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COMPLEX ANALYSIS

The Pluripolar Hull and the Fine Topology

by

Armen EDIGARIAN

Presented by Józef SICIAK

Summary. We show that the projections of the pluripolar hull of the graph of an analytic function in a subdomain of the complex plane are open in the fine topology.

1. Introduction. Let $\Omega \subset \mathbb{C}^n$ be an open set and let $E \subset \Omega$ be any subset. We say that E is *pluripolar* in Ω if for all $z \in E$ there exist a connected neighborhood U_z of z in Ω and a plurisubharmonic function $u(z,w) \not\equiv -\infty$ defined on U_z such that

$$E \cap U_z \subset \{(z,w) \in U_z : u(z,w) = -\infty\}.$$

By Josefson's theorem (see [Jos]), a set $E \subset \mathbb{C}^N$ is pluripolar if and only if there exists a globally defined plurisubharmonic function $u(z,w)$ such that

$$E \subset \{(z,w) \in \mathbb{C}^N : u(z,w) = -\infty\}.$$

By the *pluripolar hull* E^*_Ω (see [LePo]) of a pluripolar subset $E \subset \Omega$, we mean

$$E^*_\Omega := \bigcap \{z \in \Omega : u(z) = -\infty\},$$

where the intersection is taken over *all* plurisubharmonic functions u in Ω which equal $-\infty$ on E. In general, it is difficult to describe the pluripolar hull of a given set E. The following theorem, recently proved in [EdWi3], gives some information about E^*_Ω.

THEOREM 1. *Let Ω be a pseudoconvex open set in $\mathbb{C}^N$ and let $E \subset \Omega$ be an F_σ pluripolar subset. If E is connected then so is E^*_Ω.*

The following main result of the paper gives another property of E^*_Ω.

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Theorem 2. *Let Ω be a pseudoconvex open set in $\mathbb{C}^N$ and let $E \subset \Omega$ be an F_σ pluripolar subset. Assume that $U \subset \Omega$ is an open neighborhood of E^*_Ω and that $f : U \to \mathbb{C}$ is a non-constant holomorphic function. Then for any $p \in f(E^*_\Omega) \setminus f(E)$ the set $\mathbb{C} \setminus f(E^*_\Omega)$ is thin at p.*

Moreover, if $f(E)$ is open in the fine topology then so is $f(E^*_\Omega)$.

A set $F \subset \mathbb{C}$ is *thin* at a point ξ if either $\xi \notin \overline{F}$, or $\xi \in \overline{F}$ and there exists a subharmonic function h in a neighborhood of ξ such that $\overline{\lim}_{z\in F,\, z\to\xi} h(z) < h(\xi)$. One can always choose h in such a way that the upper limit equals $-\infty$ (see e.g. [Ran]).

Starting from a paper of Sadullaev [Sad] the pluripolar hull of graphs of certain analytic functions has been studied in a number of papers (see e.g. [EdWi1]–[EdWi3], [E-J2], [LePo], [Sic], [Wie1], [Wie2], and [Zwo]).

For a subset A of the complex plane $\mathbb{C}$ and a complex-valued function f on A we denote by $\Gamma_f(A)$ the graph of f over A,

$$\Gamma_f(A) = \{(z, w) \in \mathbb{C}^2 : z \in A,\ w = f(z)\}.$$

Let f be a holomorphic function in a domain $D \subset \mathbb{C}$. It is immediate that $\Gamma_f(D)$ is a pluripolar set. Supported by several examples, in [LeMaPo] Levenberg, Martin and Poletsky conjectured that if f is analytic in D and the domain of existence of f is D, then $\Gamma_f(D)$ is complete pluripolar. This conjecture was disproved in [EdWi2] (in case of the unit disc) and in [EdWi1] (in case of a domain $D = \mathbb{C} \setminus K$, where K is a compact polar set).

Denote by π_j the projection onto the jth coordinate plane in $\mathbb{C}^2$, $\pi_j(z) = z_j$ for $z = (z_1, z_2) \in \mathbb{C}^2$, $j = 1, 2$. As a corollary of Theorem 2 we get the following result, which is a positive answer to Problem 1 posed in [E-J1] (in a revised version [E-J2], the authors get independently the first part of the corollary).

Corollary 3. *Let $D \subset \mathbb{C}$ be a domain and let f be an analytic function in D. Then $\pi_1((\Gamma_f(D))^*_{\mathbb{C}^2})$ is open in the fine topology. Moreover, if f is non-constant then $\pi_2((\Gamma_f(D))^*_{\mathbb{C}^2})$ is also open in the fine topology.*

The *fine topology* is the weakest topology for which all subharmonic functions are continuous. A neighborhood basis of a point in this topology consists of sets which differ from a Euclidean neighborhood of this point by a set which is thin at this point (see e.g. [Bre]). Hence, if $A \subset \mathbb{C}$ is any set then A is open in the fine topology if and only if $\mathbb{C} \setminus A$ is thin at each $p \in A$.

