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There Are No Essential Phantom Mappings from 1-dimensional CW-complexes

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Summary. A phantom mapping h from a space Z to a space Y is a mapping whose restrictions to compact subsets are homotopic to constant mappings. If the mapping h is not homotopic to a constant mapping, one speaks of an essential phantom mapping. The definition of (essential) phantom pairs of mappings is analogous. In the study of phantom mappings (phantom pairs of mappings), of primary interest is the case when Z and Y are CW-complexes. In a previous paper it was shown that there are no essential phantom mappings (pairs of phantom mappings) between CW-complexes if $\dim Y \leq 1$. In the present paper it is shown that there are no essential phantom mappings between CW-complexes if $\dim Z \leq 1$. In contrast, there exist essential phantom pairs of mappings between CW-complexes where $\dim Z = 1$ and $\dim Y = 2$. Moreover, there exist essential phantom mappings with $\dim Z = \dim Y = 1$ where Y is a CW-complex, but Z is not.

1. Introduction. In this paper a mapping between topological spaces $h\colon Z \to Y$ is called a *phantom mapping* provided the restriction $h|C$ to any compact subset $C \subseteq Z$ is homotopic to a constant, i.e., to a mapping whose only value is a point $y_0 \in Y$. We say that a phantom mapping $h\colon Z \to Y$ is *essential* provided it is not homotopic to a constant mapping. Analogously, two mappings $h, h'\colon Z \to Y$ form a *phantom pair* if for any compact subset $C \subseteq Z$, $h|C \simeq h'|C$. The phantom pair (h, h') is *essential* provided $h \not\simeq h'$. Usually, one considers phantom mappings and phantom pairs of mappings when Z is a CW-complex. In that case compact subsets $C \subseteq Z$ can be replaced by finite subcomplexes of Z. In the literature phantom mappings, as just described, are called phantom mappings of the second kind [5]. Most work on phantom mappings refers to phantom mappings

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of the first kind, where the subsets $C \subseteq Z$ are finite-dimensional skeleta of Z.

In a recent paper [4] the author proved the following result.

Proposition 1. *For CW-complexes Z, Y with $\dim Y \leq 1$, there are no essential phantom pairs of mappings $h, h'\colon Z \to Y$. In particular, there are no essential phantom mappings $h\colon Z \to Y$.*

In this paper we are primarily interested in phantom mappings $h\colon Z \to Y$ and pairs of phantom mappings $h, h'\colon Z \to Y$ when Z and Y are CW-complexes and $\dim Z \leq 1$. In [1] the authors have studied shape properties of the Cartesian product $X \times P$ of the dyadic solenoid X and the wedge $P = S^1 \vee S^1 \vee \cdots$ of a sequence of copies of the 1-dimensional sphere $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. They exhibited a connected 2-dimensional CW-complex Y and an essential phantom pair of mappings $h, h'\colon P \to Y$ (see [1, Remark 2]). In the present paper we prove that, for phantom mappings, the analogous phenomenon is not possible, as the following theorem shows.

Theorem 1. *If $h\colon Z \to Y$ is a phantom mapping between CW-complexes and $\dim Z \leq 1$, then h is homotopic to a constant mapping, i.e., h is not essential.*

Theorem 1 is a consequence of the following one.

Theorem 2. *Let Y be a CW-complex, let $y_0 \in Y$ and let $Z = \bigvee_{i \in J} Z_i$ be the wedge of a collection of compact CW-complexes Z_i, $i \in J$, having a common vertex $z_0 \in Z_i \subseteq Z$ as its base point. If $h\colon Z \to Y$ is a phantom mapping and $h(z_0) = y_0$, then h is homotopic to y_0.*

Theorem 2 is a consequence of the following result.

Theorem 3. *Let Z be a CW-complex and let z_0 be a vertex of Z. Let $h\colon Z \to Y$ be a mapping to a space Y and let $y_0 = h(z_0)$. If h is homotopic to the constant mapping y_0, then there is a homotopy $H\colon Z \times I \to Y$ which connects h to y_0 and $H(z_0, s) = y_0$ for all $s \in I$, i.e., $h \simeq y_0$ (rel z_0).*

In the last section of the paper we will exhibit an example which shows that in Proposition 1 and Theorem 1 one cannot omit the assumption that Z is a CW-complex.

