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Central Automorphisms of Veblenian Nearaffine Planes

by

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Summary. The paper deals with nearaffine planes described by H. A. Wilbrink. We consider their central automorphisms, i.e. automorphisms satisfying the Veblen condition, which become central collineations in connected projective planes. Moreover, a concept of central pseudo-automorphism is considered, i.e. some bijections in a nearaffine plane are not automorphisms but they become central collineations in the related projective planes.

1. Basic concepts. The paper deals with nearaffine planes considered by H. A. Wilbrink in [5]. Some definitions and properties concerning affine and projective planes given in [2, pp. 115–116, 120–121] and [4, pp. 62–65] will also be used. We wish to study automorphisms and other bijections of nearaffine planes which become central collineations in the related projective planes. We apply the terminology of [5] but we use the notation from [3]. In all structures considered, points will be denoted by capital Latin letters and blocks by small Latin or Greek letters. The extension of an automorphism φ of an affine plane $\mathbf{A}$ to the projective extension $\overline{\mathbf{A}}$ will be denoted by $\overline{\varphi}$. The following statements will be used:

THEOREM 1.1 ([2, p. 120]). *If φ and ψ are collineations in a projective plane such that φ has center A and axis a, and ψ has center B and axis b, then*:

(1) $\varphi\psi = \psi\varphi$ *if and only if* $A \in b$ *and* $B \in a$.
(2) *If* $a \neq b$ *and* $A \neq B$ *then* $\varphi\psi$ *is a central collineation if and only if* φ *and* ψ *are homologies such that* $A \in b$, $B \in a$, *and* $\varphi(X) = \psi^{-1}(X)$

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for every $X \in AB$. If this is the case, then $\varphi\psi$ is a homology with center ab and axis AB.

We also recall the so-called *Veblen condition*:

(V) Let g be a straight line in a nearaffine plane; P, Q, R distinct points on g; and b a line different from g with base point P and $S \in b \setminus \{P\}$. Then $(R \equiv Q \rhd S) \cap b \neq \emptyset$.

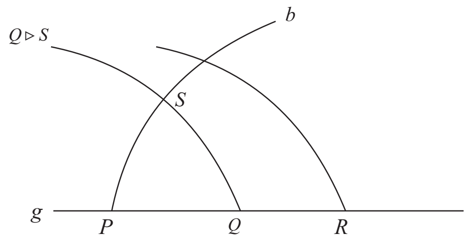

Fig. 1

THEOREM 1.2 ([5, p. 55]). *Let* $\mathbf{NA} = (\Omega, \Xi, \rhd, \equiv)$ *be a nearaffine plane satisfying* (V) *and let g be any straight line. Set*

$$\mathcal{L}_g := \{X \rhd Y \in \Xi;\ X \in g\} \cup \{h;\ h \equiv g\}.$$

Then $\mathbf{NA}(g) = (\Omega, \mathcal{L}_g)$ *is an affine plane.*

COROLLARY 1.1. *If g is a straight line in* $\mathbf{NA}$, *$P, Q \in g$ and $P \rhd R \equiv Q \rhd S$, then PR is parallel to QS in* $\mathbf{NA}(g)$.

2. Central automorphisms in nearaffine planes satisfying (V)**.** For any straight line g the point set of $\mathbf{NA}(g)$ coincides with the point set of $\mathbf{NA}$, but the set of all lines of $\mathbf{NA}(g)$ is a proper subset of the set of lines of $\mathbf{NA}$. An automorphism φ of $\mathbf{NA}$ becomes an automorphism of $\mathbf{NA}(g)$ iff $\varphi(g) = g$. In this case the notation φ_g means the restriction of φ to the lines of $\mathbf{NA}(g)$.

DEFINITION 2.1. An automorphism φ of a nearaffine plane $\mathbf{NA}$ is *central* if there exists a straight line g such that $\overline{\varphi}_g$ is a central collineation in $\overline{\mathbf{NA}(g)}$. The center and axis of $\overline{\varphi}_g$ are called the center and axis of φ.

