Some Remarks on Indicatrices of Measurable Functions

by

Marcin KYSIAK

Presented by Czesław RYLL-NARDZEWSKI

Summary. We show that for a wide class of σ-algebras $\mathcal{A}$, indicatrices of $\mathcal{A}$-measurable functions admit the same characterization as indicatrices of Lebesgue-measurable functions. In particular, this applies to functions measurable in the sense of Marczewski.

Let $f : X \to Y$ be a function. The function $s(f) : Y \to \text{CARD}$ defined by the formula $s(f)(y) = |f^{-1}([y])|$ is called the (Banach) indicatrix of $f$. For $f, g : X \to Y$, we say that $f$ is equivalent to $g$ if there exists a bijection $\varphi : X \to X$ such that $f = g \circ \varphi$. Obviously, this is equivalent to saying that $s(f) = s(g)$.

Morayne and Ryll-Nardzewski show in [5] that a function $f : [0, 1] \to [0, 1]$ is equivalent to a Lebesgue-measurable one if, and only if, either $s(f) > 0$ on a perfect set $P \subseteq [0, 1]$ or there exists $y \in [0, 1]$ such that $s(f)(y) = \infty$. In fact, they prove a more general statement. Namely, the same is true for the class of functions which are measurable with respect to the $\sigma$-algebra $\mathcal{A}$ generated by the Borel sets and a $\sigma$-ideal $\mathcal{I}$ with Borel base containing an uncountable set. They also ask about a characterization of indicatrices of other important classes of functions.

A characterization of indicatrices of continuous functions was given by Kwiatkowska in [4]. Also, Komisarski, Michalewski and Milewski in [3] characterized (under the axiom of $\Sigma^1_1$-determinacy) indicatrices of Borel functions.

The purpose of this note is to generalize the characterization of Morayne and Ryll-Nardzewski to other classes of measurable functions. We say that a set $X \subseteq [0, 1]$ is Marczewski-measurable if for every perfect set $P \subseteq [0, 1]$ $s(1_P) = 0$. We show that $f$ is Marczewski-measurable if and only if $s(1_f) = 0$. 

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there exists a perfect set $Q \subseteq P$ such that $Q \subseteq X$ or $Q \cap X = \emptyset$. The Marczewski-measurable sets form a $\sigma$-algebra; a function $f : [0,1] \to [0,1]$ is Marczewski-measurable if the pre-image of every open set is Marczewski-measurable. By Marczewski’s theorem (see [7]) this is equivalent to saying that for every perfect set $P$ there exists a perfect set $Q \subseteq P$ such that $f|Q$ is continuous.

We begin by showing that indicatrices of Marczewski-measurable functions admit the same characterization as those of Lebesgue-measurable ones. It is known that the algebra of Marczewski-measurable sets is not of the form considered in [5]. Then we try to isolate the properties of Marczewski-measurable sets and functions used in the proof to obtain a more general result.

For a family $\mathcal{A}$ of sets, let $\mathcal{H}(\mathcal{A}) = \{A \in \mathcal{A} : \forall B \subseteq A \ B \in \mathcal{A}\}$. Observe that if $\mathcal{A}$ is a $\sigma$-algebra, then $\mathcal{H}(\mathcal{A})$ is a $\sigma$-ideal.

The following lemma is a slight modification of an argument from [5]. The main difference is that we do not use the assumption of Borel base of the ideal.

**Lemma 1.** Let $\mathcal{A}$ be a $\sigma$-algebra containing $\text{Bor}$ such that $\mathcal{H}(\mathcal{A})$ contains a set of size $\mathfrak{c}$. Let $f : [0,1] \to [0,1]$ be a function such that $f|[[0,1]]$ contains a perfect set. Then $f$ is equivalent to an $\mathcal{A}$-measurable function.

**Proof.** Let $P$ be a perfect set contained in the image of $f$; we may always assume that $|f|[[0,1]] \setminus P| = \mathfrak{c}$. Let $\Psi : [0,1] \to P$ be a Borel isomorphism and let $M \in \mathcal{H}(\mathcal{A})$ be a set of cardinality $\mathfrak{c}$ such that $|[0,1] \setminus M| = \mathfrak{c}$. Observe that $\Psi$ is $\mathcal{A}$-measurable.

Let $s(f) : [0,1] \to \text{CARD}$ be the indicatrix of $f$ and let $\{M_y : y \in [0,1]\}$ be a partition of $M$ such that $|M_y| = s(f)(y) - 1$ for $y \in \Psi[[0,1] \setminus M]$ (this is meaningful, because $s(f)(y) > 0$ for $y \in P$ and we allow $M_y$ to be empty) and $|M_y| = s(f)(y)$ otherwise. Such a partition can be found because for continuum many $y \in [0,1]$ we stipulate that $|M_y| > 0$, so $\sum_{y \in [0,1]} |M_y| = \mathfrak{c}$. Define $g : [0,1] \to [0,1]$ in the following way:

$$g(x) = \begin{cases} 
\Psi(x) & \text{for } x \notin M, \\
y & \text{for } x \in M_y.
\end{cases}$$

Clearly, $g$ is equivalent to $f$ because they have the same indicatrix, and $g$ is $\mathcal{A}$-measurable, as

$$\{x \in [0,1] : g(x) \neq \Psi(x)\} \subseteq M \in \mathcal{H}(\mathcal{A}).$$

Using exactly the same argument as in [5], one can prove the following.

