The Pluripolar Hull and the Fine Topology

by

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Summary. We show that the projections of the pluripolar hull of the graph of an analytic function in a subdomain of the complex plane are open in the fine topology.

1. Introduction. Let $\Omega \subset \mathbb{C}^n$ be an open set and let $E \subset \Omega$ be any subset. We say that $E$ is pluripolar in $\Omega$ if for all $z \in E$ there exist a connected neighborhood $U_z$ of $z$ in $\Omega$ and a plurisubharmonic function $u(z, w) \neq -\infty$ defined on $U_z$ such that

$$E \cap U_z \subset \{(z, w) \in U_z : u(z, w) = -\infty\}.$$

By Josefson’s theorem (see [Jos]), a set $E \subset \mathbb{C}^N$ is pluripolar if and only if there exists a globally defined plurisubharmonic function $u(z, w)$ such that

$$E \subset \{(z, w) \in \mathbb{C}^N : u(z, w) = -\infty\}.$$

By the pluripolar hull $E^*_\Omega$ (see [LePo]) of a pluripolar subset $E \subset \Omega$, we mean

$$E^*_\Omega := \bigcap\{z \in \Omega : u(z) = -\infty\},$$

where the intersection is taken over all plurisubharmonic functions $u$ in $\Omega$ which equal $-\infty$ on $E$. In general, it is difficult to describe the pluripolar hull of a given set $E$. The following theorem, recently proved in [EdWi3], gives some information about $E^*_\Omega$.

Theorem 1. Let $\Omega$ be a pseudoconvex open set in $\mathbb{C}^N$ and let $E \subset \Omega$ be an $F_\sigma$ pluripolar subset. If $E$ is connected then so is $E^*_\Omega$.

The following main result of the paper gives another property of $E^*_\Omega$.

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Theorem 2. Let $\Omega$ be a pseudoconvex open set in $\mathbb{C}^N$ and let $E \subset \Omega$ be an $F_\alpha$ pluripolar subset. Assume that $U \subset \Omega$ is an open neighborhood of $E^*_\Omega$ and that $f : U \to \mathbb{C}$ is a non-constant holomorphic function. Then for any $p \in f(E^*_\Omega) \setminus f(E)$ the set $\mathbb{C} \setminus f(E^*_\Omega)$ is thin at $p$.

Moreover, if $f(E)$ is open in the fine topology then so is $f(E^*_\Omega)$.

A set $F \subset \mathbb{C}$ is thin at a point $\xi$ if either $\xi \not\in \overline{F}$, or $\xi \in F$ and there exists a subharmonic function $h$ in a neighborhood of $\xi$ such that $\lim_{z \to \xi} h(z) < h(\xi)$. One can always choose $h$ in such a way that the upper limit equals $-\infty$ (see e.g. [Ran]).

Starting from a paper of Sadullaev [Sad] the pluripolar hull of graphs of certain analytic functions has been studied in a number of papers (see e.g. [EdWi1]−[EdWi3], [E-J2], [LePo], [Sic], [Wie1], [Wie2], and [Zwo]).

For a subset $A$ of the complex plane $\mathbb{C}$ and a complex-valued function $f$ on $A$ we denote by $\Gamma_f(A)$ the graph of $f$ over $A$,

$\Gamma_f(A) = \{(z, w) \in \mathbb{C}^2 : z \in A, w = f(z)\}$.

Let $f$ be a holomorphic function in a domain $D \subset \mathbb{C}$. It is immediate that $\Gamma_f(D)$ is a pluripolar set. Supported by several examples, in [LeMaPo] Levenberg, Martin and Poletsky conjectured that if $f$ is analytic in $D$ and the domain of existence of $f$ is $D$, then $\Gamma_f(D)$ is complete pluripolar. This conjecture was disproved in [EdWi2] (in case of the unit disc) and in [EdWi1] (in case of a domain $D = \mathbb{C} \setminus K$, where $K$ is a compact polar set).