It follows from Definition 2.1 and Theorem 1.2 that every central automorphism of $\mathbf{NA}$ preserves two lines of $\overline{\mathbf{NA}(g)}$: the ideal line and $g \cup \{[g]_\equiv\}$. The collection of all central automorphisms of a nearaffine plane may be divided into three classes. We consider two of them, since the third one (translations) is described in [5]. We omit easy proofs.

2.1. *φ is a homothety*

PROPOSITION 2.1. *If φ is an automorphism of* $\mathbf{NA}$ *then the following conditions are equivalent*:

(a) *φ is a homothety of* $\mathbf{NA}$ *with center A.*
(b) *For every straight line g through A, φ_g is a homothety of* $\mathbf{NA}(g)$ *with center A.*
(c) *For some straight line g through A, φ_g is a homothety of* $\mathbf{NA}(g)$ *with center A.*

One can easily verify that the transitivity of the group of homotheties with fixed center U implies the following versions of Desargues' postulate [1, p. 72]:

(D2) If $U, X, X', Y, Y', Z, Z' \in \Omega$ are pairwise distinct, $U \rhd X$ is straight and $U \rhd Y$, $U \rhd Z$ are lines different from each other and from $U \rhd X$, then $X' \in U \rhd X$, $Y' \in U \rhd Y$, $Z' \in U \rhd Z$, $X \rhd Y \equiv X' \rhd Y'$ and $X \rhd Z \equiv X' \rhd Z'$ imply $Y \rhd Z \equiv Y' \rhd Z'$.

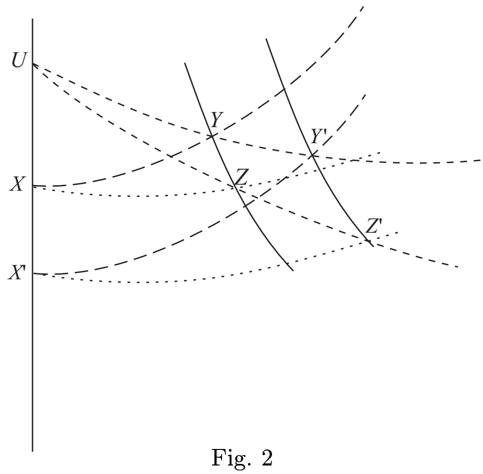

Fig. 2

(D3) If $U, X, X', Y, Y', Z, Z' \in \Omega$ are pairwise distinct, $U \rhd X$, $U \rhd Y$ and $U \rhd Z$ are pairwise distinct lines, and $X' \in U \rhd X$, $Y' \in U \rhd Y$, $Z' \in U \rhd Z$, and if $X \rhd Y \equiv X' \rhd Y'$ and $X \rhd Z \equiv X' \rhd Z'$ are straight then $Y \rhd Z \equiv Y' \rhd Z'$.

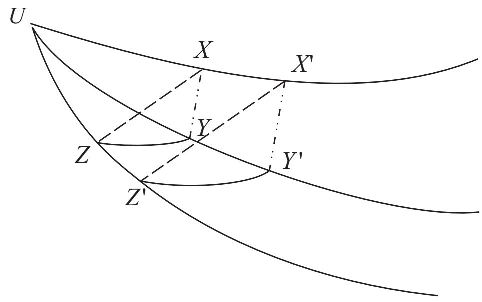

Fig. 3

2.2. *Some nonideal line is the axis of φ.* Let Υ denote the set of all straight lines and g the axis of φ. We know that the center must be ideal. The number of fixed classes of straight lines is either 1 or 2: if g is proper then the class of straight lines must be the center.

THEOREM 2.1. *If* **NA** *contains at least three classes of straight lines and φ is an automorphism with nonideal axis g, then the following conditions are equivalent*:

(i) *φ is an involution.*
(ii) *If $X, Y \in g$, $X \neq Y$, $X \rhd A \in \Upsilon$, $X \rhd A \equiv Y \rhd \varphi(A)$, then $X \rhd \varphi(A) \equiv Y \rhd A$ and $Y \rhd A \in \Upsilon$.*

Figure 4 presents all possibilities for the pair (axis g, center P): $g \in \Upsilon$ and $P = [g]_\equiv$; $g \in \Upsilon$ and $P = [a]_\equiv$, where $a \in \Xi \setminus \Upsilon$; $g \in \Upsilon$ and $P = [a]_\equiv$, where $a \in \Upsilon$ and $a \not\equiv g$; $g \in \Xi \setminus \Upsilon$ and $P = [a]_\equiv$ for some $a \in \Upsilon$.

