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DIFFERENTIAL INEQUALITIES

Direct and Reverse Gagliardo–Nirenberg Inequalities from Logarithmic Sobolev Inequalities

by

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Summary. We investigate the connection between certain logarithmic Sobolev inequalities and generalizations of Gagliardo–Nirenberg inequalities. A similar connection holds between reverse logarithmic Sobolev inequalities and a new class of reverse Gagliardo–Nirenberg inequalities.

0. Introduction. The main concern of this paper is to investigate the connections between *logarithmic Sobolev inequalities* (LSI) and generalizations of *Gagliardo–Nirenberg inequalities* (GNI). The typical LSI inequality we shall be concerned with in the first part of the paper will be of the form

$$\int_X \log\left[\frac{|u(x)|}{\|u\|_p}\right]\frac{|u(x)|^p}{\|u\|_p^p}\,d\mu(x) \le c_1 \log\left[c_2\frac{\|\nabla u\|_p}{\|u\|_p}\right] \quad \forall f \in C_{\mathrm{c}}^{\infty}(X), \tag{0.1}$$

μ being a positive Radon measure on a Riemannian manifold X, ∇ the Riemannian gradient, c_1, c_2 positive constants, and $\|\cdot\|_p$ the L^p norm. The manifold setting is chosen for the sake of notational simplicity only and could be generalized in many respects: for example, the role of the operator ∇ could be taken by a (vector-valued) derivation (see e.g. [13] for details), but certain discrete settings could be discussed as well (see [2]).

Such inequalities have a long history since the pioneering work of Gross [17], who proved the equivalence between a weaker form of such inequalities in the case $p = 2$ and hypercontractivity of the linear heat semigroup. It was proved later that (0.1) is indeed equivalent to ultracontractivity of the heat semigroup (see also [1]). This can be seen for example by noticing that,

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by applying the numerical inequality $\log x \le \varepsilon x + \log(1/\varepsilon)$, $\forall x, \varepsilon > 0$, to the r.h.s. of (0.1), one proves a family of LSI of the form considered in [14]. The proof of ultracontractivity then follows by methods which are by now standard.

More recently, it has been shown in [12], [7], [8] that the validity of such a LSI for $p \ge 2$ implies ultracontractive-like bounds of the form

$$\|u(t)\|_\infty \le \frac{C}{t^\alpha}\,\|u(0)\|_2^\beta$$

for the solutions $u(t)$ to suitable classes of *nonlinear* evolution equations including the porous media equation and the heat equation driven by the p-Laplacian (see also [9] and [15] for a generalization to a doubly nonlinear evolution equation).

Since on the other hand it is known that, in the linear case, ultracontractivity for the heat semigroup is equivalent to the usual Sobolev inequality $\|u\|_{2d/(d-2)} \le C\|\nabla u\|_2$ (and to the Nash inequalities as well), it is not surprising that (0.1) is connected to Sobolev inequalities involving the p-energy functional $\|\nabla u\|_p$, or to inequalities of Nash type involving that functional.

This is indeed a consequence of the results of [4]–[6], in which it is proved that logarithmic Sobolev inequalities imply Nash-type inequalities (which are a special case of GNI), and of [2] (see also [18]), in which it is shown that any single GNI implies a whole class of them; in Section 2 we discuss this point with a few more details.

Our aim here is to further investigate this connection. We first show that the entropy functional

$$J(p,u) = \int_X \log\left[\frac{|u(x)|}{\|u\|_p}\right]\frac{|u(x)|^p}{\|u\|_p^p}\,d\mu(x)$$

can be used to bound both from below and from above the variation of the convex function $p \mapsto \log\|u\|_p^p$. This is the content of our first result, Theorem 1.1. This will allow us first to prove the following inequality, which we call the 4-*norms inequality*. It is a generalization of the GNI and reads

$$\|u\|_q^{1/d}\|u\|_p^{1/s-1/q} \le C\|\nabla u\|_p^{1/s-1/q}\|u\|_s^{1/d}, \tag{0.2}$$

where $0 < s \le q \le p$ and $d \ge 1$, d being a parameter having the role of dimension. GNI inequalities can then be proved with the help of the results of [2].

We next prove *reverse* analogues of the above 4-norms inequalities, as a consequence of reverse LSI which we prove in Section 3. In fact we first discuss the consequences of a reverse LSI in the form given by [16], [19], [20] adapted to the real case. We shall mostly specialize to the Euclidean case, the underlying measure being the Gaussian measure. This is because that is the main case in which we are able to prove a reverse LSI for suitable classes of functions. We hope that such inequalities can be used in order to study reverse hypercontractivity for suitable classes of data. We shall then use Theorem 1.1 to prove a new *reverse* analogue of (0.2), from which reverse GNI (and in particular reverse Sobolev inequalities) will then follow, again for a suitable class of functions.

