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SET THEORY

Marczewski–Burstin Representations of Boolean Algebras Isomorphic to a Power Set

by

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Summary. The paper contains some sufficient conditions for Marczewski–Burstin representability of an algebra $\mathcal{A}$ of sets which is isomorphic to $\mathcal{P}(X)$ for some X. We characterize those algebras of sets which are inner MB-representable and isomorphic to a power set. We consider connections between inner MB-representability and hull property of an algebra isomorphic to $\mathcal{P}(X)$ and completeness of an associated quotient algebra. An example of an infinite universally MB-representable algebra is given.

1. Introduction. Let Y be a nonempty set and let $\mathcal{F}$ be a family of subsets of Y. Following the idea of Burstin and Marczewski we define

$$S(\mathcal{F}) = \{A \subset Y : (\forall P \in \mathcal{F})(\exists Q \in \mathcal{F})(Q \subset A \cap P \text{ or } Q \subset P \setminus A)\}$$

and

$$S_0(\mathcal{F}) = \{A \subset Y : (\forall P \in \mathcal{F})(\exists Q \in \mathcal{F})(Q \subset P \setminus A)\}.$$

Then $S(\mathcal{F})$ is an algebra of subsets of Y, and $S_0(\mathcal{F})$ is an ideal on Y. Note that $Y \in S(\mathcal{F})$ so $S(\mathcal{F})$ is a field of sets. (See [12], [4].)

We say that an algebra $\mathcal{A}$ (respectively, a pair $\langle \mathcal{A}, \mathcal{I} \rangle$, where $\mathcal{I}$ is an ideal contained in an algebra $\mathcal{A}$) of subsets of Y has a *Marczewski–Burstin representation* (for short, an *MB-representation*) if there exists a family $\mathcal{F}$ of subsets of Y such that $\mathcal{A} = S(\mathcal{F})$ (respectively, $\langle \mathcal{A}, \mathcal{I} \rangle = \langle S(\mathcal{F}), S_0(\mathcal{F}) \rangle$). If additionally $\mathcal{F} \subset \mathcal{A}$ (respectively, $\mathcal{F} \cap \mathcal{A} = \emptyset$) then we say that $\langle \mathcal{A}, \mathcal{I} \rangle$ is *inner* (respectively, *outer*) *MB-representable*. Observe that if $\mathcal{F}$ is empty or if the empty set belongs to $\mathcal{F}$ then $\langle S(\mathcal{F}), S_0(\mathcal{F}) \rangle = \langle \mathcal{P}(Y), \mathcal{P}(Y) \rangle$. We exclude this case from our considerations.

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The operations S and S_0 were introduced by Marczewski [15] who applied them to the family of all perfect subsets of a Polish topological space Y. Thus he obtained a new pair of a σ-algebra and a σ-ideal of sets, latter studied by several authors. An old result of Burstin [9] states that the pair consisting of the σ-algebra of Lebesgue measurable sets in $\mathbb{R}$ and the σ-ideal of Lebesgue null sets in $\mathbb{R}$ is of the form $\langle S(\mathcal{F}), S_0(\mathcal{F})\rangle$, where $\mathcal{F}$ consists of the perfect sets of positive measure. (Burstin worked earlier than Marczewski and he did not use the operations S and S_0 explicitly.) MB-representations of several algebras and ideals of sets were recently considered in [4], [1], [8], [10], [13], [16].

Certain algebras of sets have rather natural MB-representations (e.g. the sets with the Baire property or the sets with nowhere dense boundary). On the other hand, the constructions of collections $\mathcal{F}$ MB-representing the interval algebra or the algebra of Borel sets are nontrivial and need (in the case of Borel sets) some special set-theoretical assumptions [4], [1].

We know only two ideas leading to a construction of a non-MB-representable algebra [1], [3], and only one example of such an algebra is given in ZFC ([3]). On the other hand, for every Boolean algebra $\mathcal{A}$ there exists a set Y and a family $\mathcal{F} \subset \mathcal{P}(Y)$ such that $S(\mathcal{F})$ is isomorphic to $\mathcal{A}$ and $S_0(\mathcal{F}) = \{\emptyset\}$ (see [3]). P. Koszmider [11] has proposed the following definition. A Boolean algebra $\mathcal{A}$ is called *universally MB-representable* if whenever $\mathcal{B} \subset \mathcal{P}(Y)$ is an algebra of sets isomorphic to $\mathcal{A}$, then $\mathcal{B} = S(\mathcal{F})$ for some $\mathcal{F} \subset \mathcal{P}(Y)$. It is easy to see that a finite Boolean algebra $\mathcal{A}$ is universally MB-representable. For a family $\mathcal{F}$ of MB-generators we can take $\mathcal{B} \setminus \{\emptyset\}$ for an algebra $\mathcal{B}$ isomorphic to $\mathcal{A}$, or (what is equivalent) the family of atoms of $\mathcal{B}$.

The following problem seems natural: "Is the algebra $\mathcal{P}(X)$ of all subsets of some infinite set X universally MB-representable?" We discuss several aspects of this question in this paper. We find some sufficient conditions for the MB-representability of an algebra of sets which is isomorphic to $\mathcal{P}(X)$, and we obtain a characterization of such algebras which are inner MB-representable. This characterization seems to be the most useful result of this paper and enables us to study connections between properties of pairs $\langle \mathcal{A}, \mathcal{H}(\mathcal{A})\rangle$ (where $\mathcal{H}(\mathcal{A})$ is the ideal of hereditary sets of $\mathcal{A}$) such as: hull property, inner MB-representability, and the completeness of the quotient algebra $\mathcal{A} / \mathcal{H}(\mathcal{A})$. We also give two examples: of an MB-representable algebra of sets which is neither inner nor outer MB-representable, and of an outer MB-representable algebra which is not strongly outer MB-representable. Although the full answer to the problem of universal MB-representability of algebras $\mathcal{P}(X)$ remains open, we give a new example of an infinite universally MB-representable algebra.

