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*ALGEBRAIC TOPOLOGY*

# The Brouwer Fixed Point Theorem for Some Set Mappings

by

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**Summary.** For some classes $X \subset 2^{\mathbb{B}_n}$ of closed subsets of the disc $\mathbb{B}_n \subset \mathbb{R}^n$ we prove that every Hausdorff-continuous mapping $f : X \to X$ has a fixed point $A \in X$ in the sense that the intersection $A \cap f(A)$ is nonempty.

**1. Introduction.** The aim of fixed point theory is to study the equation $f(x) = x$ for mappings $f : T \to T$ of a topological space $T$. Many generalizations of theorems of the classical fixed point theory were obtained for *set-valued mappings* (also called *multivalued mappings*) $F : T \to 2^T$ and their fixed points $x \in F(x)$, [2].

The idea of this paper is to consider the remaining two possibilities:

(1) a mapping $\phi : 2^T \to T$, where $A \in 2^T$ is called a fixed point of $\phi$ if $A \ni \phi(A)$, and
(2) a mapping $\Phi : 2^T \to 2^T$, where $A \in 2^T$ is called a fixed point of $\Phi$ if $A \cap \Phi(A) \neq \emptyset$.

Of course in both cases the set $2^T$ of all subsets of $T$ can be replaced by some class $X \subset 2^T$ of subsets of $T$: we have $\phi : X \to T$ in (1) and $\Phi : X \to X$ in (2). (Observe that in the case $X = 2^T$ the mappings $\phi$ and $\Phi$ have an obvious fixed point $A = T$.)

In what follows we assume that the class $X$ contains no singleton $\{t\} \subset T$. In this way our considerations fall within the domain of mereology. We recall that *mereology* is a set theory founded by S. Leśniewski [1886–1939]. In this theory the notion of a point $y \in Z$ is replaced by the notion of a region $Y \sqsubseteq Z$ [3]. We stress that this paper does not belong to mereology—we use the standard ZFC-mathematics.

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In this paper we focus on the Brouwer case:

$$T = \mathbb{B}_n = \Big\{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{j=1}^{n} (x_j)^2 \le 1\Big\}.$$

To justify the notion of fixed point from (2) let us consider the following

Example. Let $T = \mathbb{B}_2$, $X$ the set of all regular polygons with $n$ vertices in $\mathbb{B}_2$, and $\Phi$ rotation of angle $\pi/n$ about 0. Of course there is no $A \in X$ with $A = \Phi(A)$, but $A \cap \Phi(A)$ is nonempty e.g. for every regular polygon $A$ with center at 0.

**2. A simple fixed point schema.** Denote by $d$ the Hausdorff metric $d_{\mathrm{H}}$ or the Borsuk continuity metric $d_{\mathrm{B}}$ ([1]) in the set of all nonempty closed subsets of the disc $\mathbb{B}_n$. By a *simple fixed point schema* we understand a situation described in the assumptions of the following

Theorem 1. *Let $X \subset 2^{\mathbb{B}_n}$ be such that*

(i) *Every continuous mapping $F : \mathbb{B}_n \to (X, d)$ has a fixed point $x \in F(x)$.*

*Assume also that*

(ii) *There exists a continuous mapping $G : \mathbb{B}_n \to (X, d)$ such that every point $x \in \mathbb{B}_n$ is a fixed point of $G$.*

*Then every continuous mapping $\Phi : (X, d) \to (X, d)$ has a fixed point.*

*Proof.* The mapping $F := \Phi \circ G$ has a fixed point $x \in \Phi(G(x))$ by (i) and $x \in G(x)$ by (ii). Thus $\Phi(G(x)) \cap G(x) \ne \emptyset$, so $G(x)$ is a fixed point of $\Phi$. ■

Example. Let $X$ be the set of all closed convex subsets of $\mathbb{B}_n$ with nonempty interior and diameter $\le \epsilon$. Then every continuous mapping $\Phi : (X, d_{\mathrm{H}}) \to (X, d_{\mathrm{H}})$ has a fixed point. Assume that $\epsilon < 2$ to eliminate the obvious fixed point $A = \mathbb{B}_n$. To prove the existence of a fixed point of $\Phi$ we use the simple fixed point schema with (i) being the special case of the well known Kakutani [5] fixed point theorem. The mapping $G$ in (ii) is constructed as follows. Fix $x \in \mathbb{B}_n$. Then $x$ lies on a segment $OP$ with $O = 0$, $P \in \partial\mathbb{B}_n = \mathbb{S}^{n-1}$ and divides $OP$ in the ratio of $t : (1-t) = |Ox| : |xP|$. Define $G(x)$ to be the closed ball with radius $\epsilon/2$ and center $C \in OP$ such that for $\{D\} = CP \cap \partial G(x)$ we have $x \in CD$ and $|Cx| : |xD| = t : (1-t)$.