This paper is organized as follows. In the first section we prove the main property of the entropy functional which will be used in the following, including the aforementioned Theorem 1.1. In Section 2 we prove that a suitable LSI implies (0.2), and then make some remarks on the connections with the GNI. Section 3 is devoted to the proof of reverse LSI, Sobolev and Gagliardo–Nirenberg inequalities.

1. Basic entropy inequalities. In this section we prove the basic inequalities concerning the functional $J(p,u)$, defined with respect to a general positive Radon measure μ, which will be the starting point for proving both direct and reverse Gagliardo–Nirenberg inequalities.

THEOREM 1.1. *We have*

$$\|u\|_p e^{\frac{q-p}{qp}J(1,u^p)} \le \|u\|_q \le \|u\|_p e^{\frac{q-p}{qp}J(1,u^q)} \tag{1.1}$$

for any $0 < p \le q$ *and* $u \in \mathrm{L}^p(X,\mu) \cap \mathrm{L}^q(X,\mu)$.

Proof. It is well known that the functional

$$N(r,u) = \log \|u\|_r^r = \log \int_X |u(x)|^r \, d\mu(x)$$

defined over $(0,\infty) \times \bigcap_{p>0} \mathrm{L}^p(X,\mu)$ is convex with respect to $r > 0$, and its first derivative

$$\frac{d}{dr}N(r,u) = J(r,u) + \frac{1}{r}\,N(r,u)$$

is nondecreasing with respect to $r > 0$. For more details one can refer to Section (2.4) of [8].

By the convexity of N one has, for $0 < p \le q$,

$$N'(p) \le \frac{N(q,u) - N(p,u)}{q-p} \le N'(q)$$

which becomes

$$(q-p)[J(p,u) + \log \|u\|_p] \le \log \frac{\|u\|_q^q}{\|u\|_p^p} \le (q-p)[J(q,u) + \log \|u\|_q]$$

or equivalently

$$e^{(q-p)J(p,u)}\|u\|_p^{q-p} \le \frac{\|u\|_q^q}{\|u\|_p^p} \le e^{(q-p)J(q,u)}\|u\|_q^{q-p}.$$

The latter inequalities are clearly equivalent to the assertion. ∎

We now collect some useful properties of the entropy functional which will be of help later on.

PROPOSITION 1.2. *The functional J has the following properties*:

$$J(r,u^\gamma)=\gamma J(\gamma r,u) \quad \text{for all } \gamma,r>0; \tag{1.2}$$

$$J(r,u^{s+h})\ge J(r,u^s) \quad \text{for all } r,s>0,\ h\ge 0. \tag{1.3}$$

Proof. The first statement is an immediate consequence of the definition of J. For the second, we first prove that the map $\beta\mapsto J(1,u^\beta)$ is nondecreasing. In fact, it is well known that the map

$$\alpha\mapsto \log\|u\|_{1/\alpha}$$

is convex (see e.g. [2]). By taking derivatives, the map

$$\alpha\mapsto -\frac{1}{\alpha}\,J\left(\frac{1}{\alpha},u\right)$$

is nondecreasing. Thus, the map $\beta\mapsto\beta J(\beta,u)$ is increasing, as also is, by the previous result, the functional $\beta\mapsto J(1,u^\beta)$. Finally,

$$sJ(s,u^{r+h})=J(1,u^{s(r+h)})\ge J(1,u^{sr})=sJ(s,u^r). \ ■$$

2. 4-norms inequalitites via entropy and LSI. In this section we draw the main consequences of the lower bound in (1.1), by making use of the inequalities of Proposition 1.2.

We shall first prove

PROPOSITION 2.1. *For any $0<p<\varrho$ the p-LSI implies the ϱ-LSI.*

Proof. We compute

$$\begin{aligned}\varrho J(\varrho,u)&=p\,\frac{\varrho}{p}\,J\left(p\,\frac{\varrho}{p},u\right)=pJ(p,u^{\varrho/p})\le\frac{d}{p}\log\left[\mathcal{L}_p\,\frac{\|\nabla(|u|^{\varrho/p})\|_p^p}{\|\,|u|^{\varrho/p}\|_p^p}\right]\\&=\frac{d}{p}\log\left[\mathcal{L}_p\left(\frac{\varrho}{p}\right)^p\frac{\int_X|u|^{p(\varrho/p-1)}|\nabla u|^p\,d\mu}{\|u\|_\varrho^\varrho}\right]\\&\le\frac{d}{p}\log\left[\mathcal{L}_p\left(\frac{\varrho}{p}\right)^p\frac{\|\,|u|^{\varrho-p}\|_{\sigma'}\|\,|\nabla u|^p\|_\sigma}{\|u\|_\varrho^\varrho}\right]\\&=\frac{d}{p}\log\left[\mathcal{L}_p\left(\frac{\varrho}{p}\right)^p\frac{\|u\|_\varrho^{\varrho-p}\|\nabla u\|_\varrho^p}{\|u\|_\varrho^\varrho}\right]=d\log\left[\mathcal{L}_p^{1/p}\,\frac{\varrho}{p}\,\frac{\|\nabla u\|_\varrho}{\|u\|_\varrho}\right]\end{aligned}$$

where we have also used the Hölder inequality with conjugate exponents $\sigma=\varrho/p>1$ and $\sigma'=\varrho/(\varrho-p)>1$. ■