**Lemma 2.** Let $\mathcal{A}$ be a $\sigma$-algebra containing $\text{Bor}$ such that $\mathcal{H}(\mathcal{A})$ contains a set of size $\mathfrak{c}$. Let $f : [0,1] \to [0,1]$ be a function constant on a set of cardinality $\mathfrak{c}$. Then $f$ is equivalent to an $\mathcal{A}$-measurable function.
Theorem 3. A function \( f : [0, 1] \to [0, 1] \) is equivalent to a Marczewski-measurable one if, and only if, either \( f([0, 1]) \) contains a perfect set, or there exists \( y \in [0, 1] \) such that \( |f^{-1}([y])| = c \). In particular, each Lebesgue measurable function is equivalent to a Marczewski-measurable one, and vice versa.

Proof. It is folklore that the algebra of Marczewski-measurable sets satisfies the assumptions of Lemmas 1 and 2, which shows sufficiency of this condition.

To prove the necessity, we can assume that \( f \) is itself Marczewski-measurable. Then there exists a perfect set \( P \) such that \( f[P] \) is continuous. If \( f[P] \) is uncountable, then it contains a perfect set. Otherwise, there exists \( y \in f[P] \) such that the set \( f^{-1}([y]) \) is of size continuum. ■

One can easily see that the argument above is more general than for Marczewski-measurable functions. The assumptions needed for sufficiency of the characterization (i.e. the assumptions of Lemmas 1 and 2) are very general (as long as the extensions of \( \text{Bor} \) are concerned). To prove the necessity, we only used the fact that a Marczewski-measurable function is continuous on a perfect set.

Let us say that a class of functions \( \mathcal{F} \) from a Polish space to \([0, 1]\) has the Weak Continuous Restriction Property (WCRP for short) if every \( f \in \mathcal{F} \) is continuous on a perfect set. This is a weaker property than the Continuous Restriction Property considered in [6], where the perfect set is required not to belong to a given \( \sigma \)-ideal. It is also a weaker version of a suitable instance of the Sierpiński condition considered in [1].

Let us point out that some natural reformulations of the WCRP are in fact equivalent.

Proposition 4 (folklore). The following conditions are equivalent for \( f : X \to [0, 1] \), where \( X \) is a Polish space:

1. \( f|P \) is continuous for some perfect set \( P \),
2. \( f|B \) is continuous for some uncountable Borel set \( B \),
3. \( f|P \) is Borel for some perfect set \( P \),
4. \( f|B \) is Borel for some uncountable Borel set \( B \). ■

As an immediate generalization of Theorem 3 we obtain the following.

Theorem 5. Let \( \mathcal{A} \) be a \( \sigma \)-algebra of subsets of a Polish space \( X \) containing \( \text{Bor}(X) \) such that \( \mathcal{H}(\mathcal{A}) \) contains a set of size \( c \). Assume that the class of \( \mathcal{A} \)-measurable functions has WCRP. Then a function \( f : X \to X \) is equivalent to an \( \mathcal{A} \)-measurable one if, and only if, either \( f[X] \) contains a perfect set, or there exists \( y \in X \) such that \( |f^{-1}([y])| = c \).

Proof. Analogous to the proof of Theorem 3. ■
An important class of algebras satisfying the assumptions of Theorem 5 are the algebras of sets decided by popular forcing notions. We can interpret the Marczewski-measurable sets as sets decided by the Sacks forcing $\mathbb{S}$ (i.e. sets $X$ such that the set of conditions in $\mathbb{S}$ which either miss $X$ or are included in $X$ is dense). It is folklore that if we replace the Sacks forcing by the forcing notion of Laver, Mathias, Miller or Silver, the functions measurable with respect to the corresponding $\sigma$-algebra have WCRP. Also, each of the respective ideals \(^{(1)}\) contains a set of size $c$ (this follows from the results of [2]). In particular, in the case of Mathias forcing, we obtain the following.

**Corollary 6.** Let $\mathcal{A}$ be the $\sigma$-algebra of completely Ramsey subsets of $2^\omega$. Then a function $f : 2^\omega \to 2^\omega$ is equivalent to an $A$-measurable one if, and only if, either $f[2^\omega]$ contains a perfect set, or there exists $y \in 2^\omega$ such that $|f^{-1}([y])| = c$. ■

**References**


Marcin Kysiak

Institute of Mathematics

Warsaw University

Banacha 2

02-097 Warszawa, Poland

E-mail: mkysiak@mimuw.edu.pl

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\(^{(1)}\) In the case of these forcing notions, the ideal of hereditarily measurable sets coincides with the ideal of sets missed by a dense set of conditions.