Denote by $\pi_j$ the projection onto the $j$th coordinate plane in $\mathbb{C}^2$, $\pi_j(z) = z_j$ for $z = (z_1, z_2) \in \mathbb{C}^2$, $j = 1, 2$. As a corollary of Theorem 2 we get the following result, which is a positive answer to Problem 1 posed in [E-J1] (in a revised version [E-J2], the authors get independently the first part of the corollary).

Corollary 3. Let $D \subset \mathbb{C}$ be a domain and let $f$ be an analytic function in $D$. Then $\pi_1((\Gamma_f(D))^{*2})$ is open in the fine topology. Moreover, if $f$ is non-constant then $\pi_2((\Gamma_f(D))^{*2})$ is also open in the fine topology.

The fine topology is the weakest topology for which all subharmonic functions are continuous. A neighborhood basis of a point in this topology consists of sets which differ from a Euclidean neighborhood of this point by a set which is thin at this point (see e.g. [Bre]). Hence, if $A \subset \mathbb{C}$ is any set then $A$ is open in the fine topology if and only if $\mathbb{C} \setminus A$ is thin at each $p \in A$.

2. Preliminary results. Let $\Omega$ be a domain in $\mathbb{C}^N$. In [LePo] the negative pluripolar hull is defined as

$E^-_\Omega := \bigcap\{z \in \Omega : u(z) = -\infty\}$,
where the intersection is taken over all negative plurisubharmonic functions \( u \) in \( \Omega \) that are \(-\infty\) on \( E \). The following relation between the negative pluripolar hull and the pluripolar hull holds (see [LePo]).

**Theorem 4.** Let \( \Omega \) be a pseudoconvex domain in \( \mathbb{C}^N \). Let \( \{\Omega_j\} \) be an increasing sequence of relatively compact subdomains of \( \Omega \) with \( \bigcup_j \Omega_j = \Omega \). Let \( E \subset \Omega \) be pluripolar. Then

\[
E^*_\Omega = \bigcup_j (E \cap \Omega_j)^-_\Omega_j.
\]

For a subset \( E \subset \Omega \), the pluriharmonic measure at a point \( z \in \Omega \) of \( E \) relative to \( \Omega \) is defined as

\[
(2.1) \quad \omega(z, E, \Omega) = -\sup\{u(z) : u \text{ is plurisubharmonic in } \Omega \text{ and } u \leq -\chi_E\},
\]

where \( \chi_E \) is the characteristic function of \( E \). The relation between the negative pluripolar hull and the pluriharmonic measure is given in the following theorem (see [LePo]).

**Theorem 5.** Let \( \Omega \) be a domain in \( \mathbb{C}^N \) and let \( E \subset \Omega \) be pluripolar. Then

\[
E^-_\Omega = \{z \in \Omega : \omega(z, E, \Omega) > 0\}.
\]

From Theorem 5 we get the following

**Corollary 6.** Let \( \Omega \) be a pseudoconvex domain in \( \mathbb{C}^N \) and let \( E \subset \Omega \) be an \( F_\sigma \) pluripolar subset. Then \( E^*_\Omega \) is also an \( F_\sigma \) set.

**Proof.** Let \( E = \bigcup_j K_j \), where \( K_1 \subset K_2 \subset \cdots \) are compact sets. Then \( E^*_\Omega = \bigcup_j (K_j^*)_\Omega \). So, it sufficient to show that \( K^*_\Omega \) is an \( F_\sigma \) set for any compact pluripolar set \( K \). Take an increasing sequence of relatively compact hyperconvex domains \( \Omega_j \) so that \( K \subset \Omega_1 \) and that \( \Omega = \bigcup_j \Omega_j \). Then \( K^*_\Omega = \bigcup_{j=1}^\infty K^-_j \) and \( K^-_j = \bigcup_{k=1}^\infty \{z \in \Omega_j : \omega(z, K, \Omega_j) \geq 1/k\} \). Recall that \( \omega(\cdot, K, \Omega_j) \) is an upper semicontinuous function.