Remark. The same proof based on the simple fixed point schema with the Kakutani theorem replaced by the Eilenberg–Montgomery fixed point theorem shows that if $Y$ is the set of all $\mathbb{Q}$-acyclic nonsingleton subsets of $\mathbb{B}_n$ of diameter $\le \epsilon$ then every continuous mapping $\Phi : (Y, d_{\mathrm{H}}) \to (Y, d_{\mathrm{H}})$ has a fixed point.

Let us note that the only assumption (i) of the simple fixed point schema does not guarantee the existence of a fixed point of $\Phi$. Consider the following

Example. Let $X$ be the set of all right-angled triangles in $\mathbb{B}_2$. Then (i) is satisfied for $n = 2$ and $d = d_{\mathrm{H}}$ by the Kakutani fixed point theorem. Nevertheless we will construct a fixed point free continuous mapping $\Phi : (X, d) \to (X, d)$. Let $ABC \in X$ be a triangle with hypotenuse $AB$. Consider the perpendicular bisector $s$ of the segment $AB$ and $\{P_1, P_2\} = s \cap \partial\mathbb{B}_2$. Choose $P := P_i$ from $\{P_1, P_2\}$ so that $P$ and $C$ lie on the opposite sides of the line $AB$. Let $j : \mathbb{R}^2 \to \mathbb{R}^2$ be the map

$$j(z) = P + k \cdot (z - P)$$

with $k = k(A, B) > 0$ continuous in $(A, B)$ and small enough to make the sets $ABC$ and $j[ABC]$ disjoint. One can check that these conditions are satisfied e.g. by

$$k = \frac{1 - \sqrt{1 - (|AB|/2)^2}}{1 - \sqrt{1 - (|AB|/2)^2} + |AB|}.$$

Then $\Phi(ABC) := j[ABC]$ is Hausdorff-continuous and fixed point free.

We now derive the above formula for $k$. Let $\{Q\} = PC \cap AB$. It is sufficient to take $k < |PQ| : |PC|$. If $H$ is the center of the segment $AB$ then $|PQ| \geq |PH|$ and $|PC| \leq |PH| + |HC|$, $|HC| < \max\{|AC|, |BC|\} < |AB|$. It is enough to take $k \leq |PH| : (|PH| + |AB|)$. The right hand side of this inequality is increasing in $|PH|$. Given the length $|AB|$, the minimal value of $|PH|$ is achieved for $A, B \in \partial\mathbb{B}_2$. Thus $\min\{|PH|\} = 1 - \sqrt{1 - (|AB|/2)^2}$, which yields the formula for $k$.

Another benefit from the above proof is the corollary that (ii) is not satisfied: there is no Hausdorff-continuous set-valued mapping $G : \mathbb{B}_2 \to 2^{\mathbb{B}_2}$ taking right-angled triangles as values with $x \in G(x)$ for every $x \in \mathbb{B}_2$. This seems to be of independent interest.

For another example of this type take $X$ to be the class of all two-point sets in $\mathbb{B}_n$ and construct a Hausdorff-continuous fixed point free mapping $\Phi : X \to X$ as follows. Fix $\{A, B\} \in X$, $C = (A + B)/2$, $j : \mathbb{R}^2 \to \mathbb{R}^2$ with $j(z) = C + (z - C)/2$, $\Phi(\{A, B\}) = j[\{A, B\}]$. The condition (i) is guaranteed by the fact that every Hausdorff-continuous mapping $F : \mathbb{B}_n \to X$ splits into two continuous selectors. This follows because the graph

$$\Gamma_F = \{(x, y) \in \mathbb{B}_n \times \mathbb{B}_n : y \in F(x)\}$$

of $F$ covers $\mathbb{B}_n$ and $\mathbb{B}_n$ is simply connected [4]. In this case we use a monodromy property [4, 15.2, p. 87], analogous to the well known fact on extensions of analytic functions.

**3. Two more examples.** In this section we give examples of classes $X \subset 2^{\mathbb{B}_n}$ which do not satisfy the condition (ii) in the simple fixed point schema but nevertheless guarantee the existence of fixed points of Hausdorff-continuous mappings $\Phi : X \to X$.