An early version of this paper is [6]. Some applications of the main results have recently been obtained in [7].

2. Useful facts and notation

DEFINITION 1. Let $\mathcal{F}_1, \mathcal{F}_2 \subset \mathcal{P}(Y)$. We say that $\mathcal{F}_1, \mathcal{F}_2$ are *mutually coinitial* if $\mathcal{F}_1$ is dense in $\mathcal{F}_2$ and vice versa, i.e., any set in one of the families $\mathcal{F}_1$, $\mathcal{F}_2$ has a subset belonging to the other one.

FACT 1 ([4]). *If $\mathcal{F}_1, \mathcal{F}_2$ are mutually coinitial then $\langle S(\mathcal{F}_1), S_0(\mathcal{F}_1)\rangle = \langle S(\mathcal{F}_2), S_0(\mathcal{F}_2)\rangle$. Conversely, if $\langle S(\mathcal{F}_1), S_0(\mathcal{F}_1)\rangle = \langle S(\mathcal{F}_2), S_0(\mathcal{F}_2)\rangle$ and $\mathcal{F}_i \subset S(\mathcal{F}_i)$ for $i = 1, 2$ then $\mathcal{F}_1, \mathcal{F}_2$ are coinitial.*

FACT 2 ([2]). *$\langle \mathcal{A}, \mathcal{I}\rangle$ is inner MB-representable if and only if $\langle \mathcal{A}, \mathcal{I}\rangle = \langle S(\mathcal{A}\setminus\mathcal{I}), S_0(\mathcal{A}\setminus\mathcal{I})\rangle$. Moreover, if $\langle \mathcal{A}, \mathcal{I}\rangle$ is inner MB-representable then so is $\langle \mathcal{A}, \mathcal{H}(\mathcal{A})\rangle$.*

Consider now a Boolean algebra $\mathcal{A} \subset \mathcal{P}(Y)$ with maximal element Y, isomorphic to a power set algebra $\mathcal{P}(X)$. The isomorphism means that there exists a monomorphism $\Phi : \mathcal{P}(X) \to \mathcal{P}(Y)$ such that $\Phi(X) = Y$ and $\mathcal{A} = \mathrm{im}(\Phi)$ (i.e. $\mathcal{A} = \Phi(\mathcal{P}(X))$). (By a morphism from one Boolean algebra to another one we understand a function preserving the Boolean algebra operations.)

Denote by Z the union of all atoms of $\mathcal{A}$, i.e.

$$Z = \bigcup_{x\in X} \Phi(\{x\}). \tag{1}$$

Then we have two homomorphisms of Boolean algebras

$$\Phi_1 : \mathcal{P}(X) \to \mathcal{P}(Z) \quad \text{and} \quad \Phi_2 : \mathcal{P}(X) \to \mathcal{P}(Y \setminus Z)$$

defined by

$$\Phi_1(A) = \Phi(A) \cap Z, \tag{2}$$

$$\Phi_2(A) = \Phi(A) \setminus Z, \tag{3}$$

for any $A \in \mathcal{P}(X)$. Observe that Φ_1 is a monomorphism and $\Phi_1(X) = Z$, $\Phi_2(X) = Y \setminus Z$. We can describe Φ_1 by the formula

$$\Phi_1(A) = \bigcup_{x\in A} \Phi(\{x\}) \quad \text{for } A \in \mathcal{P}(X).$$

Denote by $\mathcal{J}$ the kernel of Φ_2 (in symbols, $\mathcal{J} = \mathrm{Ker}\,\Phi_2$). Then

$$\mathcal{J} = \Big\{A \in \mathcal{P}(X) : \Phi(A) = \bigcup_{x\in A} \Phi(\{x\})\Big\}. \tag{4}$$

Note that $\mathcal{J}$ contains all finite subsets of X. Moreover $\mathcal{J} = \mathcal{P}(X)$ if and only if $Z = Y$ or (what is equivalent) if Φ_2 is the zero-homomorphism. By standard algebraic considerations, the algebra $\mathcal{B}$ of subsets of $Y \setminus Z$ defined by

$$\mathcal{B} = \mathrm{im}(\Phi_2) \tag{5}$$

is isomorphic to the quotient algebra $\mathcal{P}(X)/\mathcal{J}$. The symbols $Z, \Phi_1, \Phi_2, \mathcal{J}, \mathcal{B}$ defined respectively by (1)–(5) retain their meaning throughout the paper. If $\Phi(A) = \Phi_1(A) \cup \Phi_2(A)$, we will write $\Phi = \langle \Phi_1, \Phi_2 \rangle$.

For any $A \in \mathcal{P}(X)$ denote by $[A]$ the equivalence class of A in the quotient Boolean algebra $\mathcal{P}(X)/\mathcal{J}$.

FACT 3. *For any ideal $\mathcal{J} \subset \mathcal{P}(X)$ which contains all finite subsets of X there exists a set Y and a monomorphism $\Phi : \mathcal{P}(X) \to \mathcal{P}(Y)$ such that $\Phi = \langle \Phi_1, \Phi_2 \rangle$, $\Phi(X) = Y$ and* $\operatorname{Ker} \Phi_2 = \mathcal{J}$.