PROPOSITION 2.2. *Suppose a straight line g is the axis of an automorphism φ.*

(1) *If φ is an involution then*: *some class of straight lines nonparallel to g is the center of φ if and only if the number of classes of straight lines is even.*
(2) *If* **NA** *contains exactly two classes of straight lines then the class $[h]_\equiv$ must be the center*, *where $h \in \Upsilon$ and $h \not\equiv g$.*
(3) *If* **NA** *contains exactly three* (*resp. four*) *classes of straight lines and $[g]_\equiv$ or some class $[a]_\equiv$ of proper lines* (*resp. some class $[h]_\equiv$ of straight lines*, *where $h \not\equiv g$*) *is the center of φ*, *then φ is an involution.*

From Proposition 2.2 and Theorem 2.1 the following is immediate.

COROLLARY 2.1.

(a) *In an affine plane of even order every involution pointwise fixing one line determines configurations of parallelograms with parallel diagonals* (*Fano configurations*; *see the upper left part of Figure* 4).
(b) *In an affine plane of odd order every involution pointwise fixing one line determines configurations of trapeziums with parallel diagonals* (*the axis and a line through the center are arms of such a trapezium*; *see the lower left part of Figure* 4).
(c) *In an affine plane of order* 2 *or* 3 *every automorphism pointwise fixing a line is an involution.*

PROPOSITION 2.3. *Suppose a proper line g is the axis of an automorphism φ.*

(1) *If φ is an involution then the number of classes of straight lines is odd.*

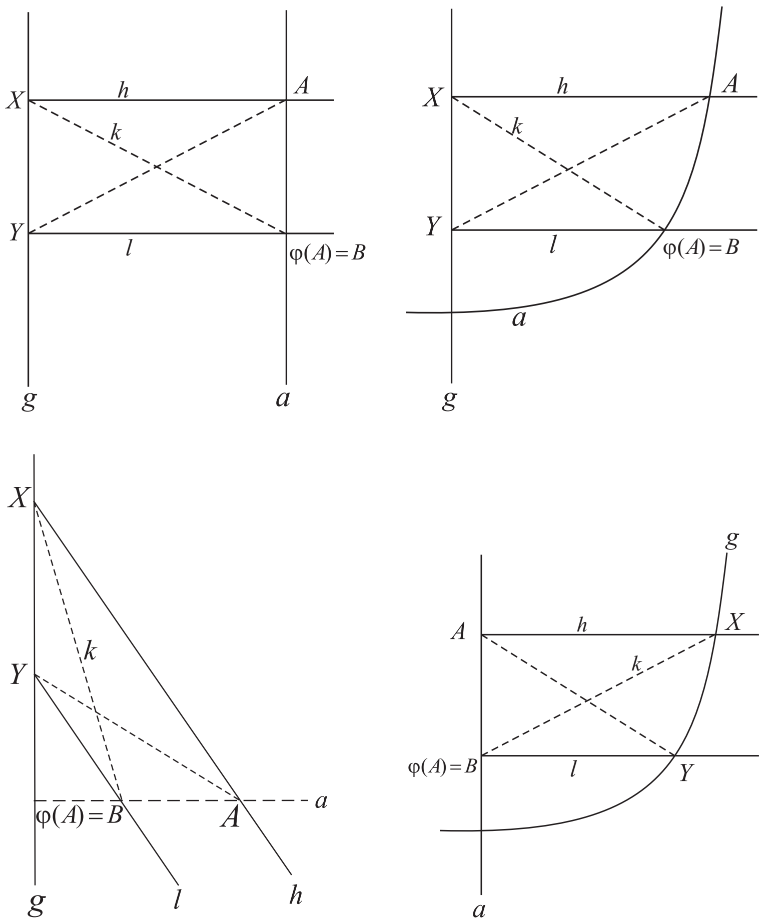

Fig. 4

(2) *The number of classes of straight lines is not two.*

(3) *If there exist exactly three classes of straight lines then φ is an involution.*