The following result is well known. For the sake of completeness we give the proof.

**Proposition 7.** Let \( E \) be a Borel polar set in \( \mathbb{C} \). Then \( E^*_\mathbb{C} = E \).

**Proof.** Fix \( z_0 \notin E \). By Choquet’s theorem there exists a sequence of open sets \( U_1 \supset U_2 \supset \cdots \supset E \) such that \( z_0 \notin U_j \) and \( c(U_j) \to 0 \) when \( j \to \infty \). Here \( c \) is the logarithmic capacity (see e.g. [Ran]). Put \( \tilde{E} = \bigcap_j U_j \). Then \( c(\tilde{E}) = 0 \) (so \( \tilde{E} \) is polar), \( \tilde{E} \) is a \( G_\delta \) set, \( \tilde{E} \subset E \), and \( z_0 \notin \tilde{E} \). Hence, \( \tilde{E} \) is complete polar and \( z_0 \notin E^*_\mathbb{C} \).

Recall the following result (see [Anc]).
THEOREM 8 (Ancona’s theorem). Let \( K \) be a compact non-polar set in \( \mathbb{C} \). Then for any \( \varepsilon > 0 \) there exists a compact set \( K' \subset K \) such that \( c(K\setminus K') < \varepsilon \) and \( K' \) is regular at any point of itself.

As a corollary we get the following useful result.

COROLLARY 9. Let \( E \) be a Borel set in \( \mathbb{C} \). Assume that \( E \) is non-polar. Then there exists a sequence of compact sets \( K_1 \subset K_2 \subset \cdots \subset E \), regular at any point of each of them, and a polar Borel set \( P \) such that \( E = E_1 \cup P_1 \). We have \( E_1 = \bigcup_j \tilde{K}_j \), where \( \tilde{K}_j \) is an increasing sequence of compact sets. Now, it suffices to use Theorem 8. \( \blacksquare \)

3. Proof of the main result. Recall the following localization principle [EdWi3].

THEOREM 10. Let \( \Omega \subset \mathbb{C}^n \) be an open set and let \( E \) be an \( F_\sigma \) pluripolar subset of \( \Omega \). Then for any open set \( \Omega' \Subset \Omega \) and any open set \( U \) such that \( \partial U \cap E^*_\Omega = \emptyset \) we have

\[
\omega(z, E \cap U \cap \Omega', \Omega') = \omega(z, E \cap U \cap \Omega' \cap \Omega), \quad z \in U \cap \Omega'.
\]

Proof of Theorem 3. Let \( p \in f(E^*_\Omega) \setminus f(E) \) and let \( z_0 \in f^{-1}(p) \cap E^*_\Omega \). Put \( F = \mathbb{C} \setminus f(E^*_\Omega) \). Then \( F \) is Borel (\( G_\delta \)). Assume that \( F \) is not thin at \( p \). Hence, there exists a sequence of compact sets \( K_1 \subset K_2 \subset \cdots \subset F \), regular at any point of each of them, and a polar Borel set \( P \) such that \( F \setminus \{p\} = P \cup \bigcup_j \tilde{K}_j \).

Put \( U_j = f^{-1}(\mathbb{C} \setminus K_j) \cap U \). Since \( E \) is an \( F_\sigma \) set, there exists a sequence of compact sets \( E_1 \subset E_2 \subset \cdots \subset F \) such that \( E = \bigcup_j E_j \). Then \( E^*_\Omega = \bigcup_j (E_j)^*_\Omega \). Hence, \( p \in \bigcup_j f((E_j)^*_\Omega) \). Put \( L_j = f(E_j) \).

First, assume that \( f(E) \) is non-polar. Then without loss of generality, we may assume that \( L_1 \) is non-polar.