Proof. Set $Y = X \cup W$ where W is the Stone space for the quotient algebra $\mathcal{P}(X)/\mathcal{J}$ or the space described in [3, Th. 4] (assume that $X \cap W = \emptyset$). Then we can define $\Phi(A) = A \cup \Psi([A])$ for $A \subset X$, where Ψ is an isomorphism between $\mathcal{P}(X)/\mathcal{J}$ and the corresponding Stone algebra of clopen sets, or the algebra constructed in [3, Th. 4], respectively. We have $\Phi_1(A) = A$ and $\Phi_2(A) = \Psi([A])$. ∎

REMARK 1. Observe that if the quotient Boolean algebra $\mathcal{P}(X)/\mathcal{J}$ is atomic, we do not need the Stone representation in our construction. For W we can take $\operatorname{At}(\mathcal{P}(X)/\mathcal{J})$, i.e. the set of atoms of $\mathcal{P}(X)/\mathcal{J}$, and $\Phi_2(A) = \{a \in \operatorname{At}(\mathcal{P}(X)/\mathcal{J}) : a < [A]\}$ where $<$ denotes the natural order in the Boolean algebra.

DEFINITION 2. A set of the form $\Phi(\{x\})$ (belonging to $\operatorname{im}(\Phi_1)$) will be called a *multipoint atom* of $\mathcal{A} = \operatorname{im}(\Phi)$ if its cardinality $|\Phi(\{x\})|$ is greater than 1.

3. Results.

The following theorem gives a sufficient condition for MB-representability of an algebra $\mathcal{A}$ isomorphic to $\mathcal{P}(X)$.

THEOREM 1. *Assume that $\Phi : \mathcal{P}(X) \to \mathcal{P}(Y)$ is a monomorphism, $\mathcal{A} = \operatorname{im}(\Phi)$ and $\Phi = \langle \Phi_1, \Phi_2 \rangle$. Let $\mathcal{B} = \operatorname{im}(\Phi_2)$. If $\langle \mathcal{B}, \{\emptyset\} \rangle$ has an MB-representation $\langle S(\mathcal{F}_0), S_0(\mathcal{F}_0) \rangle$ for some $\mathcal{F}_0 \subset \mathcal{P}(Y \setminus Z)$ then $\mathcal{A}$ is MB-representable.*

Proof. Suppose that $\mathcal{B} = S(\mathcal{F}_0)$. Let s be a selector of the family $\{\Phi(\{x\}) \colon x \in X\}$. Put $f(x) = s(\Phi(\{x\}))$. Denote by $\mathcal{F}_1$ the family of multipoint atoms of $\mathcal{A}$. Let

$$\mathcal{F}_2 = \{f(A) \cup K : A \notin \mathcal{J},\ K \subset \Phi_2(A),\ K \in \mathcal{F}_0\}.$$

(For any $A \notin \mathcal{J}$ there exists a $K \in \mathcal{F}_0$ included in $\Phi_2(A)$ since $S_0(\mathcal{F}_0) = \{\emptyset\}$.) Take $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$. We claim that $\mathcal{A} = S(\mathcal{F})$. First let us prove $\mathcal{A} \subset S(\mathcal{F})$. Let $P \in \mathcal{A}$. Consider two cases.

1° If $P = \Phi(A)$ for $A \in \mathcal{J}$, then $P = \Phi_1(A)$. Let $F \in \mathcal{F}$. If $F \in \mathcal{F}_1$, we have either $F \subset P$ or $F \subset P^{c}$. Assume that $F \in \mathcal{F}_2$. Thus $F = f(B) \cup K$

for some $B \notin \mathcal{J}$, and $K \in \mathcal{F}_0$ with $K \subset \Phi_2(B)$. Hence

$$F \setminus P = (f(B) \setminus P) \cup K = (f(B) \setminus \Phi_1(A)) \cup K = f(B \setminus A) \cup K \in \mathcal{F}_2$$

because $[B \setminus A] = [B]$ and $K \subset \Phi_2(B \setminus A) = \Phi_2(B)$.

2° Suppose that $P = \Phi(A)$ for some $A \notin \mathcal{J}$. Then $P = \Phi_1(A) \cup \Phi_2(A)$. Let $F \in \mathcal{F}$. For $F \in \mathcal{F}_1$ we have either $F \subset P$ or $F \subset P^c$. If $F \in \mathcal{F}_2$ then $F = f(B) \cup K$ for some $B \notin \mathcal{J}$ and $K \in \mathcal{F}_0$ with $K \subset \Phi_2(B)$. Consider the set $\Phi_2(A) \cap \Phi_2(B)$ (maybe empty). Then either there exists a $K_1 \in \mathcal{F}_0$ such that $K_1 \subset K \cap \Phi_2(A \cap B) \subset \Phi_2(A \cap B)$, or there exists a $K_2 \in \mathcal{F}_0$ such that $K_2 \subset K \setminus \Phi_2(A \cap B) \subset \Phi_2(B \setminus A)$. In the first case $B \cap A \notin \mathcal{J}$ and $Q_1 = f(A \cap B) \cup K_1 \subset F \cap P$. In the second case $B \setminus A \notin \mathcal{J}$ and $Q_2 = f(B \setminus A) \cup K_2 \subset F \setminus P$. So the inclusion $\mathcal{A} \subset S(\mathcal{F})$ has been proved.

To show that $S(\mathcal{F}) \subset \mathcal{A}$ assume that $P \notin \mathcal{A}$. There are the following three possibilities:

(I) P separates the points of some multipoint atom $\Phi(\{x\})$ of $\mathcal{A}$. Then for $F = \Phi(\{x\})$ there is no $Q \in \mathcal{F}$ such that either $Q \subset F \cap P$ or $Q \subset F \setminus P$.

