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PROBABILITY THEORY AND STOCHASTIC PROCESSES

On the Maximal Lévy–Ottaviani Inequality for Sums of Independent and Dependent Random Vectors

by

Zbigniew S. SZEWCZAK

Presented by Stanisław KWAPIEŃ

Summary. We prove that the sums S_k of independent random vectors satisfy

$$P\left(\max_{1\le k\le n}\|S_k\|>3t\right)\le 2\max_{1\le k\le n}P(\|S_k\|>t),\qquad t\ge 0.$$

1. Introduction and result. Let $\{X_k\}_{k\in\mathbb{N}}$, $\mathbb{N}=\{1,2,3,\ldots\}$, be a sequence of independent random vectors defined on a probability space $(\Omega,\mathcal{F},P)$. Set $S_n=\sum_{k=1}^n X_k, S_0=0$. The main result of this paper is inspired by the following statement on p. 23 in [13]:

"It is known that *if the inequality*

$$(*)\qquad P\Big(\max_{1\le i\le n}\|S_i\|>t\Big)\le K\max_{1\le i\le n}P\Big(\|S_i\|>\frac{t}{L}\Big)$$

holds true for each $n=1,2,\ldots,$ *for each* $t>0$, *and for all sequences of independent random vectors with values in an arbitrary Banach space* $\mathbf{F}$, *then* $L\ge 3$ *and* $K\ge 2$. We do not know if

$$P\Big(\max_{1\le i\le n}\|S_i\|>t\Big)\le 2\max_{1\le i\le n}P\Big(\|S_i\|>\frac{t}{3}\Big)."$$

In this paper we prove

THEOREM. *If* $X_1,\ldots,X_n$ *are independent random vectors, then for each* $t\ge 0$,

$$(1.1)\qquad P\Big(\max_{1\le k\le n}\|S_k\|>3t\Big)\le 2\max_{1\le k\le n}P(\|S_k\|>t).$$

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The inequality $(*)$ with $L = K = 4$ is stated in [5, Theorem 1(i)]. In [16, pp. 3–4] the proof of $(*)$ with $L = K = 3$ by S. Kwapień can be found (see also [13, p. 15]). This proof (see also [13, p. 16], [11, p. 149]) uses the following variant of the Lévy–Ottaviani inequality (cf. [14], [15]):

$$P\Big(\max_{1\le k\le n}\|S_k\| > s+t\Big) \le \frac{P(\|S_n\|>t)}{1-\max_{1\le k\le n}P(\|S_n-S_k\|>s)},\qquad s,t\ge 0, \tag{1.2}$$

where s is such that $\max_{1\le k\le n}P(\|S_n-S_k\|>s)<1$. It should be pointed out that $(*)$ with $K=(3+\sqrt{5})/2$, $L=3$ is stated without proof in [13, p. 23] (on p. 28 this result is attributed to J. Sawa).

There are many important applications of the Lévy–Ottaviani inequality (cf. [2, p. 296], [7]), for example it proved to be useful for verification of the Anscombe condition (cf. [4, p. 103], [6, p. 17]) and relative stability (cf. [1], [17, Theorem 2]).

The paper is organized as follows: in the next section we prove (1.1), while in the last section we generalize it to dependent vectors and give applications to moment inequalities.

2. Proof of Theorem. We use Etemadi's approach (cf. [5, p. 215], [2, p. 288]), instead of (1.2) as in [13]. For fixed $s,t\ge 0$ and $k=1,\dots,n$ consider the sets

$$C_k=\{\|S_n-S_{n-j}\|\le s+t \text{ for all } 0\le j<k,\ \|S_n-S_{n-k}\|>s+t\}.$$

The sets C_k, $k=1,\dots,n$, are disjoint and

$$\bigcup_{k=1}^n C_k=\Big\{\max_{1\le k\le n}\|S_n-S_{n-k}\|>s+t\Big\}=\Big\{\max_{0\le k<n}\|S_n-S_k\|>s+t\Big\}.$$

Since $\{\|S_n\|\le t;\ \|S_n-S_{n-k}\|>s+t\}\subseteq\{\|S_{n-k}\|>s\}$ we obtain

$$\begin{aligned}P\Big(\max_{0\le k<n}\|S_n-S_k\|>s+t;\ \|S_n\|\le t\Big)&=\sum_{k=1}^n P(C_k;\ \|S_n\|\le t)\\&\le\sum_{k=1}^n P(C_k;\ \|S_{n-k}\|>s)=\sum_{k=1}^n P(C_k)P(\|S_{n-k}\|>s)\\&\le\max_{1\le k<n}P(\|S_{n-k}\|>s)P\Big(\max_{0\le k<n}\|S_n-S_k\|>s+t\Big).\end{aligned}$$