Example. Let $X$ be the set of all non-degenerate segments in $\mathbb{B}_n$ (i.e., of positive length). We prove that condition (ii) is not satisfied. Suppose for contradiction that there is a Hausdorff-continuous mapping $G : \mathbb{B}_n \to X$ such that $x \in G(x)$ for every $x \in \mathbb{B}_n$. The end-points of the segment $G(x)$ for $x \in \mathbb{B}_n$ are two continuous selectors of $G$. Denote these selectors by $A_1, A_2 : \mathbb{B}_n \to \mathbb{B}_n$. Of course, there is $i \in \{1,2\}$ with $A_i(x) = x$ for every $x \in \partial\mathbb{B}_n$. Then the mappings $A_1$ and $A_2$ have a coincidence point $c \in \mathbb{B}_n$ such that $A_1(c) = A_2(c)$. This is a contradiction because our segments are not singletons. (If $A_1$ and $A_2$ had no coincidence point then the formula

$$\{r(x)\} := \partial\mathbb{B}_n \cap \{A_j(x) + t(A_i(x) - A_j(x)) : t > 0\} \quad \text{for } i \neq j \in \{1,2\}$$

would provide a retraction $r : \mathbb{B}_n \to \partial\mathbb{B}_n$, which is impossible.)

Theorem 2. *Let $X$ be the set of all segments in $\mathbb{B}_n$ of positive length. Then every Hausdorff-continuous mapping $\phi : X \to \mathbb{B}_n$ has a fixed point.*

*Proof.* Every segment in $\mathbb{B}_n$ can be parallel translated to have center at 0. In this way the set $X_1$ of all segments in $\mathbb{R}^n$ with center 0 and length $\leq 2$ is a strong deformation retract of $(X, d_{\mathrm{H}})$. The space of all segments with center 0 and length 2 in $\mathbb{R}^n$ is the real projective space $\mathbb{RP}^{n-1}$. We have the homotopy equivalence $e : X \to \mathbb{RP}^{n-1}$:

$$X \simeq X_1 \approx \mathbb{RP}^{n-1} \times (0,1] \simeq \mathbb{RP}^{n-1}.$$

Let $X_2$ be the space of all segments with end-points $P$, $P/2$ for $P \in \partial\mathbb{B}_n$. The mapping $h : \mathbb{S}^{n-1} \to X_2$, $h(P) = PP'$ with $P' = P/2$, is a homeomorphism.

Suppose for a contradiction that there exists a fixed point free continuous mapping $\phi : X \to \mathbb{B}_n$. Projecting the segment $J \in X$ from the point $\phi(J)$ into the sphere $\partial\mathbb{B}_n$ we obtain an arc $L(J)$ with center $c(J) \in L(J)$. We thus get a continuous function $c : X \to \partial\mathbb{B}_n$. Consider the commuting diagram

$$\begin{array}{ccccc} X_2 & \xrightarrow{\ i\ } & X & \xrightarrow{\ c\ } & \mathbb{S}^{n-1} \\ \uparrow & & \downarrow & & \\ \mathbb{S}^{n-1} & \xrightarrow{\ \pi\ } & \mathbb{RP}^{n-1} & & \end{array}$$

with $i$ the inclusion and $\pi$ the projection, and the sequence

$$H_{n-1}(\mathbb{S}^{n-1}, \mathbb{Z}_2) \xrightarrow{h_\star} H_{n-1}(X_2, \mathbb{Z}_2) \xrightarrow{i_\star} H_{n-1}(X, \mathbb{Z}_2) \xrightarrow{c_\star} H_{n-1}(\mathbb{S}^{n-1}, \mathbb{Z}_2).$$

Since $\pi_\star = 0$ and $\pi = e \circ i \circ h$, we have $i_\star = 0$. On the other hand, we will prove that $c_\star i_\star h_\star = \mathrm{id}$, which is a contradiction. Indeed, observe that

$$c \circ i \circ h(P) = c(PP') \in L(PP') \quad \text{and} \quad P \in L(PP').$$

We can move the point $P$ along the arc $L(PP')$ to the point $c(PP')$, getting a homotopy

$$c \circ i \circ h \simeq \mathrm{id}_{\mathbb{S}^{n-1}}.$$

Thus $c_\star \circ i_\star \circ h_\star = \mathrm{id} \neq 0$, a contradiction. ∎

Corollary. *For $X$ from Theorem 2, every Hausdorff-continuous mapping $\Phi : X \to X$ has a fixed point.*

*Proof.* Denote by $s(J)$ the center of the segment $J$. The mapping $\phi := s \circ \Phi$ has a fixed point $J$ by Theorem 2. Thus $s(\Phi(J)) \in J$ and $s(\Phi(J)) \in \Phi(J) \cap J \neq \emptyset$, so $J$ is a fixed point of $\Phi$.