THEOREM 2.2 (4-norms inequality). *Suppose that the following LSI holds true for some $p,d>0$:*

$$pJ(p,u)\le d\log\left[\mathcal{L}_p^{1/p}\,\frac{\|\nabla u\|_p}{\|u\|_p}\right]. \tag{2.1}$$

Then

$$\|u\|_q^{1/d}\|u\|_p^{1/s-1/q} \le \mathcal{L}_p^{(1/p)1/s-1/q}\|\nabla u\|_p^{(1/s-1/q)}\|u\|_s^{1/d} \quad \text{for } 0<s\le q\le p. \tag{2.2}$$

Moreover if $\varrho \ge p$ *then* (2.1) *implies*

$$\|u\|_q^{1/d}\|u\|_\varrho^{1/s-1/q} \le \left(\mathcal{L}_p^{1/p}\,\frac{\varrho}{p}\right)^{1/s-1/q}\|\nabla u\|_\varrho^{1/s-1/q}\|u\|_s^{1/d} \quad \text{for } 0<s\le q\le p\le\varrho. \tag{2.3}$$

Proof. We will prove (2.2) by combining the right-hand inequality of (1.1) and the p-LSI (2.1) rewritten in the form

$$e^{pJ(p,u)} \le \mathcal{L}_p^{d/p}\,\frac{\|\nabla u\|_p^d}{\|u\|_p^d}. \tag{2.4}$$

To this end we need the monotonicity property (1.3) that we recall here:

$$qJ(q,u) = J(1,u^q) \le pJ(p,u) = J(1,u^p) \quad \text{for any } p\ge q>0.$$

Using this together with the right-hand inequality of (1.1) one obtains

$$\frac{\|u\|_q}{\|u\|_s} \le e^{\frac{q-s}{sq}J(1,u^q)} \le e^{\frac{q-s}{sq}J(1,u^p)}$$

for any $p\ge q\ge s>0$. Now we combine this last inequality with (2.4) to obtain

$$\frac{\|u\|_q}{\|u\|_s} \le e^{\frac{q-s}{sq}J(1,u^p)} \le \exp\left[\frac{q-s}{sq}\log\left(\mathcal{L}_p^{d/p}\,\frac{\|\nabla u\|_p^d}{\|u\|_p^d}\right)\right]$$

or equivalently

$$\frac{\|u\|_q}{\|u\|_s} \le \mathcal{L}_p^{(d/p)(1/s-1/q)}\,\frac{\|\nabla u\|_p^{d(1/s-1/q)}}{\|u\|_p^{d(1/s-1/q)}} \tag{2.5}$$

for any $p\ge q\ge s>0$. This is clearly equivalent to (2.2). The last part follows from the first and from Proposition 2.1 ∎

Given the above result, the GNI is a consequence of the results of [2]. Although the following results are known from [2], for completeness and for the reader's convenience we recall concisely how to proceed in this direction from our starting point.

• *p-Nash inequalities.* Fix $p,d>0$. The first consequence of (2.2) (just by letting $q=p$) is the following family of p-Nash inequalities:

$$\|u\|_p^{1+ps/d(p-s)} \le \mathcal{L}_p^{1/p}\|\nabla u\|_p\|u\|_s^{ps/d(p-s)} \quad \text{whenever } 0<s<p. \tag{2.6}$$

Similarly inequality (2.3) implies a family of ϱ-Nash inequalities with $\varrho\ge p$ and with proportionality constant $\mathcal{L}_p^{1/p}\varrho/p$. We first stress that the above result holds for any $p>0$. Also, the term *p-Nash inequality* is due to similarity to the celebrated Nash inequality:

$$\|u\|_2^{1+2/d} \le C\|\nabla u\|_2 \|u\|_1^{2/d}.$$

The above remark does not distinguish the cases of p larger and smaller than d. The following remarks deal with some more detailed consequences of the above results, which take into account such differences.

• *Gagliardo–Nirenberg inequalities.* In the previous remark we proved that a p-LSI implies a 4-norms inequality such as (2.2) and then a family of p-Nash inequalities, which are a special case of GNI:

$$\|u\|_r \le \mathcal{C}_p^{\vartheta/p} \|\nabla u\|_p^{\vartheta} \|u\|_s^{1-\vartheta} \tag{2.7}$$

for any $p, r, s, d > 0$ and $\vartheta \in [0,1]$ such that

$$\frac{1}{r} = \vartheta\left(\frac{1}{p} - \frac{1}{d}\right) + (1-\vartheta)\,\frac{1}{s} \tag{2.8}$$

where $\mathcal{C}_p \propto \mathcal{L}_p$, $\mathcal{L}_p$ being the constant in the p-LSI.