2. Reducing the proof of Theorem 1 to Theorem 2. In the proofs of Theorems 1 and 2 we use the well-known fact that every pair (A, B) consisting of a CW- complex A and a subcomplex B has the homotopy extension property for arbitrary spaces S, i.e., every mapping $G : (A \times 0) \cup (B \times I) \to S$, $I = [0, 1]$, extends to a mapping $\tilde{G} : A \times I \to S$ (see e.g. [3, Theorem 7.2] or [2, Proposition 0.16]).

Proof of Theorem 1 using Theorem 2. We will first show that it suffices to prove Theorem 1 in the special case when Z is connected. Indeed, if Z is a CW-complex, $\dim Z \le 1$, and $h\colon Z \to Y$ is a phantom mapping, then every component Q of Z is a connected CW-complex with $\dim Q \le 1$ and the restriction $h_Q = h|Q\colon Q \to Y$ is a phantom mapping. Applying Theorem 1 in the special case, one concludes that h_Q is homotopic to a constant, i.e., there is a point $y_Q \in Y$ and a homotopy $G_Q\colon Q \times I \to Y$ which connects h_Q to y_Q. Denote by R_Q the component of Y which contains the point y_Q.

We claim that there is a component R of Y such that $R = R_Q$ for all Q. Indeed, choose a point $z_Q \in Q$ and note that $G_Q|(z_Q \times I)$ is a path in Y which connects the point $h(z_Q)$ to y_Q. Since $y_Q \in R_Q$, it follows that also $h(z_Q) \in R_Q$. For components Q, Q' of Y, the two-point set $\{z_Q, z_{Q'}\}$ is a compact subset of Z, and thus $h|\{z_Q, z_{Q'}\}$ is homotopic to a constant $y_{QQ'} \in Y$. In particular, there are paths η, η' in Y which connect $h(z_Q)$ to $y_{QQ'}$ and $h(z_{Q'})$ to $y_{QQ'}$, respectively. Clearly, the concatenation of η with the inverse of η' is a path in Y which connects $h(z_Q)$ to $h(z_{Q'})$. Since $h(z_Q) \in R_Q$ and R_Q is a component of Y, it follows that $\eta\eta^{-1}(1) \subseteq R_Q$, and thus

$$h(z_{Q'}) = \eta\eta^{-1}(1) \in R_Q.$$

One also has $h(z_{Q'}) \in R_{Q'}$ and we see that $R_Q \cap R_{Q'} \ne \emptyset$. However, if two components of Y intersect, they coincide. Denote by R the component of Y which coincides with every R_Q, and thus R_Q does not depend on Q and can be denoted by R. Clearly, R contains all points y_Q.

Now choose an arbitrary point $y_0 \in R$. For each Q there is a path ξ_Q in Y which connects the points y_Q and y_0. Consider the homotopy $H_Q\colon Q \times I \to Y$ which is the concatenation of the homotopy G_Q and the path ξ_Q. Clearly, H_Q connects $h_Q = h|Q$ to y_0. Since the components Q of Z are open subsets of Z, the homotopies $H_Q\colon Q \times I \to Y$ determine a homotopy $H\colon Z \times I \to Y$ such that $H|(Q \times I) = H_Q$. Clearly, H connects h to y_0, as desired.

We will now prove Theorem 1 under the additional assumption that Z is connected. It is well known that every connected 1-dimensional CW-complex Z contains a maximal tree T, i.e., a tree containing all vertices (0-cells) of Z (see e.g. [2, Proposition 1A.1]). Since the pair (Z, T) has the homotopy extension property and T is contractible, the quotient mapping $q\colon Z \to Z/T$ is a homotopy equivalence, and thus admits a homotopy inverse $r\colon Z/T \to Z$ (see e.g. [2, Proposition 0.17]). If $Z = T$, then Z contracts to a point $z_0 \in Z$. Consequently, every mapping $h\colon Z \to Y$ is homotopic to the constant $y_0 = h(z_0)$. If $Z \ne T$, then $Z' = Z/T$ is a connected 1-dimensional CW-complex having the point $z_0' = T$ as its only 0-cell, i.e., Z' is of the form $Z' = \bigvee_{i \in J} Z_i$, where each Z_i is a copy of the 1-sphere S^1 and z_0' is the base point of Z'. The assumption that h is a phantom mapping implies that also $h' = hr\colon Z' \to Y$

is a phantom mapping, because $r(C) \subseteq Z$ is compact whenever $C \subseteq Z'$ is compact. Being a vertex of Z, z_0 belongs to T, and thus $q(z_0) = z_0'$. Since Z is connected, there is no loss of generality in assuming that $r(z_0') = z_0$, and thus $h'(z_0') = y_0$. We can now apply Theorem 2 to h', z_0' and y_0 and conclude that $h' \simeq y_0$. It follows that also $h'q \simeq y_0$, and thus $hrq \simeq y_0$. Since $rq \simeq 1$, we conclude that indeed $h \simeq y_0$. ∎