2. Preliminary results. Let Ω be a domain in $\mathbb{C}^N$. In [LePo] the *negative pluripolar hull* is defined as

$$E^-_\Omega := \bigcap \{z \in \Omega : u(z) = -\infty\},$$

where the intersection is taken over all *negative* plurisubharmonic functions u in Ω that are $-\infty$ on E. The following relation between the negative pluripolar hull and the pluripolar hull holds (see [LePo]).

THEOREM 4. *Let Ω be a pseudoconvex domain in $\mathbb{C}^N$. Let $\{\Omega_j\}$ be an increasing sequence of relatively compact subdomains of Ω with $\bigcup_j \Omega_j = \Omega$. Let $E \subset \Omega$ be pluripolar. Then*

$$E^*_\Omega = \bigcup_j (E \cap \Omega_j)^-_{\Omega_j}.$$

For a subset $E \subset \Omega$, the *pluriharmonic measure* at a point $z \in \Omega$ of E relative to Ω is defined as

$$(2.1) \quad \omega(z, E, \Omega) = -\sup\{u(z) : u \text{ is plurisubharmonic in } \Omega \text{ and } u \le -\chi_E\},$$

where χ_E is the characteristic function of E. The relation between the negative pluripolar hull and the pluriharmonic measure is given in the following theorem (see [LePo]).

THEOREM 5. *Let Ω be a domain in $\mathbb{C}^N$ and let $E \subset \Omega$ be pluripolar. Then*

$$E^-_\Omega = \{z \in \Omega : \omega(z, E, \Omega) > 0\}.$$

From Theorem 5 we get the following

COROLLARY 6. *Let Ω be a pseudoconvex domain in $\mathbb{C}^N$ and let $E \subset \Omega$ be an F_σ pluripolar subset. Then E^*_Ω is also an F_σ set.*

Proof. Let $E = \bigcup_j K_j$, where $K_1 \subset K_2 \subset \cdots$ are compact sets. Then $E^*_\Omega = \bigcup_j (K_j)^*_\Omega$. So, it sufficient to show that K^*_Ω is an F_σ set for any compact pluripolar set K. Take an increasing sequence of relatively compact hyperconvex domains Ω_j so that $K \subset \Omega_1$ and that $\Omega = \bigcup_j \Omega_j$. Then $K^*_\Omega = \bigcup_{j=1}^\infty K^-_{\Omega_j}$ and $K^-_{\Omega_j} = \bigcup_{k=1}^\infty \{z \in \Omega_j : \omega(z, K, \Omega_j) \ge 1/k\}$. Recall that $\omega(\cdot, K, \Omega_j)$ is an upper semicontinuous function. ∎

The following result is well known. For the sake of completeness we give the proof.

PROPOSITION 7. *Let E be a Borel polar set in $\mathbb{C}$. Then $E^*_{\mathbb{C}} = E$.*

Proof. Fix $z_0 \notin E$. By Choquet's theorem there exists a sequence of open sets $U_1 \supset U_2 \supset \cdots \supset E$ such that $z_0 \notin U_j$ and $c(U_j) \to 0$ when $j \to \infty$. Here c is the logarithmic capacity (see e.g. [Ran]). Put $\widetilde{E} = \bigcap_j U_j$. Then $c(\widetilde{E}) = 0$ (so $\widetilde{E}$ is polar), $\widetilde{E}$ is a G_δ set, $\widetilde{E} \supset E$, and $z_0 \notin \widetilde{E}$. Hence, $\widetilde{E}$ is complete polar and $z_0 \notin E^*_{\mathbb{C}}$. ∎

Recall the following result (see [Anc]).

THEOREM 8 (Ancona's theorem). *Let K be a compact non-polar set in $\mathbb{C}$. Then for any $\varepsilon > 0$ there exists a compact set $K' \subset K$ such that $c(K \setminus K') < \varepsilon$ and K' is regular at any point of itself.*

As a corollary we get the following useful result.

COROLLARY 9. *Let E be a Borel set in $\mathbb{C}$. Assume that E is non-polar. Then there exists a sequence of compact sets $K_1 \subset K_2 \subset \cdots \subset E$, regular at any point of each of them, and a polar Borel set P such that $E = P \cup \bigcup_j K_j$.*

Proof. First note that there exists an F_σ set E_1 and a polar set P_1 so that $E = E_1 \cup P_1$. We have $E_1 = \bigcup_j \widetilde{K}_j$, where $\widetilde{K}_j$ is an increasing sequence of compact sets. Now, it suffices to use Theorem 8. ∎

3. Proof of the main result. Recall the following localization principle [EdWi3].

THEOREM 10. *Let $\Omega \subset \mathbb{C}^n$ be an open set and let E be an F_σ pluripolar subset of Ω. Then for any open set $\Omega' \Subset \Omega$ and any open set U such that $\partial U \cap E^*_\Omega = \emptyset$ we have*

$$\omega(z, E \cap U \cap \Omega', \Omega') = \omega(z, E \cap U \cap \Omega', U \cap \Omega'), \qquad z \in U \cap \Omega'. \tag{3.1}$$

Proof of Theorem 3. Let $p \in f(E^*_\Omega) \setminus f(E)$ and let $z_0 \in f^{-1}(p) \cap E^*_\Omega$. Put $F = \mathbb{C} \setminus f(E^*_\Omega)$. Then F is Borel (G_δ). Assume that F is not thin at p. Hence, there exists a sequence of compact sets $K_1 \subset K_2 \subset \cdots \subset F$, regular at any point of each of them, and a polar Borel set P such that $F \setminus \{p\} = P \cup \bigcup_j K_j$.