We now recall that this fact actually guarantees the validity of all the GNI above, once the relative position of p and d is fixed, see also [2, Th. 10.2]. To this end we will need some results of [2].

The subcritical case: $0 < p < d$. By Theorem 3.1 of [2] it is known that a single GNI of the form (2.7) implies the other GNI inequalities corresponding to $0 < p < d$ fixed, while $\vartheta \in [0,1]$ and $r, s > 0$ are related as in (2.8). Then a p-LSI of the form (2.1) implies the whole family of GNI (2.7) mentioned above, via a p-Nash inequality. This family also contains as a special case the classical p-Sobolev inequality:

$$\|u\|_{pd/(d-p)} \le \mathcal{C}_p \|\nabla u\|_p.$$

The critical case: $p = d$. By Theorem 3.3 of [2], a single GNI of the form (2.7) implies the other GNI corresponding to $p = d > 0$, $0 < s < r < \infty$, $\vartheta = 1 - s/r$.

With the help of Theorem 3.2.6 of [18], we can also show that the above mentioned family of GNI implies some versions of Moser–Trudinger inequalities. See [2, Theorem 3.4] for details.

The supercritical case: $p > d$. By Theorem 3.2 of [2], a single GNI of the form (2.7) implies the other GNI inequalities corresponding to $p > d > 0$ fixed, while $0 < s < r \le \infty$, $\vartheta \in [0,1]$ are related as in (2.8). In particular by letting $r \to \infty$ we get

$$\|u\|_\infty \le \mathcal{C}_p^{\vartheta/p} \|\nabla u\|_p^{\vartheta} \|u\|_s^{1-\vartheta} \tag{2.9}$$

for all $0 < s < \infty$. This last family contains a version of the well known Morrey inequality.

• *Other Gagliardo–Nirenberg inequalities.* In this last remark we focus our attention on the main consequences of the second 4-norms inequality (2.3). We proved in a previous remark, using Theorem 2.2, that a suitable p-LSI implies a 4-norms inequality such as (2.3) and then a ϱ-Nash inequality, provided $\varrho \geq p$. This fact leads us to prove that a p-LSI implies a larger family of GNI:

$$\|u\|_r \leq \mathcal{G}_\varrho^\vartheta \|\nabla u\|_\varrho^\vartheta \|u\|_s^{1-\vartheta} \tag{2.10}$$

whenever $0 < p \leq \varrho$, $\vartheta \in [0,1]$, with $\mathcal{G}_q \propto \mathcal{L}_p^{1/p} q/p$ and

$$\frac{1}{r} = \vartheta\left(\frac{1}{\varrho} - \frac{1}{d}\right) + (1-\vartheta)\,\frac{1}{s}.$$

Thus we can extend the above remarks simply by replacing p with ϱ, and $\mathcal{L}_p^{1/p}$ with $\mathcal{L}_p^{1/p}\varrho/p$. Informally speaking, we recalled that, for fixed $p > 0$, a single p-LSI implies a family of ϱ-GNI of the type (2.10), with $\varrho \geq p$ (this being the content of [2, Section 8]), and hence all the ϱ-versions of Sobolev, Moser–Trudinger and Morrey inequalities.

3. Reverse inequalities. In this section we start by proving a new family of reverse logarithmic Sobolev inequalities in a general setting. These reverse LSI will give as a direct consequence a reverse Sobolev inequality, while put together with a reverse 4-norms inequality will give a family of reverse Gagliardo–Nirenberg inequalities as well.

As far as we know, reverse LSI first appeared in the works of S. B. Sontz [19], [20] in the setup of Segal–Bargmann spaces. After these pioneering works, a paper [16] of F. Galaz-Fontes, L. Gross and S. B. Sontz gave a generalization of reverse LSI over complex manifolds and investigated the connection between reverse LSI and reverse hypercontractivity.

Although reverse LSI are in a sense typical of the complex setting, we shall show that they have some real analogue. A reverse LSI of a different type appears in [11]. Hereafter, we always deal with spaces of *real-valued* functions.

We start by proving the main theorems of this section, concerning reverse inequalities with respect to a positive measure, absolutely continuous with respect to a reference measure, indicated by dx:

$$d\mu(x) = m(x)\,dx,$$

m being a function for which Δm makes sense as a locally integrable function (hereafter, Δ denotes the Laplace–Beltrami operator). In this section we shall indicate explicitly the measure in the notation of L^q norms (writing, e.g., $\|\cdot\|_{q,\mu}$), since we shall make more than one possible choice of measure.