(II) $P = \Phi_1(A) \cup \Phi_2(B)$ where $[B] \neq [A]$ in $\mathcal{P}(X)/\mathcal{J}$. Then $A \triangle B \notin \mathcal{J}$. Assume that $A \setminus B \notin \mathcal{J}$. Let $F = f(A \setminus B) \cup K$, $K \in \mathcal{F}_0$, $K \subset \Phi_2(A \setminus B)$. We have $F \cap P = f(A \setminus B)$ and $F \setminus P = K$. None of these sets contains a set $Q \in \mathcal{F}$. For $B \setminus A \notin \mathcal{J}$ the argument is quite similar.

(III) $P = \Phi_1(A) \cup S$ where $S \subset Y \setminus Z$ and $S \notin \mathcal{B}$. Then there exists a set $K \in \mathcal{F}_0$ such that any $K_1 \in \mathcal{F}_0$ is contained neither in $K \cap S$ nor in $K \setminus S$. Set $F = f(X) \cup K$. Then neither $F \cap P$ nor $F \setminus P$ includes any set from $\mathcal{F}$. ■

Let us make some comments on the above proof. Observe that without the sets from $\mathcal{F}_1$ we cannot show that for any $P \in S(\mathcal{F})$ the set $P \cap Z$ is of the form $\Phi_1(A)$ for some $A \in \mathcal{P}(X)$. On the other hand, if we do not use the selector s for the sets $\Phi(\{x\})$, then any set $P = \bigcup_{x \in A} \Phi(\{x\})$ with $|\Phi(\{x\})| > 1$ for $x \in A$ belongs to $S(\mathcal{F})$ (even though A does not belong to $\mathcal{J}$). Note that, for any ideal $\mathcal{J} \subset \mathcal{P}(X)$ containing all finite sets, there exists a monomorphism Φ such that $\mathcal{A} = \mathrm{im}(\Phi)$ satisfies the assumptions of Theorem 2 and $\mathcal{J} = \mathrm{Ker}\,\Phi_2$. This follows from [3, Thm. 4] applied to $\mathcal{P}(X)/\mathcal{J}$ and from Fact 3.

The following theorem gives a characterization of inner MB-representability of an algebra $\mathcal{A}$ isomorphic to $\mathcal{P}(X)$.

THEOREM 2. *Let $\mathcal{A} = \mathrm{im}(\Phi)$ where $\Phi : \mathcal{P}(X) \to \mathcal{P}(Y)$ is a monomorphism, $\Phi = \langle \Phi_1, \Phi_2 \rangle$ and let $\mathcal{B} = \mathrm{im}(\Phi_2)$. Then $\mathcal{A}$ is inner MB-representable if and only if the following two conditions are satisfied simultaneously*:

- (∗) *the set of all $x \in X$ such that $\Phi(\{x\})$ is a multipoint atom of $\mathcal{A}$, belongs to $\mathcal{J}$,*
- (∗∗) *the algebra $\mathcal{B}$ is atomic and the atoms of $\mathcal{B}$ cover $Y \setminus Z$.*

Proof. Suppose that $\mathcal{A} = S(\mathcal{F})$ where $\mathcal{F} \subset \mathcal{A}$. Then $\mathcal{A} = S(\mathcal{A} \setminus \mathcal{H}(\mathcal{A}))$ (Fact 2) where $\mathcal{H}(\mathcal{A})$ is the ideal of hereditary sets in $\mathcal{A}$, that is,

$$\mathcal{H}(\mathcal{A}) = \{A \in \mathcal{A} : (\forall B \subset A)(B \in \mathcal{A})\}.$$

For $\mathcal{A} = \mathrm{im}(\Phi)$ we have $\mathcal{H}(\mathcal{A}) = \{\Phi_1(A) \colon A \in \mathcal{J} \;\&\; (\forall x \in A)(|\Phi(\{x\})| = 1)\}$. Hence the families $\mathcal{A} \setminus \mathcal{H}(\mathcal{A})$ and $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ are mutually coinitial where $\mathcal{F}_1$ is the set of multipoint atoms of $\mathcal{A}$, and $\mathcal{F}_2$ consists of the sets of the form $F = \Phi_1(B) \cup \Phi_2(B)$ for $B \notin \mathcal{J}$. Thus $S(\mathcal{A} \setminus \mathcal{H}(\mathcal{A})) = S(\mathcal{F})$ (Fact 1).

Denote by A_0 the set of all $x \in X$ such that $\Phi(\{x\})$ contains more than one point. Consider $\Phi_1(A_0) = \bigcup_{x \in A_0} \Phi(\{x\})$. Then $\Phi_1(A_0) \in \mathcal{A}$ if and only if $\Phi_1(A_0) = \Phi(A_0)$, which means that $A_0 \in \mathcal{J}$. On the other hand, we have:

- $F \subset \Phi_1(A_0)$ for any multipoint atom F,
- for any $F = \Phi_1(B) \cup \Phi_2(B)$ with $B \in \mathcal{P}(X)$, either the set $A_0 \cap \Phi_1(B)$ contains a multipoint atom of $\mathcal{A}$, or $F \cap \Phi_1(A_0) = \emptyset$.

So $\Phi_1(A_0) \in S(\mathcal{A} \setminus \mathcal{H}(\mathcal{A})) = \mathcal{A}$ and consequently $A_0 \in \mathcal{J}$. Condition $(*)$ has been proved.

To show $(**)$, consider an arbitrary $y \in Y \setminus Z$. The singleton $\{y\}$ does not belong to $\mathcal{A}$. Hence there exists a set $F \in \mathcal{F}$ such that $y \in F$ and $F \setminus \{y\}$ has no subsets from $\mathcal{F}$.