Put $U_j = f^{-1}(\mathbb{C} \setminus K_j) \cap U$. Since E is an F_σ set, there exists a sequence of compact sets $E_1 \subset E_2 \subset \cdots \subset E$ such that $E = \bigcup_j E_j$. Then $E^*_\Omega = \bigcup_j (E_j)^*_\Omega$. Hence, $p \in \bigcup_j f\big((E_j)^*_\Omega\big)$. Put $L_j = f(E_j)$.

First, assume that $f(E)$ is non-polar. Then without loss of generality, we may assume that L_1 is non-polar.

Fix a hyperconvex domain $\Omega' \Subset \Omega$. We want to estimate $\omega(z_0, E_j \cap \Omega', \Omega')$. By the localization principle we have

$$\omega(z_0, E_j \cap \Omega', \Omega') = \omega(z_0, E_j \cap \Omega', \Omega' \cap U_k) \le \omega(p, L_j, \widehat{\mathbb{C}} \setminus K_k).$$

We claim that

$$\lim_{k \to \infty} \omega(p, L_j, \widehat{\mathbb{C}} \setminus K_k) = 0. \tag{3.2}$$

Fix $j \ge 1$. For each natural number k we let D_k be the connected component of $\widehat{\mathbb{C}} \setminus K_k$ which contains p. We have

$$\omega(p, L_j, \widehat{\mathbb{C}} \setminus K_k) = \omega(p, L_j \cap D_k, D_k).$$

Note that D_k is a regular domain (see [Ran, Theorem 4.2.4]). Put $h_k(z) = \omega(z, L_j \cap D_k, D_k)$. Then h_k is a harmonic function on D_k. Moreover, it extends subharmonically to $\widehat{\mathbb{C}} \setminus L_j$ (we put $h_k = 0$ on $\widehat{\mathbb{C}} \setminus D_k$). Hence

$h(z) = \lim_{k\to\infty} h_k(z)$ is non-negative and subharmonic on $\widehat{\mathbb{C}} \setminus L_j$ (being the decreasing limit of a sequence of subharmonic functions). Moreover, $h = 0$ on $\bigcup_k K_k$. Since $\bigcup_k K_k$ is non-thin at p, p is an accumulation point of $\bigcup_k K_j$ and $h(p) = 0$. Hence, we have proved (3.2).

So, we have $\omega(z_0, E_j \cap \Omega', \Omega') = 0$. Hence, $z_0 \notin (E_j \cap \Omega')^-_{\Omega'}$. Since $\Omega' \Subset \Omega$ is an arbitrary hyperconvex domain, we get $z_0 \notin (E_j)^*_\Omega$ and $z_0 \notin \bigcup_j (E_j)^*_\Omega$. But we know that $z_0 \in f^{-1}(p) \cap E^*_\Omega$. A contradiction.

Now, assume that $f(E)$ is polar. Note that $f(E^*_\Omega) \subset f(E)^*_{\mathbb{C}} = f(E)$.

Assume that $f(E)$ is open in the fine topology. Take $p \in f(E^*_\Omega)$. Note that there are two cases: $p \in f(E^*_\Omega) \setminus f(E)$ and $p \in f(E)$. In both cases we see that $\mathbb{C} \setminus f(E^*_\Omega)$ is thin at p. ∎

Proof of Corollary 3. Note that $\pi_1(\Gamma_f(D)) = D$ is open and, therefore, open in the fine topology. If f is non-constant then $\pi_2(\Gamma_f(D)) = f(D)$ is also open. ∎

4. Example. Note that in Corollary 3 we cannot state, in general, that $\pi_1((\Gamma_f(D))^*_{\mathbb{C}^2})$ is open. Indeed, take $a_n = 1/n$ and $c_n = e^{-n^2}/n^2$, $n \in \mathbb{N}$. Put

$$f(z) = \sum_{n=1}^{\infty} \frac{c_n}{z - a_n}.$$

Note that f is a holomorphic function on the domain $D = \mathbb{C} \setminus \{a_n : n \in \mathbb{C}\} \cup \{0\}$. By [EdWi1], $(\Gamma_f(D))^*_{\mathbb{C}^2} = \Gamma_f(D) \cup \{(0, f(0))\}$. So, $\pi_1((\Gamma_f(D))^*_{\mathbb{C}^2}) = D \cup \{0\}$.

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Armen Edigarian
Institute of Mathematics
Jagiellonian University
Reymonta 4
30-059 Kraków, Poland
E-mail: Armen.Edigarian@im.uj.edu.pl

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