Fix a hyperconvex domain \( \Omega' \Subset \Omega \). We want to estimate \( \omega(z_0, E_j \cap \Omega', \Omega') \). By the localization principle we have

\[
\omega(z_0, E_j \cap \Omega', \Omega') = \omega(z_0, E_j \cap \Omega' \cap U_k) \leq \omega(p, L_j, \widehat{\mathbb{C}} \setminus K_k).
\]

We claim that

\[
\lim_{k \to \infty} \omega(p, L_j, \widehat{\mathbb{C}} \setminus K_k) = 0.
\]

Fix \( j \geq 1 \). For each natural number \( k \) we let \( D_k \) be the connected component of \( \widehat{\mathbb{C}} \setminus K_k \) which contains \( p \). We have

\[
\omega(p, L_j, \widehat{\mathbb{C}} \setminus K_k) = \omega(p, L_j \cap D_k, D_k).
\]

Note that \( D_k \) is a regular domain (see [Ran, Theorem 4.2.4]). Put \( h_k(z) = \omega(z, L_j \cap D_k, D_k) \). Then \( h_k \) is a harmonic function on \( D_k \). Moreover, it extends subharmonically to \( \widehat{\mathbb{C}} \setminus L_j \) (we put \( h_k = 0 \) on \( \widehat{\mathbb{C}} \setminus D_k \)). Hence
\[ h(z) = \lim_{k \to \infty} h_k(z) \] is non-negative and subharmonic on \( \hat{\mathbb{C}} \setminus L_j \) (being the decreasing limit of a sequence of subharmonic functions). Moreover, \( h = 0 \) on \( \bigcup_k K_k \). Since \( \bigcup_k K_k \) is non-thin at \( p \), \( p \) is an accumulation point of \( \bigcup_k K_j \) and \( h(p) = 0 \). Hence, we have proved (3.2).

So, we have \( \omega(z_0, E_j \cap \Omega', \Omega') = 0 \). Hence, \( z_0 \notin (E_j \cap \Omega')_\Omega \). Since \( \Omega' \subseteq \Omega \) is an arbitrary hyperconvex domain, we get \( z_0 \notin (E_j)_\Omega^* \) and \( z_0 \notin \bigcup_j (E_j)_\Omega^* \).

But we know that \( z_0 \in f^{-1}(p) \cap E_{\Omega}^* \). A contradiction.

Now, assume that \( f(E) \) is polar. Note that \( f(E_\Omega^*) \subset f(E)_\Omega^* = f(E) \).

Assume that \( f(E) \) is open in the fine topology. Take \( p \in f(E_{\Omega}^*) \). Note that there are two cases: \( p \in f(E_{\Omega}^*) \setminus f(E) \) and \( p \in f(E) \). In both cases we see that \( \mathbb{C} \setminus f(E_{\Omega}^*) \) is thin at \( p \).

**Proof of Corollary 3.** Note that \( \pi_1(\Gamma_f(D)) = D \) is open and, therefore, open in the fine topology. If \( f \) is non-constant then \( \pi_2(\Gamma_f(D)) = D \) is also open.

**4. Example.** Note that in Corollary 3 we cannot state, in general, that \( \pi_1((\Gamma_f(D))_{\mathbb{C}^2})^* \) is open. Indeed, take \( a_n = 1/n \) and \( c_n = e^{-n^2}/n^2, n \in \mathbb{N} \). Put

\[
f(z) = \sum_{n=1}^{\infty} \frac{c_n}{z - a_n}.
\]

Note that \( f \) is a holomorphic function on the domain \( D = \mathbb{C} \setminus \{a_n : n \in \mathbb{C}\} \cup \{0\} \). By [EdWi1], \( (\Gamma_f(D))_{\mathbb{C}^2}^* = \Gamma_f(D) \cup \{(0, f(0))\} \). So, \( \pi_1((\Gamma_f(D))_{\mathbb{C}^2}) = D \cup \{0\} \).

**References**


B. Josefson, On the equivalence between locally polar and globally polar sets for plurisubharmonic functions on $\mathbb{C}^n$, ibid. 16 (1978), 109–115.


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