(4) *If there exist exactly four classes of straight lines then*:

  (a) *φ is a mapping of order* 3, *i.e. if $A \notin g$ then $A \neq \varphi(A) \neq \varphi(\varphi(A)) \neq \varphi(\varphi(\varphi(A))) = A$.*

  (b) *If $X, Y, Z \in g$, $A \notin g$, $X \rhd A \in \Upsilon$ and $X \rhd A \equiv Y \rhd \varphi(A) \equiv Z \rhd \varphi(\varphi(A))$, then $X \rhd \varphi(A) \in \Upsilon$ and $X \rhd \varphi(A) \equiv Y \rhd \varphi(\varphi(A)) \equiv Z \rhd A$.*

3. Examples. Proper nearaffine planes with more than two classes of straight lines are not well known. Also our examples are given for nearaffine planes with exactly two classes of straight lines. We shall consider some nearaffine planes related to ordered fields [3]. Here the set of proper lines is given by $\{\{(p,q)\} \cup \{(x,y);\ (x-p)(y-q) = r\};\ p,q,r \in F,\ -r > 0\} \cup \{\{(p,q)\} \cup \{(x,y);\ (f(x)-p)(g(y)-q) = r\};\ p,q,r \in F,\ -r < 0\}$, where f, g are some bijections satisfying the condition

$$(u-v)(w-z)(f(u)-f(v))(g(w)-g(z)) > 0$$

for $u,v,w,z \in F$, $u \neq v$, $w \neq z$.

EXAMPLE 3.1. Let F be an Euclidean ordered field. For $s \in F$, $0 < s \neq 1$, take the functions

$$f(x) = \begin{cases} x & \text{for } x \geq 0, \\ sx & \text{for } x \leq 0, \end{cases} \qquad g(y) = y.$$

Consider the mapping

$$\varphi(x,y) = \begin{cases} (-x,y) & \text{for } x \geq 0, \\ (-sx,y) & \text{for } x \leq 0. \end{cases}$$

We obtain

$$\begin{aligned} &\varphi(\{(x,y);\ (x-p)(y-q) = r,\ x \geq 0\}) \\ &\qquad\qquad = \{(x,y);\ (x+p)(y-q) = -r,\ x \leq 0\}; \\ &\varphi(\{(x,y);\ (x-p)(y-q) = r,\ x \leq 0\}) \\ &\qquad\qquad = \{(x,y);\ (x/s+p)(y-q) = -r,\ x \geq 0\}; \\ &\varphi(\{(x,y);\ (sx-p)(y-q) = r,\ x \leq 0\}) \\ &\qquad\qquad = \{(x,y);\ (x+p)(y-q) = -r,\ x \geq 0\}. \end{aligned}$$

In ordered fields $r > 0 \Leftrightarrow -r < 0$. Therefore φ is an automorphism. Of course the straight line given by $x = 0$ is its axis and the class $[a]_{\equiv}$ is its center, where a is described by $y = 0$.

For every $z \in F$ and $1 \neq u > 0$ the mapping $h_{z,u}(x,y) = (x/u, u(y-z)+z)$ is a homothety with center $(0,z)$ [3, p. 356]. Note that the center of $h_{z,u}$ is on the axis of φ and vice versa. Thus we have

$$h_{z,u}(x,y) \circ \varphi = \varphi \circ h_{z,u}(x,y) = \begin{cases} (-x/u, u(y-z)+z) & \text{for } x \geq 0, \\ (-sx/u, u(y-z)+z) & \text{for } x \leq 0. \end{cases}$$

It is not a central automorphism since for the straight line $y = z$ joining both centers we have

$$\varphi \circ h_{z,u}(x,z) = \begin{cases} (-x/u, u(z-z)+z) \neq (x,z) & \text{for some } x \geq 0, \\ (-sx/u, u(z-z)+z) \neq (x,z) & \text{for some } x \leq 0. \end{cases}$$

For every $w \in F$ there exists a straight translation $\tau_w(x,y) = (x, y+w)$, τ_w and $h_{z,u}$ are dilatations, so $\tau_w \circ h_{z,u}$ and $h_{z,u} \circ \tau_w$ also must have the ideal axis. We obtain $\tau_w \circ h_{z,u} = (x/u, u(y-z)+z+w)$ with center $(0, z+t/(1-u))$ and $h_{z,u} \circ \tau_w(x,y) = (x/u, u(y-z+w)+z)$ with center $(0, z+ut/(1-u))$.