3. Proof of Theorem 2 using Theorem 3. Since h is a phantom mapping and Z_i is a compact subset of Z, there exists a homotopy $G_i\colon Z_i \times I \to Y$ which connects $h_i = h|Z_i$ to a constant $y_i \in Y$. Since $z_0 \in Z_i$, $G_i|(z_0 \times I)$ is a path in Y which connects $y_0 = h(z_0)$ to y_i. The concatenation of G_i with the inverse of that path yields a homotopy on Z_i which connects h_i to y_0. Therefore, there is no loss of generality in assuming that already G_i connects h_i to y_0. Applying Theorem 3, one obtains a homotopy $H_i\colon Z_i \times I \to Y$ which connects h_i to y_0 and has the additional property that $H_i(z_0, s) = y_0$ for $s \in I$. Since $(Z_i \times I) \cap (Z_{i'} \times I) = z_0 \times I$ and $H_i|(z_0 \times I) = y_0 = H_{i'}|(z_0 \times I)$, we conclude that there is a unique homotopy $H\colon Z \times I \to Y$ such that $H|(Z_i \times I) = H_i$ for $i \in J$. Clearly, H connects h to the constant y_0. ∎

4. Proof of Theorem 3. In the proof we will use the following lemma.

Lemma 1. *Let Z be a CW-complex and let z_0 be a vertex of Z. Let $h, h'\colon Z \to Y$ be mappings to a space Y such that $h(z_0) = h'(z_0) = y_0$. Let $G\colon Z \times I \to Y$ be a homotopy which connects h to h', and let $u\colon I \to Y$ be the loop based at y_0 given by $u(s) = G(z_0, s)$, $s \in I$. If $u \simeq y_0$* (rel ∂I), *then there is a homotopy $H\colon Z \times I \to Y$ which connects h to h' and $H(z_0, s) = y_0$ for $s \in I$, i.e., H is a homotopy* (rel z_0).

Proof. By assumption, there is a homotopy $K\colon I \times I \to Y$ such that $K(s, 0) = u(s)$, $K(s, 1) = y_0$ for $s \in I$ and $K(0, t) = K(1, t) = y_0$ for $t \in I$. Endow $A = Z \times I$ with the CW-structure whose cells are products of cells of Z and I, respectively. Then $B = (Z \times 0) \cup (Z \times 1) \cup (z_0 \times I)$ is a subcomplex of A. Define a mapping L from $(A \times 0) \cup (B \times I) \subseteq Z \times I \times I$ to Y by

$$L(z, s, 0) = G(z, s), \quad s \in I, \tag{1}$$

$$L(z, 0, t) = h(z), \quad s \in I, \tag{2}$$

$$L(z, 1, t) = h'(z), \quad s \in I, \tag{3}$$

$$L(z_0, s, t) = K(s, t), \quad s \in I. \tag{4}$$

It is readily verified that formulae (1)–(4) are compatible. Since (A, B) is a pair consisting of a CW-complex A and a subcomplex B, the homotopy extension theorem applies and yields an extension $\tilde{L}$ of L, defined on $A \times I = Z \times I \times I$. We now define the desired homotopy $H\colon Z \times I \to Y$ by putting

$H(z,s) = L(z,s,1)$. Formulae (2)–(4) show that indeed $H(z,0) = h(z)$, $H(z,1) = h'(z)$ for $z \in Z$ and $H(z_0,s) = y_0$ for $s \in I$. ∎

Proof of Theorem 3. By assumption, there is a homotopy $G\colon Z \times I \to Y$ such that $G(z,0) = h(z)$ and $G(z,1) = y_0$ for $z \in Z$. Consider the loop $u\colon I \to I$ given by $u(s) = G(z_0,s)$. Define a homotopy $G'\colon Z \times I \to Y$ by putting

$$G'(z,t) = \begin{cases} G(z,2t), & 0 \le t \le 1/2, \\ u(2(1-t)), & 1/2 \le t \le 1. \end{cases} \tag{5}$$