We denote by $\mathcal{C}$ the class of functions v for which the following integration by parts formula holds:

$$\int_X v(x)(\Delta m)(x)\,dx = \int_X (\Delta v)(x)m(x)\,dx. \tag{3.1}$$

We notice that if $f \in H^1(X,\mu)$ (i.e. $f, \nabla f \in \mathrm{L}^2(X,\mu)$) and $f^2 \in \mathcal{C}$ then the Leibniz rule implies that $\int_X f\Delta f\,d\mu$ is finite.

THEOREM 3.1 (Reverse logarithmic Sobolev inequality). *Let f be a measurable, real-valued function such that*:

(i) $f \in H^1(X,\mu)$, $f^2 \in \mathcal{C}$;
(ii) *f satisfies the inequality*

$$\int_X f(x)(\Delta f)(x)\,d\mu(x) \geq 0. \tag{3.2}$$

(iii) *There exists $c > 0$ such that*

$$B(c,m) = \int_X e^{(\Delta m)(x)/cm(x)}\,d\mu(x) < \infty. \tag{3.3}$$

Then the following reverse LSI holds true:

$$2\|\nabla f\|_{2,\mu}^2 \leq c\int_X f^2\log\left(\frac{f^2}{\|f\|_{2,\mu}^2}\right)d\mu + c\log(B(c,m))\|f\|_{2,\mu}^2. \tag{3.4}$$

Proof. We adapt to our setting a method of L. Gross and S. B. Sontz [20]. We first recall Young's inequality:

$$st \leq s\log(s) - s + e^t \quad \text{for } s > 0 \text{ and } t \in \mathbb{R}.$$

The choice $s = cf(x)^2 > 0$, $t = (\Delta m)(x)/cm(x)$ together with integration over (X,μ) leads to

$$\int_X cf(x)^2\,\frac{(\Delta m)(x)}{cm(x)}\,d\mu(x) \leq \int_X cf(x)^2\log(cf(x)^2)\,d\mu(x) - \int_X cf(x)^2\,d\mu(x) + \int_X e^{(\Delta m)(x)/cm(x)}\,d\mu(x).$$

Noticing that $\log(cf(x)^2) = \log(c) + \log(f(x)^2)$, $d\mu(x) = m(x)dx$ and $B(c,m) = \int_X e^{(\Delta m)(x)/cm(x)}d\mu(x) < \infty$ by hypothesis, we obtain

$$\begin{aligned}&\int_X f(x)^2(\Delta m)(x)\,dx \\ &\quad\leq c\int_X f(x)^2\log(f(x)^2)\,d\mu(x) + (c\log(c) - c)\int_X f(x)^2\,d\mu(x) + B(c,m).\end{aligned} \tag{3.5}$$

Integration by parts, allowed by our assumptions, then gives

$$\begin{aligned}\int_X f(x)^2(\Delta m)(x)\,dx &= \int_X (\Delta f^2)(x)m(x)\,dx\\ &= 2\int_X |\nabla f(x)|^2 m(x)\,dx + 2\int_X f(x)(\Delta f)(x)m(x)\,dx\\ &\geq 2\int_X |\nabla f(x)|^2 m(x)\,dx.\end{aligned}$$

The last inequality follows from our assumption $\int_X f(x)(\Delta f)(x)m(x)dx \geq 0$. Now letting $\lambda > 0$ and replacing f by λf in (3.5) gives

$$\begin{aligned}2\lambda^2\int_X |\nabla f(x)|^2\,d\mu(x) \leq c\lambda^2 &\int_X f(x)^2\log(\lambda^2 f(x)^2)\,d\mu(x)\\ &+ (c\log(c)-c)\lambda^2\int_X f(x)^2\,d\mu(x) + B(c,m).\end{aligned}$$

Divide both members by λ^2 to obtain

$$\begin{aligned}2\|\nabla f\|_{2,\mu}^2 &\leq c\int_X f(x)^2\log(f(x)^2)\,d\mu(x)\\ &\quad + [c\log(\lambda^2)+c\log(c)-c]\|f\|_{2,\mu}^2 + \frac{B(c,m)}{\lambda^2}.\end{aligned}\tag{3.6}$$

Optimizing with respect to λ^2 gives

$$\lambda^2 = \frac{B(c,m)}{c\|f\|_{2,\mu}^2}.$$

Substituting this value in (3.6) leads to

$$\begin{aligned}2\|\nabla f\|_{2,\mu}^2 \leq c\int_X f^2\log(f^2)\,d\mu &+ \left[c\log\left(\frac{B(m,c)}{c\|f\|_{2,\mu}^2}\right) + c\log(c) - c\right]\|f\|_{2,\mu}^2\\ &+ c\|f\|_{2,\mu}^2,\end{aligned}$$

which is the claim. ■

In what follows it will be useful to rewrite (3.4) in the form

$$\begin{aligned}2\frac{\|\nabla f\|_{2,\mu}^2}{\|f\|_{2,\mu}^2} &\leq c\int_X \log\left(\frac{f^2}{\|f\|_{2,\mu}^2}\right)\frac{f^2}{\|f\|_{2,\mu}^2}\,d\mu + c\log(B(c,m))\\ &= cJ_\mu(1,f^2)+K.\end{aligned}\tag{3.7}$$