In the same way we obtain the noncentral automorphism $\tau_w \circ \varphi = \varphi \circ \tau_w$.

EXAMPLE 3.2. Let F be the field of reals with standard order and take $f(x) = x^3$ and $f(y) = y$. Then $h_{z,u} = (x/u, u(y-z)+z)$ is a noncentral automorphism for $u \neq -1$ [3, p. 356]. It is a homothety with center $(0,z)$ for every $z \in F$ and $u = -1$. For every $w \in F$ the mapping $\tau_w(x,y) = (x, y+w)$ is a straight translation again. As before, we obtain distinct noncentral automorphisms $\tau_w \circ h_{z,u}$ and $h_{z,u} \circ \tau_w$ (they are also distinct for $u = -1$).

Consider the bijections $\varphi_s(x,y) = (sx, y)$ and $\psi_t(x,y) = (x, ty)$. They are automorphisms if $s > 0$ and $t > 0$. The straight line $y = 0$ (respectively $x = 0$) is the axis of ψ_t (resp. φ_s) and the ideal point corresponding to the straight line $x = 0$ ($y = 0$) is the center of ψ_t (resp. φ_s). Of course the center of φ_s is on the axis of ψ_t and vice versa, so $\varphi_s \circ \psi_t(x,y) = \psi_t \circ \varphi_s(x,y) = (sx, ty)$. The line joining both centers is ideal in $\overline{\mathbf{NA}(g)}$, where g is given by $x = 0$. Only the ideal line may be the axis of $\varphi_s \circ \psi_t$. If this is the case, then every line with base point on g is mapped onto some parallel line. But $\varphi_s \circ \psi_t$ maps the proper line $x(y-q) = r$ with $r < 0$ onto the line $x(y-tq) = str$. The line $x^3(y-q) = r$ with $r > 0$ is mapped onto the line $x^3(y-tq) = s^3tr$. Thus $str = r$ for $r < 0$ and $s^3tr = r$ for $r > 0$. This is possible only for $s = t = 1$ or $s = t = -1$. But $s, t > 0$ and then $\varphi_s = \psi_t = \mathrm{id}$. Note that $\varphi_s \circ \psi_t$ may be a central automorphism although φ_s and ψ_t are not automorphisms. This happens if $s = t = -1$, but then $\varphi_s \circ \psi_t = \varphi_{-1} \circ \psi_{-1} = h_{0,-1}$.

In general the center $(0,z)$ of $h_{z,-1}$ is not on the axis $y = 0$ of ψ_t. Therefore $h_{z,-1} \circ \psi_t \neq \psi_t \circ h_{z,-1}$. But $h_{0,-1} \circ \psi_t = \psi_t \circ h_{0,-1}$ since $(0,0)$ is on the line $y = 0$. However, $h_{z,-1} \circ \psi_t$ is a noncentral automorphism.

In contrast to ψ_t, the center of $h_{z,-1}$ is on the axis of φ_s. Hence we obtain $h_{z,-1} \circ \varphi_s = \varphi_s \circ h_{z,-1}$. The product $h_{z,-1} \circ \varphi_s$ is a noncentral automorphism.