Consequently, there exists $F = \Phi_2(A) \cup \Phi_1(A)$ such that $y \in \Phi_2(A)$ and no proper subset of $\Phi_2(A)$ belongs to $\mathcal{B}$. (If not, i.e. if $\Phi_2(B) \subsetneq \Phi_2(A)$, then either $F_1 = \Phi_2(B) \cup \Phi_1(B) \subset F \setminus \{y\}$ or $F_2 = \Phi_2(A \setminus B) \cup \Phi_1(A \setminus B) \subset F \setminus \{y\}$.) Hence $\Phi_2(A)$ is an atom of $\mathcal{B}$ and $y \in \Phi_2(A)$.

Conversely, suppose that $(*)$ and $(**)$ are satisfied. Denote by $\mathrm{At}(\mathcal{B})$ the family of all atoms of $\mathcal{B}$. Let $\Phi_2(A) \in \mathrm{At}(\mathcal{B})$. Consequently, $[A] \in \mathrm{At}(\mathcal{P}(X)/\mathcal{J})$. It follows that for any $B \in \mathcal{P}(X)$ exactly one of the sets $A \cap B$ and $A \setminus B$ belongs to $\mathcal{J}$. Denote by $\mathcal{F}_1$ the family of multipoint atoms of $\mathcal{A}$, and by $\mathcal{F}_2$ the family of sets of the form $F = \Phi_1(A) \cup \Phi_2(A)$ where $\Phi_2(A) \in \mathrm{At}(\mathcal{B})$. We will show that $\mathcal{A} = S(\mathcal{F})$. First we prove that $\mathcal{A} \subset S(\mathcal{F})$. We have $\mathcal{A} \subset S(\mathcal{A} \setminus \mathcal{I})$ for any proper ideal $\mathcal{I}$ in $\mathcal{A}$ ([4]). Since $\mathcal{F}$ and $\mathcal{A} \setminus \mathcal{H}(\mathcal{A})$ are mutually coinitial, we have $\mathcal{A} \subset S(\mathcal{A} \setminus \mathcal{H}(\mathcal{A})) = S(\mathcal{F})$.

To prove $S(\mathcal{F}) \subset \mathcal{A}$ suppose that $P \notin \mathcal{A}$. Then we have one of the following possibilities:

(i) P separates the points of some atom $\Phi(\{x\})$ of $\mathcal{A}$. Then $F = \Phi(\{x\})$ contains no subsets of $F \cap P$ and of $F \setminus P$ which belong to $\mathcal{F}$.

(ii) P separates the points of some atom $T \in \mathrm{At}(\mathcal{B})$. Then we can choose a set A such that $T = \Phi_2(A)$ and $\Phi_1(A)$ does not contain any multipoint atom (by $(*)$). The set $F = \Phi_1(A) \cup T = \Phi_1(A) \cup \Phi_2(A)$ is a "bad" set in $\mathcal{F}$ for P (i.e., $P \cap F$ and $F \setminus P$ do not contain any set from $\mathcal{F}$).

(iii) $P = \Phi_1(A) \cup D$ where D does not separate points of any atom of $\mathcal{B}$ but $D \neq \Phi_2(A)$. Then one of the sets $\Phi_2(A) \setminus D$ or $D \setminus \Phi_2(A)$ contains a $T \in \mathrm{At}(\mathcal{B})$. Let $T = \Phi_2(B)$. Then we can choose a set B so that either $\Phi_1(B) \subset \Phi_1(A)$ or $\Phi_1(B) \cap \Phi_1(A) = \emptyset$, respectively, and B does not contain any multipoint atom of $\mathcal{A}$. Then the set $F = \Phi_1(B) \cup T = \Phi_1(B) \cup \Phi_2(B)$ is "bad" for P. ∎

REMARK 2. If the set $\Phi_2(X) = Y \setminus Z$ is empty, then $\mathcal{A} = S(\mathrm{At}(\mathcal{A}))$. This representation is evidently inner.

THEOREM 3. *Let $\Phi : \mathcal{P}(X) \to \mathcal{P}(Y)$ be a monomorphism and $\mathcal{A} = \mathrm{im}(\Phi)$. Let $\Phi = \langle \Phi_1, \Phi_2 \rangle$ and $\mathcal{B} = \mathrm{im}(\Phi_2)$. If $\mathcal{B}$ is atomic and the atoms of $\mathcal{B}$ cover the set $Y \setminus Z$ then $\mathcal{A}$ is MB-representable.*

Proof. Let $\mathcal{F}_1$ be the family of multipoint atoms of $\mathcal{A}$. Let s be a selector of the family $\{\Phi(\{x\}) \colon x \in X\}$ and $f(x) = s(\Phi(\{x\})$. Put

$$\mathcal{F}_2 = \{f(A) \cup \Phi_2(A) \colon A \in \mathcal{P}(X),\ \Phi_2(A) \in \mathrm{At}(\mathcal{B})\}.$$

Take $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$. Combining the reasonings from the proofs of Theorems 1 and 2 we obtain $\mathcal{A} = S(\mathcal{F})$. ∎

Now we give examples of pairs $\langle \mathcal{A}, \mathcal{I} \rangle$ where $\mathcal{A}$ is an algebra isomorphic to a power set and $\mathcal{I} \subset \mathcal{A}$ is an ideal with some interesting properties.

DEFINITION 3. Let $\mathcal{I} \subset \mathcal{A} \subset \mathcal{P}(Y)$ where $\mathcal{I}$ is an ideal and $\mathcal{A}$ is an algebra. We say that:

(i) the pair $\langle \mathcal{A}, \mathcal{I} \rangle$ has the *hull property* provided for every $U \subset Y$ there is a $V \in \mathcal{A}$ (called a *hull* of U) such that $U \subset V$ and for every $W \in \mathcal{A}$ if $U \subset W$ then $V \setminus W \in \mathcal{I}$;

(ii) $\langle \mathcal{A}, \mathcal{I} \rangle$ is *complete* provided the quotient algebra $\mathcal{A}/\mathcal{I}$ is complete.