Thus we get

$$\begin{aligned}P\Big(\max_{0\le k<n}\|S_n-S_k\|>s+t;\ \|S_n\|\le t\Big)\\ \le\max_{1\le k<n}P(\|S_k\|>s)P\Big(\max_{0\le k<n}\|S_n-S_k\|>s+t\Big).\end{aligned}\tag{2.1}$$

The above inequality and the obvious inclusion

$$\Big\{\max_{1\le k\le n}\|S_k\| > s+2t\Big\} \subseteq \{\|S_n\| > t\} \cup \Big\{\max_{0\le k<n}\|S_n - S_k\| > s+t;\ \|S_n\| \le t\Big\}$$

yield

$$P\Big(\max_{1\le k\le n}\|S_k\| > s+2t\Big) \le P(\|S_n\| > t) + \max_{1\le k<n} P(\|S_k\| > s). \tag{2.2}$$

Thus (1.1) follows from (2.2) for $s = t$.

3. Dependent random vectors. Let $\{X_k\}_{k\in\mathbb{N}}$ be a sequence of random vectors defined on a probability space $(\Omega, \mathcal{F}, P)$. Define

$$\psi_n^* = \sup_{k\in\mathbb{N}} \sup\Big\{\frac{P(A\cap B)}{P(A)P(B)} : P(A)P(B) > 0,\ A \in \mathcal{F}_1^k,\ B \in \mathcal{F}_{n+k}^\infty\Big\},$$

where $\mathcal{F}_k^m$ is the σ-field generated by $X_k, X_{k+1}, \ldots, X_{k+m}$, $m \in \mathbb{N}$. For properties of the coefficient ψ_n^*, see [3, Vol. I, Chapter 5]. In this section we provide some inequalities for sequences of dependent random vectors satisfying $\psi_1^* < \infty$. Among such sequences are digits of the regular continued fraction expansion of irrational numbers. In this case $\psi_1^* \le 2\ln 2 < 1.39$ (with respect to the Gauss measure) and $\psi_1^* \le \frac{12\ln 2}{12-\pi^2\ln 2} < 1.62$ (with respect to the Lebesgue measure) (cf. [9, Corollary 1.3.15, p. 49] and [3, Vol. I, Proposition 5.2(I)(c), p. 153]).

It is not difficult to see from the first part of the proof of the Theorem (details are left to the reader) that for $n \ge 1$ and $s, t \ge 0$,

$$P\Big(\max_{0\le k<n}\|S_n - S_k\| > s+t; \|S_n\| \le t\Big) \le \psi_1^* \max_{1\le k<n} P(\|S_k\| > s) P\Big(\max_{0\le k<n}\|S_n - S_k\| > s+t\Big), \tag{3.1}$$

and for s such that $\psi_1^* \max_{1\le k<n} P(\|S_n - S_k\| > s) < 1$,

$$P\Big(\max_{1\le k\le n}\|S_k\| > s+t\Big) \le \frac{P(\|S_n\| > t)}{1 - \psi_1^* \max_{1\le k<n} P(\|S_n - S_k\| > s)} \tag{3.2}$$

(apply (3.1) to $X_k^{\mathrm{rev}} := X_{n-k+1}$, $k = 1, \ldots, n$). In particular,

$$(1+\psi_1^*) \max_{1\le k\le n} P(\|S_k\| > t) \ge P\Big(\max_{0\le k<n}\|S_n - S_k\| > 2t\Big). \tag{3.3}$$

Further, by the second part of the proof of the Theorem,

$$P\Big(\max_{1\le k\le n}\|S_k\| > s+2t\Big) \le P(\|S_n\| > t) + \psi_1^* \max_{1\le k<n} P(\|S_k\| > s) P\Big(\max_{0\le k<n}\|S_n - S_k\| > s+t\Big), \tag{3.4}$$

and for s such that $\psi_1^* \max_{1\le k<n} P(\|S_k\| > s) < 1$,

$$P\Big(\max_{1\le k\le n} \|S_k\| > s + 2t\Big) \le \frac{P(\|S_n\| > t)}{1 - \psi_1^* \max_{1\le k<n} P(\|S_k\| > s)}. \tag{3.5}$$