As a direct consequence of this theorem we obtain the following

Theorem 3.2 (Reverse Sobolev inequality). *Let $f \in \mathrm{L}^2(X,\mu)$ satisfy a reverse LSI of the form (3.7) for some constants $c, K > 0$. Then for any*

$\varepsilon > 0$ there exists $M'_\varepsilon > 0$ such that

$$M'_\varepsilon \exp\left(\frac{2\varepsilon}{c(2+\varepsilon)}\,\frac{\|\nabla f\|^2_{2,\mu}}{\|f\|^2_{2,\mu}}\right) \le \frac{\|f\|^2_{2+\varepsilon,\mu}}{\|f\|^2_{2,\mu}}.$$

In particular, there exists $M_\varepsilon > 0$ such that the reverse Sobolev inequality

$$M_\varepsilon \|\nabla f\|_{2,\mu} \le \|f\|_{2+\varepsilon,\mu} \tag{3.8}$$

holds.

Proof. First we rewrite (3.7) as

$$\frac{1}{c}\frac{\|\nabla f\|^2_{2,\mu}}{\|f\|^2_{2,\mu}} - \frac{K}{2c} \le J_\mu(2,f) = \frac{1}{\varepsilon}\int_X \log\left(\frac{|f|^\varepsilon}{\|f\|^\varepsilon_{2,\mu}}\right)\frac{|f|^2}{\|f\|^2_{2,\mu}}\,d\mu$$
$$\le \log\int_X \frac{|f|^{2+\varepsilon}}{\|f\|^{2+\varepsilon}_{2,\mu}}\,d\mu = \frac{2+\varepsilon}{2}\log\frac{\|f\|^2_{2+\varepsilon,\mu}}{\|f\|^2_{2,\mu}},$$

where in the first line we used the property $J_\mu(1,f^2) = 2J_\mu(2,f)$ while in the second line we applied the Jensen inequality with respect to the probability measure $(|f|^2/\|f\|^2_{2,\mu})\,d\mu$. Now we use the inequality $\log(x) \le x$ to obtain

$$\log\left(\frac{2\varepsilon}{c(2+\varepsilon)}\,\frac{\|\nabla f\|^2_{2,\mu}}{\|f\|^2_{2,\mu}}\right) - \frac{\varepsilon K}{c(2+\varepsilon)} \le \frac{2\varepsilon}{c(2+\varepsilon)}\,\frac{\|\nabla f\|^2_{2,\mu}}{\|f\|^2_{2,\mu}} - \frac{\varepsilon K}{c(2+\varepsilon)}$$
$$\le \log\frac{\|f\|^2_{2+\varepsilon,\mu}}{\|f\|^2_{2,\mu}}.$$

Exponentiating the three terms gives

$$\exp\left(-\frac{\varepsilon K}{c(2+\varepsilon)}\right)\frac{2\varepsilon}{c(2+\varepsilon)}\,\frac{\|\nabla f\|^2_{2,\mu}}{\|f\|^2_{2,\mu}} \tag{3.9}$$
$$\le \exp\left(\frac{2\varepsilon}{c(2+\varepsilon)}\,\frac{\|\nabla f\|^2_{2,\mu}}{\|f\|^2_{2,\mu}} - \frac{\varepsilon K}{c(2+\varepsilon)}\right) \le \frac{\|f\|^2_{2+\varepsilon,\mu}}{\|f\|^2_{2,\mu}}. \quad ■$$

As far as we know, reverse Sobolev inequalities appeared first in the work of S. B. Sontz [19], [20] in the context of Segal–Bargmann spaces. Their connection with reverse hypercontractivity has been discussed in [16].

Theorem 3.3 (Reverse 4-norms inequality). *Let $f \in \mathrm{L}^2(X,\mu)\cap\mathrm{L}^q(X,\mu)$, with $q>2$, satisfy a reverse LSI of the form* (3.7) *for some constants $c, K>0$. Then*

$$\frac{\|\nabla f\|^2_{2,\mu}}{\|f\|^2_{2,\mu}} \le e^{K/2}\left[\frac{\|f\|_{q,\mu}}{\|f\|_{p,\mu}}\right]^{\frac{cqp}{2(q-p)}} \tag{3.10}$$

for any $2 \le p \le q$ and any $c > 2$.

