger such that $m(A_i^j \cap I^*)/m(I^*)$ is even smaller. We examine the subsequent $1/4$ of the right-hand side of I_i^j. The measure of A_i^j on this part is

$$\frac{\frac{1}{4} - \frac{1}{(i+2)^2}}{\frac{1}{4}}.$$

Since

$$1 - \frac{1}{i+1} < \frac{\frac{1}{4} - \frac{1}{(i+2)^2}}{\frac{1}{4}} = 1 - \frac{4}{(i+2)^2}$$

for $i \geq 1$, $m(A_i^j \cap I^*)/m(I^*)$ will always be larger if we include this subsequent $1/4$, and therefore we will not include it.

Thus, when I^* shares either endpoint with I_i^j, $m(A_i^j \cap I^*)/m(I^*)$ is at a minimum when I^* contains the entirety of the left-hand or right-hand half and the first $1/2$ of the right-hand or left-hand side, respectively. So the measure of I^* is $3/4$ the measure of I_i^j.

We now obtain the formula in the lemma. For the left-hand side of I_i^j, the "overall density" is

$$\frac{m(A_i^j \cap I_i^j \text{ left-hand})}{m(I_i^j \text{ left-hand})} = 1 - \frac{1}{i+1},$$

and for the first 1/2 of the right-hand side, the "overall density" of A_i^j is

$$\frac{\frac{1}{2}-\frac{1}{i+2}}{\frac{1}{2}}.$$

Since the ratio of the measure of the left-hand side of I_i^j to the first 1/2 of the right-hand side is 2 : 1, we can use a weighted average to obtain the formula of the "overall density" of A_i^j on I^*:

$$\begin{aligned}\frac{m(A_i^j \cap I^*)}{m(I^*)} &= \frac{2}{3}\cdot\left(1-\frac{1}{i+1}\right)+\frac{1}{3}\cdot\left(\frac{i}{i+2}\right)\\ &= \frac{3i^2+5i}{3(i+1)(i+2)}. \ \blacksquare\end{aligned}$$

THEOREM 6. *For any $x_0 \in C$, $x_0 \in \Phi(B)$.*

Proof. Let $x_0 \in C$. If x_0 is an endpoint of I_I^j, then it is a density point of B from one side based on how we defined each A_i^j. Thus, we will prove that x_0 is also a density point from the other side of B. This proof will also apply when x_0 is not the endpoint of any interval I_i^j. We must show

$$\lim_{h\to 0}\frac{m(B\cap(x_0-h,x_0+h))}{2h}=1.$$

Let $\varepsilon > 0$. We must find $H > 0$ such that

$$\frac{m(B\cap[x-h,x+h])}{2h} > 1-\varepsilon$$

for all $h < H$. We only examine the right side of x, as the argument will be analogous for the left-hand side. Thus, we examine the interval $(x, x + H)$.

We observe the formula laid out in Lemma 5. Since

$$\frac{3i^2+5i}{3i^2+9i+6}\to 1^{(-)}$$

as $i \to \infty$, we can find i^* such that

$$\frac{3(i^*)^2+5i^*}{3(i^*)^2+9i^*+6} > 1-\varepsilon.$$

Set $H_1 = \frac{3}{4}\cdot\frac{1}{3^{i^*}}$ because $\frac{1}{3^i}$ is the measure of I_i^j and $\frac{3}{4}$ is the ratio of the measure of I^* (the interval containing an endpoint of I_i^j and yielding the minimum overall density on I_i^j) to the measure of I_i^j. Set H_2 such that $(x_0, x_0 + H_2)$ does not contain any I_i^j such that $i > i^*$. This is possible because, x_0 not being an endpoint of any I_i^j, there is a sequence of disjoint I_i^j's whose endpoints both converge to x_0, and there are only finitely many I_i^j's such that $i < i^*$.

We claim that if $H = \min\{H_1, H_2\}$, then

$$\frac{m(B \cap [x-h, x+h])}{2h} > 1 - \varepsilon \quad \text{for } h < H.$$

CASE 1. Suppose $H = H_1$. Then $(x, x+H)$ does not include any I_i^j such that $i > i^*$, and

$$\frac{m(B \cap [x, x+H])}{H} > 1 - \varepsilon$$

by Lemma 5. It will suffice to show that this remains true for $h < H$. Since any interval about x contains infinitely many I_i^j's, the measure of every interval which has a non-empty intersection with $[x, x+H)$ is necessarily smaller than H, and so from Lemma 4, the overall density of B intersected with those intervals contained in $[x, x+H)$ will be less than B on the interval $I_{i^*}^j$. This will remain true as we restrict h, because the interval $(x, x+h)$ will continue to contain intervals whose measure is smaller than H. In addition, at most one of those intervals will intersect $(x, x+h)$ in a way in which the overall density of B on that interval is at a minimum, but even this overall density will still be larger than B on I_{i*}^j because the interval is smaller than I_{i*}^j.

CASE 2. Suppose $H = H_2$. In this case, $h < H_1$ and the interval $(x, x+h)$ does not include any I_i^j such that $i > i^*$, and therefore

$$\frac{m(B \cap [x-h, x+h])}{2h} > 1 - \varepsilon$$

by Case 1. ∎

THEOREM 7. *For any $x_0 \in C$, $x_0 \notin \Phi_f(B)$.*

Proof. This result follows from Lemma 2 and Theorem 5. ∎

Since C is uncountable, Theorems 5 and 6 show that $\Phi(B) \triangle \Phi_f(B)$ is uncountable.

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References

[1] J. Campbell and D. Swanson, *Families of density type topologies*, Int. J. Pure Appl. Math., to appear.

[2] J. C. Oxtoby, *Measure and Category*, 2nd ed., Grad. Texts in Math. 2, Springer, New York, 1971.

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