Then G' is well defined, because $G(z,1) = y_0 = G(z_0,1) = u(1)$. Note that $G'(z,0) = G(z,0) = h(z)$ and $G'(z,1) = u(0) = y_0$ for $z \in Z$. Now consider the loop $u'\colon I \to Y$ given by $u'(t) = G'(z_0,t)$. Let us show that

$$u' \simeq y_0 \,(\mathrm{rel}\ \partial I). \tag{6}$$

Indeed, by (5),

$$u'(t) = \begin{cases} u(2t), & 0 \le t \le 1/2, \\ u(2(1-t)), & 1/2 \le t \le 1. \end{cases} \tag{7}$$

Denote by $v\colon I \to I$ the inverse of the loop u, i.e. let $v(s) = u(1-s)$. Then $v(2t-1) = u(2(1-t))$ and we see that u' is the concatenation uv of the loops u and v. Therefore, $u' \simeq y_0$ (rel ∂I). A homotopy $U\colon I \times I \to Y$ which realizes the latter relation can be defined by putting

$$U(s,t) = \begin{cases} u(2st), & 0 \le t \le 1/2, \\ u(2s(1-t)), & 1/2 \le t \le 1. \end{cases} \tag{8}$$

We can now apply Lemma 1, taking h, y_0, G' and u' for h, h', G and u, respectively. We obtain a homotopy $H\colon Z \times I \to Y$ which has all the desired properties. ∎

5. An example. In this section we will exhibit a connected 1-dimensional subset Z of the Euclidean plane $\mathbb{R}^2$ and an essential phantom mapping $h\colon Z \to S^1$. Note that $\dim S^1 = 1$. The example shows that neither in Proposition 1 nor in Theorem 1 can the assumption that Z is a CW-complex be omitted.

Let us first describe the space $Z \subseteq \mathbb{R}^2$ and the mapping $h\colon Z \to S^1$. Let $r_1 = 1 > r_2 > \ldots > -1$ be a sequence of real numbers with $\lim r_i = -1$. Put $a_i = (r_{2i-1}, 0)$, $b_i = (r_{2i}, 1) \in \mathbb{R}^2$ and $* = (-1,0) \in \mathbb{R}^2$, for $i \in \mathbb{N}$. Let Z^+ be the union of $\{*\}$ and of all segments $[a_i, b_i]$ and $[a_{i+1}, b_i]$. Let S^+ and S^- be the upper and the lower halves of S^1, respectively. We define Z by putting $Z = Z^+ \cup S^-$. Let $h^+\colon Z^+ \to S^+$ be the mapping which sends $(x,y) \in Z^+$ to the only point $(x,y') \in S^+$ and let $h^-\colon S^- \to S^-$ be the identity mapping on S^-. Note that both mappings are continuous and coincide on the intersection $Z^+ \cap S^- = \{(-1,0),(1,0)\}$. Since the sets Z^+, S^- are closed subsets of Z and $Z = Z^+ \cup S^-$, there is a well-defined mapping $h\colon Z \to S^1$ such that $h|Z^+ = h^+$ and $h|S^- = h^-$. Note that $h\colon Z \to S^1$ is a bijection, $h(*) = *$ and $h(a_1) = a_1$. We will prove that h is an essential phantom mapping.

First note that Z is not compact, because it is not closed in $\mathbb{R}^2$. Indeed, (b_i) is a sequence in Z with $\lim b_i = (-1,1) \notin Z$. Therefore, every compact subset $C \subseteq Z$ is a proper subset of Z, i.e., there is a point $z_C \in Z$ which does not belong to C. Since $h\colon Z \to S^1$ is a bijection, the restriction $h|(Z \setminus \{z_C\})$ is a bijection onto $S^1 \setminus h(z_C)$. Since $S^1 \setminus h(z_C)$ is homeomorphic to $\mathbb{R}$, it is contractible. Therefore, every mapping of a space into $S^1 \setminus h(z_C)$ is homotopic to a constant. In particular, $h|(Z \setminus \{z_C\})$ is homotopic to a constant. Since $C \subseteq Z \setminus \{z_C\}$, it follows that also $h|C$ is homotopic to a constant, and thus h is a phantom mapping.