(iii) $\langle \mathcal{A}, \mathcal{I} \rangle$ is *topological* provided $\langle \mathcal{A}, \mathcal{I} \rangle = \langle S(\tau \setminus \{\emptyset\}), S_0(\tau \setminus \{\emptyset\}) \rangle$ for some topology τ on Y; then $\mathcal{I}$ forms the ideal of nowhere dense sets and $\mathcal{A}$ forms the algebra of sets with nowhere dense boundary in τ.

Baldwin [5] showed that:

(a) if $\langle \mathcal{A}, \mathcal{I} \rangle$ has the hull property then $\langle \mathcal{A}, \mathcal{I} \rangle$ has an inner MB-representation;

(b) the hull property and completeness of $\langle \mathcal{A}, \mathcal{I} \rangle$ do not follow from each other.

In [2] the authors gave an example of a pair $\langle \mathcal{A}, \mathcal{I} \rangle$ which is inner MB-representable but does not have the hull property. Any topological pair $\langle \mathcal{A}, \mathcal{I} \rangle$ is complete and has the hull property.

Now, we are in a position to prove:

THEOREM 4. (1) *There exists an algebra $\mathcal{A}$ isomorphic to $\mathcal{P}(X)$ such that $\langle \mathcal{A}, \mathcal{H}(\mathcal{A}) \rangle$ is complete but is not inner MB-representable*, *and consequently does not have the hull property.*

(2) *There exists an algebra* $\mathcal{A}$ *isomorphic to* $\mathcal{P}(X)$ *such that* $\langle \mathcal{A}, \mathcal{H}(\mathcal{A})\rangle$ *has an inner MB-representation but does not have the hull property.*
(3) *If the algebra* $\mathcal{A}$ *is isomorphic to* $\mathcal{P}(X)$ *and* $\langle \mathcal{A}, \mathcal{H}(\mathcal{A})\rangle$ *has the hull property then this pair is complete.*

Proof. (1) Let $\mathcal{A} = \mathrm{im}(\Phi)$ for a monomorphism $\Phi : \mathcal{P}(X) \to \mathcal{P}(Y)$, $\Phi = \langle \Phi_1, \Phi_2\rangle$ where:

- $Y = Z \cup T$ for $Z = (X \times \{0\}) \cup (X \times \{1\})$ and some $T \neq \emptyset$,
- $\Phi(\{x\}) = \{\{x\} \times \{0\}, \{x\} \times \{1\}\}$.

Let $\mathcal{J}$ be a maximal ideal in $\mathcal{P}(X)$. Then $\Phi(A) = \Phi_1(A) \cup \Phi_2(A)$ where $\Phi_2(A) = \emptyset$ for $A \in \mathcal{J}$ and $\Phi_2(A) = T$ for $A \in \mathcal{P}(X) \setminus \mathcal{J}$. The pair $\langle \mathcal{A}, \{\emptyset\}\rangle$ is complete, because $\mathcal{A}/\{\emptyset\}$ is isomorphic to $\mathcal{A}$ and consequently to $\mathcal{P}(X)$. On the other hand, $\mathcal{A}$ is not inner MB-representable because $\Phi(\{x\})$ is a multipoint atom for every $x \in X$. Note that, as opposed to Baldwin's example, the algebra $\mathcal{A}/\mathcal{H}(\mathcal{A}) = \mathcal{A}/\{\emptyset\}$ is atomic.

(2) Let now $\mathcal{A} = \mathrm{im}(\Phi)$ for a monomorphism $\Phi : \mathcal{P}(X) \to \mathcal{P}(Y)$ with $\Phi = \langle \Phi_1, \Phi_2\rangle$ where

- $\Phi(\{x\})$ is singleton for every $x \in X$,
- $|X| = \omega$,

and let

- $\{X_\alpha : \alpha < 2^\omega\}$ be a family of almost disjoint subsets of X with $|X_\alpha| = \omega$,
- $\mathcal{J} = \bigcap_{\alpha<2^\omega} \mathcal{J}_\alpha$, where $\{\mathcal{J}_\alpha : \alpha < 2^\omega\}$ is a family of maximal ideals in $\mathcal{P}(X)$ with the following properties:
  (a) $(\forall\alpha < 2^\omega)([X]^{<\omega} \in \mathcal{J}_\alpha)$,
  (b) $(\forall\alpha < 2^\omega)$ $(X_\alpha \notin \mathcal{J}_\alpha)$, and consequently,
  (c) $(\forall\alpha, \beta < 2^\omega)(\alpha \neq \beta \Rightarrow X_\alpha \in \mathcal{J}_\beta)$.