Proof. We combine the first part of the entropy inequality (1.1), with $2 \le p \le q$:

$$\|f\|_{p,\mu} e^{\frac{q-p}{qp} J_\mu(1,f^p)} \le \|f\|_{q,\mu},$$

rewritten in the form

$$cJ_\mu(1,f^p) \le c\,\frac{qp}{q-p} \log\biggl(\frac{\|f\|_{q,\mu}}{\|f\|_{p,\mu}}\biggr),$$

with the reverse LSI (3.4) rewritten in the form

$$2\,\frac{\|\nabla f\|_{2,\mu}^2}{\|f\|_{2,\mu}^2} - K \le cJ_\mu(1,f^2).$$

We notice that we can glue these inequalities using the monotonicity of the Young functional $J_\mu(1,u^2) \le J_\mu(1,u^p)$ for any $p \ge 2$. Hence we obtain

$$2\,\frac{\|\nabla f\|_{2,\mu}^2}{\|f\|_{2,\mu}^2} - K \le cJ_\mu(1,u^2) \le cJ_\mu(1,u^p) \le c\,\frac{qp}{q-p} \log\biggl(\frac{\|f\|_{q,\mu}}{\|f\|_{p,\mu}}\biggr).$$

Exponentiating and using the inequality $x \le e^x$ finally gives

$$\frac{\|\nabla f\|_{2,\mu}^2}{\|f\|_{2,\mu}^2}\, e^{-K/2} \le \exp\biggl(\frac{\|\nabla f\|_{2,\mu}^2}{\|f\|_{2,\mu}^2} - \frac{K}{2}\biggr) \le \biggl[\frac{\|f\|_{q,\mu}}{\|f\|_{p,\mu}}\biggr]^{\frac{cqp}{2(q-p)}}. \ ■$$

COROLLARY 3.4 (Reverse Gagliardo–Nirenberg inequalities). *Let $f \in \mathrm{L}^2(X,\mu) \cap \mathrm{L}^q(X,\mu)$, with $q > 2$, satisfy a reverse LSI of the form* (3.7) *for some constants $c, K > 0$. Then the family of reverse GNI*

$$\|\nabla f\|_{2,\mu}^{\vartheta} \|f\|_{2,\mu}^{1-\vartheta} \le e^{K\vartheta/4} \|f\|_{q,\mu} \tag{3.11}$$

holds for any $q > 2$, where $\vartheta = 2(q-2)/cq$ and $K > 0$ is the constant in (3.7).

Proof. Just let $p = 2$ in (3.10). ■

REMARK 3.5. Notice that condition (3.2) is obviously true for harmonic functions in the space (X,μ), i.e. those functions f which satisfy $\Delta f = 0$ on the support of μ. Condition (3.2) is also fulfilled when the integral appearing there is finite and when moreover either f is a non-negative subharmonic function (i.e. $f \ge 0$ and $\Delta f \ge 0$ a.e.) or f is a nonpositive superharmonic function (i.e. $f \le 0$ and $\Delta f \le 0$ a.e.). It is also satisfied, in the Euclidean case, by positive convex functions or by negative concave functions in $\mathrm{L}^2(X,\mu)$, if the corresponding integral appearing in (3.2) exists.

3.1. *The Gaussian setup.* In this section first we draw the main consequences of the above results in the Gaussian setup, i.e. when $(X,\mu) = (\mathbb{R}^d,\gamma)$ where γ is the Gaussian measure

$$d\gamma(x) = (2\pi)^{-d/2} e^{-|x|^2/2}\, dx.$$

We then prove some families of reverse Sobolev, 4-norms and Gagliardo–Nirenberg inequalities. The validity of the last inequalities depends on the reverse LSI (3.4) and hence on the condition (3.2) which reads, in the present context,

$$\int_{\mathbb{R}^d} |f(x)|^2(|x|^2 - d)\, d\gamma(x) \ge 2 \int_{\mathbb{R}^d} |\nabla f(x)|^2\, d\gamma(x). \tag{3.12}$$

In the present setting this inequality plays the role of the identities of V. Bargmann (see [3, p. 210]) and of E. A. Carlen (see [10]), which hold in the Segal–Bargmann space. Inequality (3.12) holds for a class of functions that includes harmonic functions; this class will play, in our context, the role played by the Segal–Bargmann functions in the complex case.

Although the theorem below is stated for compactly supported functions, standard approximation procedures allow extending the assertion to larger classes of functions.