It remains to prove that h is an essential mapping, i.e., it is not homotopic to a constant. The proof is by contradiction. Assume that there is a homotopy $H\colon Z \times I \to S^1$ which connects h to a constant. Since S^1 is pathwise connected, there is no loss of generality in assuming that the constant is $* = (-1,0)$, and thus $H(z,0) = h(z)$ and $H(z,1) = *$, for $z \in Z$. Let $\varphi\colon \mathbb{R} \to S^1$ be the universal covering of S^1, i.e., $\varphi(t) = (\cos 2\pi t, \sin 2\pi t)$ for $t \in \mathbb{R}$. Consider the constant mapping $1/2\colon Z \to \mathbb{R}$ and note that $\varphi(1/2) = (-1,0) = *$. Therefore, by the homotopy lifting theorem ([2, Proposition 1.30]), there exists a homotopy $\tilde{H}\colon Z \times I \to \mathbb{R}$ such that $\varphi\tilde{H} = H$ and $\tilde{H}(z,1) = 1/2$. Let $\tilde{h}\colon Z \to \mathbb{R}$ be the mapping defined by $\tilde{h}(z) = \tilde{H}(z,0)$. Note that $\varphi\tilde{h}(z) = \varphi\tilde{H}(z,0) = H(z,0) = h(z)$ and $\varphi\tilde{h}(*) = h(*) = *$.

Choose open neighborhoods V of $* = (-1,0)$ in S^1 and $\tilde{V}$ of $\tilde{h}(*)$ in $\mathbb{R}$ such that $\varphi|\tilde{V}\colon \tilde{V} \to V$ is a homeomorphism. We also require that $\tilde{V}$ is an open interval in $\mathbb{R}$ and $a_1 = (1,0) \notin V$. Then choose an open neighborhood U of $*$ in Z such that $\tilde{h}(U) \subseteq \tilde{V}$. Also choose $j \in \mathbb{N}$, $j > 1$, so large that $a_j \in U$. This is possible, because $\lim a_i = *$. Note that the set

$$A = S^- \cup \bigcup_{i=1}^{j-1} ([a_i, b_i] \cup [a_{i+1}, b_i])$$

is an arc with endpoints $*$ and a_j. Consequently, there exists a homeomorphism $u\colon [0,1] \to A \subseteq Z$ such that $u(0) = *$ and $u(1) = a_j$. Note that $hu\colon [0,1] \to S^1$ is a path with initial point $hu(0) = h(*) = *$ and terminal point $hu(1) = h(a_j)$. Consider the path $\tilde{v} = \tilde{h}u\colon [0,1] \to \mathbb{R}$. Note that $\tilde{v}(0) = \tilde{h}u(0) = \tilde{h}(*) \in \tilde{V}$ and $\tilde{v}(1) = \tilde{h}u(1) = \tilde{h}(a_j) \in \tilde{V}$, because $a_j \in U$ and $\tilde{h}(U) \subseteq \tilde{V}$. Moreover, $\varphi\tilde{v} = hu$, because $\varphi\tilde{h} = h$. Since $a_1 \in A \setminus \{*, a_j\}$ and $u(0) = *$, $u(1) = a_j$, there is an $s_0 \in [0,1]$, $0 < s_0 < 1$, such that

$a_1 = u(s_0)$. Since $\varphi\tilde{v}(s_0) = hu(s_0) = h(a_1) = a_1$, we cannot have $\tilde{v}(s_0) \in \tilde{V}$, because that would imply $a_1 = \varphi\tilde{v}(s_0) \in \varphi(\tilde{V}) = V$, contrary to the assumption that $a_1 \notin V$. Consequently, $\tilde{v}(s_0) \notin \tilde{V}$.

Now notice that $\tilde{v}\colon [0,1] \to \mathbb{R}$ is an injection. Assuming the contrary, we would have points $s_1, s_2 \in [0,1]$ such that $s_1 \neq s_2$, but $\tilde{v}(s_1) = \tilde{v}(s_2)$. This would imply $\varphi\tilde{v}(s_1) = \varphi\tilde{v}(s_2)$, or equivalently $hu(s_1) = hu(s_2)$, which is impossible, because both mappings u and h are injections, hence also hu is an injection. It is an elementary fact that an injection of $[0,1]$ into $\mathbb{R}$ is an order preserving or an order reversing function. In both cases $\tilde{v}(s_0)$ lies between the points $\tilde{v}(0)$ and $\tilde{v}(1)$. Since these points belong to $\tilde{V}$ and $\tilde{V}$ is an interval in $\mathbb{R}$, it follows that also $\tilde{v}(s_0) \in \tilde{V}$. However, this contradicts the previously established relation $\tilde{v}(s_0) \notin \tilde{V}$. ∎

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