Then for $\alpha \neq \beta$ we have $[X_\alpha] \neq [X_\beta]$ in $\mathcal{P}(X)/\mathcal{J}$, and $[X_\alpha]$ is an atom in $\mathcal{P}(X)/\mathcal{J}$ because for every $B \in \mathcal{P}(X)$ we have either $B \cap X_\alpha \in \mathcal{J}$ or $X_\alpha \setminus B \in \mathcal{J}$. So $|\mathrm{At}(\mathcal{P}(X)/\mathcal{J})| = 2^\omega$. Take W and Φ_2 as in Remark 1. Let $Y = X \cup W$ and $\Phi_1(A) = A$ for any $A \subset X$. Then for $\mathcal{B} = \mathrm{im}(\Phi_2)$ we have $|\mathrm{At}(\mathcal{B})| = 2^\omega$. Moreover, $|\mathcal{B}| \leq 2^\omega$ because $\mathcal{B}$ is a homomorphic image of $\mathcal{P}(X)$ and $|\mathcal{P}(X)| = 2^\omega$. Hence there exists a set E which is a union of some atoms of $\mathcal{B}$ but E does not belong to $\mathcal{B}$. (We have 2^{2^ω} different sets which are unions of atoms of $\mathcal{B}$.) We claim that E does not have a hull. Indeed, if $E \subset \Phi_1(B)\cup\Phi_2(B)$ then there exists an X_α such that $E \subset \Phi_1(B\setminus X_\alpha)\cup\Phi_2(B\setminus X_\alpha)$ and $\Phi_1(X_\alpha) \cup \Phi_2(X_\alpha) \notin \mathcal{H}(\mathcal{A})$.

(3) Assume that $\langle \mathcal{A}, \mathcal{H}(\mathcal{A})\rangle$ (where $\mathcal{A}$ is isomorphic to $\mathcal{P}(X)$) has the hull property. Then $\langle \mathcal{A}, \mathcal{H}(\mathcal{A})\rangle$ is inner MB-representable. So conditions $(*)$ and $(**)$ of Theorem 2 are satisfied. We claim that $\mathcal{P}(X)/\mathcal{J}$ is complete. Indeed, if not then there exists a set E which is a union of atoms of an

algebra $\mathcal{B} = \mathrm{im}(\Phi_2)$ (isomorphic to $\mathcal{P}(X)/\mathcal{J}$) and does not belong to $\mathcal{B}$. (The supremum in $\mathcal{B}$ is simply the union of a family of sets because the atoms of $\mathcal{B}$ cover all the set $Y \setminus Z$.) By the same arguments as in (2), the set E does not have a hull.

We now show that the algebra $\mathcal{A}/\mathcal{H}(\mathcal{A})$ is also complete. Recall that by $(*)$, the set

$$A_0 = \{x \in X : |\Phi(\{x\})| > 1\}$$

belongs to $\mathcal{J}$. Define $\mathcal{J}_0 = \{A \in \mathcal{J} : A \cap A_0 = \emptyset\}$. We can consider $\mathcal{J}_0$ as an ideal in $\mathcal{P}(X)$ and also in $\mathcal{P}(X \setminus A_0)$. Observe that

$$\mathcal{H}(\mathcal{A}) = \{\Phi(A) : A \in \mathcal{J} \text{ and } (\forall x \in A)(|\Phi(\{x\})| = 1\}$$

is the image of $\mathcal{J}_0$ under the monomorphism Φ. Hence by standard algebraic considerations we observe that

- $\mathcal{A}/\mathcal{H}(\mathcal{A})$ is isomorphic to $\mathcal{P}(X)/\mathcal{J}_0$.
- $\mathcal{P}(X)/\mathcal{J}_0$ is isomorphic to the direct sum of $\mathcal{P}(X \setminus A_0)/\mathcal{J}_0$ and $\mathcal{P}(A_0)$.
- $\mathcal{P}(X \setminus A_0)/\mathcal{J}_0$ is isomorphic to $\mathcal{P}(X)/\mathcal{J}$.

So $\mathcal{A}/\mathcal{H}(\mathcal{A})$ is isomorphic to the direct sum of two complete algebras $\mathcal{P}(X)/\mathcal{J}$ and $\mathcal{P}(A_0)$. Consequently, the pair $\langle \mathcal{A}, \mathcal{H}(\mathcal{A}) \rangle$ is complete. ∎

REMARK 3. The proof of part (2) of Theorem 4 shows that the implication in Theorem 1 cannot be reversed. Indeed, the constructed algebra $\mathcal{A}$ is MB-representable (even inner MB-representable) but it is not the case for the pair $\langle \mathcal{B}, \{\Theta\} \rangle$. The family $\mathcal{F}_0$ for which $\langle \mathcal{B}, \{\Theta\} \rangle = \langle S(\mathcal{F}_0), S_0(\mathcal{F}_0) \rangle$ must contain all atoms of $\mathcal{B}$ and consequently $S(\mathcal{F}_0) = \mathcal{P}(Y \setminus Z)$.

REMARK 4. Both assumptions in (3) are essential. Indeed, if we take the algebra $\mathcal{P}(\mathbb{R})$ and the ideal of all countable sets, then such a pair has the hull property but is not complete [5]. (The ideal of countable sets is not equal to $\mathcal{H}(\mathcal{P}(\mathbb{R}))$.)

On the other hand, consider the following example. Define $\mathcal{A} \subset \mathcal{P}(\omega)$ as the algebra generated by all possible unions of the sets $\{2n, 2n+1\}$, $n \in \omega$, and finite sets. Then $\mathcal{H}(\mathcal{A})$ is the ideal of finite sets. It is not difficult to see that the pair $\langle \mathcal{A}, \mathcal{H}(\mathcal{A}) \rangle$ has the hull property and that $\mathcal{A}/\mathcal{H}(\mathcal{A})$ is isomorphic to $\mathcal{P}(\omega)/\mathrm{fin}$, hence the pair $\langle \mathcal{A}, \mathcal{H}(\mathcal{A}) \rangle$ is not complete.