THEOREM 3.6 (Reverse inequalities, Gaussian case). *Let f be a smooth, compactly supported function such that*

$$\int_{\mathbb{R}^d} f(x)(\Delta f)(x)\, d\gamma(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x)(\Delta f)(x) e^{-|x|^2/2}\, dx \ge 0. \tag{3.13}$$

Then for any $c > 2$ there exists a positive constant $B(c)$ such that the following reverse LSI holds true:

$$2\|\nabla f\|_{2,\gamma}^2 \le c \int_{\mathbb{R}^d} f^2 \log\left(\frac{f^2}{\|f\|_{2,\gamma}^2}\right) d\gamma + B(c,d)\|f\|_{2,\gamma}^2 \tag{3.14}$$

where

$$B(c,d) = d\left(-1 + \frac{1}{2}\log\left(\frac{2c}{c-2}\right)\right). \tag{3.15}$$

Moreover the following inequalities hold:

(a) Reverse Sobolev inequality: *for any $\varepsilon > 0$ and $c > 2$ there exists a constant $G_{\varepsilon,c} > 0$ such that*

$$G_{\varepsilon,c}\|\nabla f\|_{2,\gamma} \le \|f\|_{2+\varepsilon,\gamma} \tag{3.16}$$

where $G_{\varepsilon,c} = \frac{2\varepsilon}{c(2+\varepsilon)} e^{-\frac{\varepsilon B(c,d)}{c(2+\varepsilon)}}$ and $B(c,d)$ is given by (3.15).

(b) Reverse GNI inequalities: *for any $c > 2$ there exists a positive constant $N(c,d,p,q)$ such that*

$$\|\nabla f\|_{r,\gamma}^{\vartheta} \|f\|_{p,\gamma}^{1-\vartheta} \le N\|f\|_{q,\gamma} \tag{3.17}$$

for any $0 < r \le 2$ and $2 \le p < q$, where $\vartheta = 4(q-p)/cqp$, $N = e^{B(c,d)(q-p)/cqp}$ and $B(c)$ is given by (3.15).

Proof. First we prove (3.14). This is a consequence of Theorem 3.1 together with some calculations. In fact, assumption (i) of that theorem is satisfied for the present class of functions. Moreover the Gaussian density $\gamma(x) = (2\pi)^{-d/2}e^{-|x|^2/2}$ on $\mathbb{R}^d$ satisfies the identity

$$(\Delta\gamma)(x) = (2\pi)^{-d/2}(|x|^2 - d)e^{-|x|^2/2}$$

for any $x \in \mathbb{R}^d$. Then we compute the constant $B(c,\gamma)$:

$$B(c,\gamma) = \int_{\mathbb{R}^d} e^{(\Delta\gamma)(x)/c\gamma(x)}\, d\gamma(x) = \int_{\mathbb{R}^d} e^{(|x|^2-d)/c} d\gamma(x)$$
$$= e^{-d/c}\left(\frac{2c}{c-2}\right)^{d/2} = B(c,d).$$

This proves (3.14).

The reverse Sobolev inequality (3.16) is just a direct consequence of (3.14), exactly as in the general case.

The reverse GNI (3.17) is a consequence of the 4-norms inequality (3.10), which holds in the present case as well, together with the Hölder inequality; in fact $\|f\|_{r,\gamma} \le \|f\|_{s,\gamma}$ whenever $0 < r < s$, since the Gaussian measure on $\mathbb{R}^d$ is a probability measure. Moreover this implies

$$\frac{\|\nabla f\|_{r,\gamma}}{\|f\|_{s,\gamma}} \le \frac{\|\nabla f\|_{2,\gamma}}{\|f\|_{2,\gamma}} \quad \text{whenever } 0 < r \le 2,\ s \ge 2,$$

which combined with the reverse 4-norms inequality (3.10) gives us

$$\frac{\|\nabla f\|_{r,\gamma}^2}{\|f\|_{s,\gamma}^2} \le e^{B(c,d)/2}\left[\frac{\|f\|_{q,\gamma}}{\|f\|_{p,\gamma}}\right]^{\frac{cqp}{2(q-p)}} \quad \text{whenever } \begin{cases} 0 < r \le 2, \\ s \ge 2, \\ 2 \le p < q. \end{cases}$$

Finally, let $s = p$ and obtain

$$\|\nabla f\|_{r,\gamma}^{\frac{4(q-p)}{cqp}} \|f\|_{p,\gamma}^{1-\frac{4(q-p)}{cqp}} \le e^{\frac{B(c,d)(q-p)}{cqp}} \|f\|_q$$

with $0 < r \le 2$, $2 \le p < q$. Letting $\vartheta = 4(q-p)/cqp$ and $N = e^{B(c,d)(q-p)/cqp}$ gives (3.17). This concludes the proof. ∎

We remark that the above class of reverse Gagliardo–Nirenberg inequalities contains, as a special case, a reverse Moser inequality, by letting $r = p = 2$ and $q > 2$:

$$\|\nabla f\|_{2,\gamma}^{\vartheta}\|f\|_{2,\gamma}^{1-\vartheta} \le N\|f\|_{q,\gamma}, \quad \vartheta = \frac{2(q-2)}{cq},\ c > 2.$$

The reverse Moser inequality is obtained letting $q = 2(1 + 1/d) > 2$.

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