Thus by (3.4) we get

COROLLARY. *Let $X_1, \dots, X_n$ be a sequence of random vectors such that $\psi_1^* < \infty$. Then*

$$P\Big(\max_{1\le k\le n} \|S_k\| > 3t\Big) \le (1 + \psi_1^*) \max_{1\le k\le n} P(\|S_k\| > t). \tag{3.6}$$

A generalization of (3.2) for the coefficient ϕ (see [3, Vol. I, Definition 3.3, p. 67]), which is weaker than ψ^* and asymmetric (cf. [3, Vol. I, Proposition 5.2(III)(a), p. 153]), is due to Iosifescu (cf. [8, Théorème 3, pp. 327–328], [10, Lemma 1.1.6, p. 8]). An extension of the latter is formula (13) in [17, p. 69], while the analogue of (3.6) is stated in [17, Proposition 8, p. 68].

As a simple application of these inequalities we consider moment inequalities. Let $q \ge 1$. Then setting in (3.5)

$$s = \Big((\psi_1^* + \sqrt{2^q \psi_1^*}) \max_{1\le k<n} E(\|S_k\|^q)\Big)^{1/q},$$

we see by Markov's inequality that for every $t > 0$,

$$P\Big(\max_{1\le k\le n} \|S_k\| > s + 2t\Big) \le (1 + \sqrt{\psi_1^*/2^q}) P(\|S_n\| > t).$$

Therefore integration yields

$$E\Big(\Big(\max_{1\le k\le n} \|S_k\| - s\Big)^+\Big)^q \le 2^q (1 + \sqrt{\psi_1^*/2^q}) E\|S_n\|^q.$$

This and the inequality

$$((a-b)^+)^q \ge 2^{1-q} a^q - b^q, \qquad a, b \ge 0,$$

with $a = \max_{1\le k\le n} \|S_k\|$ and $b = s$ give

$$2^{2q-1} (1 + \sqrt{\psi_1^*/2^q})^2 \max_{1\le k\le n} E(\|S_k\|^q) \ge E \max_{1\le k\le n} \|S_k\|^q \tag{3.7}$$

(cf. [18, p. 339]). More generally we have

PROPOSITION. *Let $X_1, \dots, X_n$ be a sequence of random vectors such that $\psi_1^* < \infty$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing function. Then*

$$E\Big(\varphi\Big(\max_{1\le k\le n} \|S_k\|\Big)\Big) \le 2(1 + \psi_1^*) \max_{1\le k\le n} E(\varphi(3\|S_k\|)). \tag{3.8}$$

We adapt the proof due to S. Kwapień (cf. [12]). Setting $s = t$ in (3.5) yields

$$P\Big(\max_{1\le k\le n} \|S_k\| > 3t\Big) \le \frac{P(\|S_n\| > t)}{1 - \psi_1^* \max_{1\le k<n} P(\|S_k\| > t)}.$$

Define

$$t_0 := \inf\Big\{t : 2\psi_1^* \max_{1\le k<n} P(\|S_k\| > t) \le 1\Big\},$$

and let k_0 be such that

$$\max_{1\le k<n} P(\|S_k\| \ge t_0) = P(\|S_{k_0}\| \ge t_0).$$

Then $2\psi_1^* P(\|S_{k_0}\| \ge t_0) \ge 1$ and

$$\frac{1}{1-\psi_1^* \max_{1\le k<n} P(\|S_k\| > t)} \le 2$$

for $t \ge t_0$, while $2\psi_1^* P(\|S_{k_0}\| > t) \ge 1$ for $t < t_0$. Therefore,

$$P\Big(\max_{1\le k\le n} \|S_k\| > 3t\Big) \le 2P(\|S_n\| > t) + 2\psi_1^* P(\|S_{k_0}\| > t),$$

for all $t \in \mathbb{R}^+$. Hence, by Proposition 0.2.1 of [13], applied to $\varphi(t/3)$,

$$E\Big(\varphi\Big(\max_{1\le k\le n} \|S_k\|\Big)\Big) \le 2E(\varphi(3\|S_n\|)) + 2\psi_1^* E(\varphi(3\|S_{k_0}\|)),$$

which concludes the proof of the Proposition.

Remark. For independent random vectors the Proposition was obtained by S. Kwapień (cf. [12]) with constant 6; here we have replaced it by 4. Observe also that for $\varphi(t) = t^4$ from (3.8) we get constant 324, while (3.7) gives 200. On the other hand, for $q = 8$ the inequality (3.7) gives 36992, while (3.8) yields 26244.

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Zbigniew S. Szewczak
Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
87-100 Toruń, Poland
E-mail: zssz@mat.uni.torun.pl

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