Recently, making use of Theorem 2 the author has obtained the following results:

THEOREM 5 ([7]). *The following two conditions are equivalent*:

(I) *there exists a set Y and an algebra $\mathcal{A} \subset \mathcal{P}(Y)$ isomorphic to $\mathcal{P}(\omega)$, with $\langle \mathcal{A}, \mathcal{H}(\mathcal{A}) \rangle$ complete, which is inner MB-representable but not topological*;

(II) *there exists an ideal* $\mathcal{J} \subset \mathcal{P}(\omega)$ *such that* $\mathcal{P}(\omega)/\mathcal{J}$ *is isomorphic to* $\mathcal{P}(\omega_1)$.

Some consequences of Steprāns' results [14] and Theorem 5 lead to the following

COROLLARY 1 ([7]). *The existence of a pair* $\langle \mathcal{A}, \mathcal{I} \rangle$ *which is complete and has the hull property but is not topological is consistent with ZFC.*

In [1] the authors strengthened the notion of outer MB-representability of an algebra of sets. We say that an algebra $\mathcal{A} \subset \mathcal{P}(X)$ is *strongly outer MB-representable* if for any family $\mathcal{C} \subset \mathcal{P}(X)$ such that $\mathcal{A} \subset \mathcal{C}$ and $|\mathcal{C}| = |\mathcal{A}|$ there exists a family $\mathcal{F} \subset \mathcal{P}(X)$ disjoint from $\mathcal{C}$ for which $\mathcal{A} = S(\mathcal{F})$. Evidently, if $\mathcal{A}$ is strongly outer MB-representable then it is outer MB-representable. Now, we show that the converse does not hold.

THEOREM 6. *There exist algebras of sets* $\mathcal{A}_1, \mathcal{A}_2$ *which are isomorphic to some power sets and additionally have the following properties*:

(a) $\mathcal{A}_1$ *is outer MB-representable but not strongly outer MB-representable.*

(b) $\mathcal{A}_2$ *is MB-representable but neither inner nor outer MB-representable.*

Proof. The proof is based on two simple observations:

OBSERVATION 1. *If an algebra* $\mathcal{A}$ *of sets has an atom* A *such that* $|A| = 2$ *then* $\mathcal{A}$ *is not outer MB-representable.*

Indeed, let $A = \{x, y\}$. Suppose that $\mathcal{A} = S(\mathcal{F})$ for some family $\mathcal{F}$ of sets. Since A does not belong to $\mathcal{H}(\mathcal{A})$, there exists an $F \in \mathcal{F}$ such that $F \subset A$. If $F = \{x\}$ or $F = \{y\}$ then $F \in \mathcal{A}$, which is impossible because A is an atom. So $F = \{x, y\}$ and the representation is not outer.

OBSERVATION 2. *If an infinite algebra of sets* $\mathcal{A}$ *has an atom* A *such that* $1 < |A| < \infty$ *then* $\mathcal{A}$ *is not strongly outer representable.*

Indeed, let $\mathcal{C} = \mathcal{A} \cup \mathcal{P}(A)$. Then $\mathcal{A} \subset \mathcal{C}$ and $|\mathcal{A}| = |\mathcal{C}|$. But if $\mathcal{A} = S(\mathcal{F})$ then there exists a set $F \in \mathcal{F}$ such that $F \subset A$ and hence $F \in \mathcal{C}$. So, the representation is not strongly outer.

Now we are in a position to construct the algebras with the desired properties.

(a) Let $\mathcal{A}_1$ be an algebra with an infinite number of atoms such that any atom A contains exactly three points. A set B belongs to $\mathcal{A}$ if and only if B is the union of atoms. (Then $\mathcal{A}$ is isomorphic to $\mathcal{P}(X)$ where $|X| = |\mathrm{At}\,\mathcal{A}|$.) By Observation 2, $\mathcal{A}_1$ is not strongly MB-representable. On the other hand, $\mathcal{A}_1 = S(\mathcal{F})$ where $\mathcal{F}$ consists of all sets which have exactly two elements and are contained in atoms of $\mathcal{A}_1$.

(b) Let X be an infinite set and $\mathcal{A}_2 = \mathrm{im}(\Phi)$ for a monomorphism $\Phi : \mathcal{P}(X) \to \mathcal{P}(Y)$ and $\Phi = \langle \Phi_1, \Phi_2 \rangle$. Assume that $|\Phi(\{x\})| = 2$ for all $x \in X$ and $\mathcal{J}$ is a proper maximal ideal in $\mathcal{P}(X)$. Then $\mathcal{A}_2$ is MB-representable by Theorem 1 but it is not inner MB-representable by Theorem 2 and it is not outer MB-representable by Observation 1. ∎

The next theorem shows that there exists an infinite universally MB-representable algebra.

THEOREM 7. *Let X be an infinite set. Then the algebra $\mathcal{A}$ consisting of the finite and cofinite subsets of X is universally MB-representable.*

Proof. Standard algebraic considerations show that any monomorphism $\Phi : \mathcal{A} \to \mathcal{P}(Y)$ is of the form $\Phi(A) = \bigcup_{x\in A} \Phi(\{x\})$ if A is finite, and $\Phi(A) = \bigcup_{x\in A} \Phi(x) \cup T$, where $T = Y \setminus \bigcup_{x\in X} \Phi(\{x\})$, if A is cofinite. Denote by $\mathcal{A}'$ the image of $\mathcal{A}$ under Φ. Let s be a selector of the family $\Phi(\{x\})$. Put $f(x) = s(\Phi(\{x\})) \in \Phi(\{x\})$ for $x \in X$. If $\mathcal{F}_1$ is the family of all multipoint atoms $\Phi(\{x\})$ and $\mathcal{F}_2$ consists of all sets of the form $f(\Phi_1(A)) \cup T$ for A cofinite, then $\mathcal{A}' = S(\mathcal{F}_1 \cup \mathcal{F}_2)$. The proof of this fact is quite similar to the proof of Theorem 2. ∎

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