Remark. The same proofs show that Theorem 2 and the Corollary are true for the class $X$ of all segments in $\mathbb{B}_n$ of positive length $\leq \epsilon$.

Now we return to the first example in this paper.

Example. Let $X$ be the set of all regular polygons in $\mathbb{B}_2$ with $n$ vertices and of diameter $\leq \epsilon$. If $\epsilon = 2$ then every Hausdorff-continuous mapping $\Phi : X \to X$ has a fixed point $A \in X$ which is a regular polygon inscribed in $\partial\mathbb{B}_2$. In fact, suppose for contradiction that $\Phi(A) \subset \mathbb{B}_2 \backslash A$ for every $A \in X$ inscribed in $\mathbb{S}^1$. Fix such an $A$. The set $\mathbb{B}_2 \backslash A$ has $n$ components $C_1, \ldots, C_n$ and

$$\Phi(A) \subset C_i \quad \text{for some } i.$$

We rotate $A$ about 0. Let $O_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ denote the rotation of angle $\alpha$ about 0 in $\mathbb{R}^2$. Of course $\Phi(O_\alpha[A]) \subset O_\alpha[C_i]$. For $\alpha = 2\pi/n$ we get a contradiction:

$$O_\alpha[A] = A \quad \text{but} \quad O_\alpha[C_i] = C_{i+1}.$$

In what follows we assume that $\epsilon < 2$.

We prove that (ii) from the simple fixed point schema is not satisfied. Suppose for contradiction that there is a Hausdorff-continuous mapping $G : \mathbb{B}_2 \to X$ such that $x \in G(x)$ for every $x \in \mathbb{B}_2$. Since $\mathbb{B}_2$ is simply connected, the vertices $w_1(x), \ldots, w_n(x)$ of the polygon $G(x)$ for $x \in \mathbb{B}_2$ are continuous selectors of $G$ (for the argument see the end of Section 2). Of course there exists $i$ such that

$$w_i(x) = x \quad \text{for every } x \in \partial\mathbb{B}_2.$$

We can assume that $i = 1$. Denote by $S^1(x)$ the circle circumscribed around the regular polygon $G(x)$. By a similarity of the type $\mathbb{R}^2 \ni P \mapsto \lambda(P - P_0) \in \mathbb{R}^2$ we identify $S^1(x)$ with $\mathbb{S}^1$. Note that this similarity involves no rotation. Since $w_1(x) \in S^1(x) \approx \mathbb{S}^1$, we have

$$\mathbb{S}^1 \overset{i}{\subset} \mathbb{B}_2 \xrightarrow{w_1} \mathbb{S}^1.$$

Denote by deg the Brouwer degree. As $\mathbb{B}_2$ is contractible, $\deg(w_1 \circ i) = 0$. On the other hand we will prove that $w_1 \circ i \simeq \mathrm{id}$, so $\deg(w_1 \circ i) = 1$, a contradiction.

To prove the last homotopy relation it is sufficient to show that $x \in \mathbb{S}^1 = \partial\mathbb{B}_2$ and $w_1(x) \in \mathbb{S}^1 \approx S^1(x)$ are antipodal points for no $x \in \mathbb{S}^1$. Denote by $C_x$ the center of the polygon $G(x)$. We should check that the angle $\alpha$ between the segments $Ox$ and $C_x w_1(x) = C_x x$ is not $180^\circ$ for any $x \in \mathbb{S}^1$. Fix $x \in \mathbb{S}^1$. The supremum of $\alpha$ over all possible regular polygons $G(x)$ in $B_2$ with vertices $w_1 = x, w_2, \ldots, w_n$ is achieved for $w_2 \in \mathbb{S}^1$ (or equivalently for $w_n \in \mathbb{S}^1$) in the limit $w_2 \to w_1$. In this way $\sup\{\alpha : \text{all possible } G(x)\} = \pi/n < \pi$, which proves the homotopy $w_1 \circ i \simeq \mathrm{id}$.

Theorem 3. *Let $X$ be the set of all regular $n$-gons in $\mathbb{B}_2$ of diameter $\le \epsilon < 2$. Then every Hausdorff-continuous mapping $\phi : X \to \mathbb{B}_2$ has a fixed point.*

*Proof.* Every regular $n$-gon in $\mathbb{B}_2$ can be parallel translated so as to have center 0. In this way the set $X_1$ of all regular $n$-gons with center at 0 and diameter $\le \epsilon$ in $\mathbb{R}^2$ is a strong deformation retract of $(X, d_{\mathrm{H}})$. The subspace $X_1^\epsilon$ of $X_1$ of polygons with diameter $\epsilon$ is homeomorphic to $\mathbb{S}^1$. We have the homotopy equivalence $e : X \to \mathbb{S}^1$:

$$X \simeq X_1 \simeq X_1^\epsilon \approx \mathbb{S}^1.$$

Let $X_2$ be the space of all polygons $W_P \in X$ with a vertex $P \in \partial\mathbb{B}_2$, with center in the segment $OP$ and diameter $\epsilon$. The mapping $h : \mathbb{S}^1 \to X_2$, $h(P) = W_P$, is a homeomorphism.

Let $i : X_2 \to X$ be the inclusion. The mapping $e \circ i \circ h$ is not an injection—it winds the circle $\mathbb{S}^1$ $n$-times around $\mathbb{S}^1$.

Suppose for a contradiction that there exists a fixed point free continuous mapping $\phi : X \to \mathbb{B}_2$. Projecting the polygon $W \in X$ from $\phi(W)$ to the circle $\partial\mathbb{B}_2$ we obtain an arc $L(W)$ with center $c(W) \in L(W)$. We thus get a continuous function $c : X \to \partial\mathbb{B}_2$. Consider the commuting diagram

$$\begin{array}{ccccc} X_2 & \xrightarrow{\;i\;} & X & \xrightarrow{\;c\;} & \mathbb{S}^1 \\ \uparrow & & \downarrow & & \\ \mathbb{S}^1 & \xrightarrow{\;p_n\;} & \mathbb{S}^1 & & \end{array}$$

with $i$ the inclusion, and $p_n(z) = z^n$ for $z \in \mathbb{S}^1 \subset \mathbb{C}$. Consider the sequence

$$H_1(\mathbb{S}^1) \xrightarrow{h_\star} H_1(X_2) \xrightarrow{i_\star} H_1(X) \xrightarrow{c_\star} H_1(\mathbb{S}^1)$$

of homology groups with integer coefficients. All groups in this sequence are isomorphic to $\mathbb{Z}$. We will show that $i_\star = \times n$ and $c_\star i_\star h_\star = \mathrm{id}$, which will give a contradiction. Fixing generators we can write

$$\mathbb{Z} \xrightarrow{h_\star} \mathbb{Z} \xrightarrow{i_\star} \mathbb{Z} \xrightarrow{c_\star} \mathbb{Z}.$$

Since $p_{n\star} = \times n$ and $p_n = e \circ i \circ h$, we have $i_\star = \times n$. On the other hand,

$$c \circ i \circ h(P) = c(W_P) \in L(W_P) \quad \text{and} \quad P \in L(W_P).$$

We can move the point $P$ along the arc $L(W_P)$ to $c(W_P)$, getting a homotopy $c \circ i \circ h \simeq \mathrm{id}_{\mathbb{S}^1}$. Thus $c_\star \circ i_\star \circ h_\star = \mathrm{id}$, a contradiction. ∎

COROLLARY. *For $X$ from Theorem 3, every Hausdorff-continuous mapping $\Phi : X \to X$ has a fixed point.*

The proof is analogous to the proof of the Corollary after Theorem 2.

**4. Résumé and open problems.** The main positive results of this paper can be summarized in

THEOREM 4. *Let $X \subset 2^{\mathbb{B}_n}$ and $X(\epsilon) = \{A \in X : \mathrm{diam}(A) \le \epsilon\}$. Then the statement:*

- *Every Hausdorff-continuous mapping $\Phi : X(\epsilon) \to X(\epsilon)$ has a fixed point $A \in X(\epsilon)$ in the sense that $A \cap \Phi(A) \neq \emptyset$*

*is true for every class $X$ from the following list:*

1. *The set of all closed convex subsets of $\mathbb{B}_n$ with nonempty interior.*
2. *The set of all nonsingleton $\mathbb{Q}$-acyclic subsets of $\mathbb{B}_n$.*
3. *The set of all segments in $\mathbb{B}_n$.*
4. *The set of all regular $n$-gons in $\mathbb{B}_2$.*

We ask if the list from Theorem 4 could be extended by the following classes:

5, 6. The set of all convex $n$-gons in $\mathbb{B}_2$.
7, 8. The set of all $n$-gons in $\mathbb{B}_2$.
9–13. The set of all cubes (regular tetrahedrons, octahedrons, dodecahedrons, icosahedrons) in $\mathbb{B}_3$.
14–15. The set of all closed curves in $\mathbb{B}_3$ (with the Borsuk continuity metric [1] in place of the Hausdorff metric).

**Addendum.** Questions 9–13 were answered in the affirmative by the author in October 2012; a paper is in preparation. The questions remain unanswered e.g. for cubes in higher dimensions.

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