EXAMPLE 3.3. Let F be the field of reals with standard order and put $f(x) = g(x) = x^3$. For every $u \in F$, $u \neq 0, 1$, $h_{0,u}(x,y) = (x/u, uy)$ is a homothety with center $(0,0)$ [3, p. 356]. We define φ_s and ψ_t as in Example 3.2. Then φ_s, ψ_t are (central) automorphisms if $s, t > 0$ and then $\varphi_s \circ \psi_t = \psi_t \circ \varphi_s$. In general $\varphi_s \circ \psi_t$ is not central. But $\varphi_s \circ \psi_t$ is central if $t = s^{-1}$. Indeed, φ_s maps the line $(x-p)(y-q) = r$ with $r < 0$ onto the line $(x-sp)(y-q) = sr$, and the line $(x^3-p)(y^3-q) = r$ with $r > 0$ onto the line $(x^3 - s^3p)(y^3-q) = s^3r$. For ψ_t the situation is analogous. Therefore $\varphi_s \circ \psi_t$ maps the line $(x-p)(y-q) = r$ onto $(x-sp)(y-tq) = str$ and the line $(x^3-p)(y^3-q) = r$ onto $(x^3-s^3p)(y^3-t^3q) = s^3t^3r$. In particular, for the straight line g described by $x = 0$ all proper lines with base points on g are given by the equations $x(y-q) = r$ with $r < 0$ and $x^3(y^3-q) = r$ with $r > 0$, and $\varphi_s \circ \psi_t$ maps them onto lines $x(y-tq) = str$ and $x^3(y^3-t^3q) = s^3t^3r$

respectively. Only the ideal line may be the axis of $\varphi_s \circ \psi_t$, since it passes through the centers of φ_s and of ψ_t. Every ideal point of $\overline{\mathbf{NA}(g)}$ is fixed by $\overline{(\varphi_s)_g} \circ \overline{(\psi_t)_g}$ if $r = str$ and $r = s^3t^3r$. This is possible if $t = s^{-1}$. In this case $\varphi_s \circ \psi_{s^{-1}}$ is a central automorphism with the ideal axis and center $(0,0)$. We obtain $\varphi_s \circ \psi_{s^{-1}}(x,y) = (sx, s^{-1}y) = h_{0,s^{-1}}(x,y)$.

Consider the remaining products. We have $h_{0,u} \circ \varphi_s(x,y) = \varphi_s \circ h_{0,u}(x,y) = (sx/u, uy)$ and $h_{0,u} \circ \psi_t(x,y) = \psi_t \circ h_{0,u}(x,y) = (x/u, uty)$. The line joining the centers of φ_s and $h_{0,u}$ is $y = 0$, so $h_{0,u} \circ \varphi_s$ is central if every point $(x,0)$ is fixed. We obtain $(sx/u, 0) = (x,0)$, hence $s = u$ and then $h_{0,u} \circ \varphi_u = \psi_u$ is a central automorphism with the ideal center corresponding to the line $x = 0$ and the axis $y = 0$.

The line joining the centers of ψ_t and $h_{0,u}$ is $x = 0$, so now $(0,y)$ should be fixed by the product and we obtain $(0, uty) = (0,y)$, i.e. $u = t^{-1}$, $h_{0,t^{-1}} \circ \psi_t(x,y) = (tx, y) = \varphi_t(x,y)$.

It is not difficult to conclude that for every $u \neq 1, 0$, the group generated by $h_{0,u}$, φ_u, ψ_u is $\Gamma_u = \{\varphi_{n,m};\ n, m \in \mathbb{Z}\}$, where $\mathbb{Z}$ is the set of integers and $\varphi_{n,m}(x,y) = (u^n x, u^m y)$

4. Pseudo-automorphisms of a nearaffine plane. There exist examples of bijections of the point sets of nearaffine planes which are not automorphisms, but become automorphisms on affine planes determined by straight lines.

Definition 4.1. A bijection φ of a nearaffine plane $\mathbf{NA}$ is a *pseudo-automorphism* if there exists a straight line g such that φ_g is an automorphism of $\mathbf{NA}(g)$. Such a pseudo-automorphism φ is called *central* if $\overline{\varphi}_g$ is a central collineation in $\overline{\mathbf{NA}(g)}$.

Of course φ is a pseudo–automorphism if φ maps a line with base point on g onto a line with base point on g, although the image of the base point need not be the base point. Moreover, lines which have base points not on g need not be mapped onto lines.

Example 4.1. Let $\mathbf{NA}$ be the classical nearaffine plane over the field of reals. Consider the proper line $a = \{(0,0)\} \cup \{(x,y)\};\ xy = 1\}$. For every point $P = (u,v)$ the straight lines through P are given by $x = u$, $y = v$. If $u \neq 0 \neq v$ then they intersect a at the points $U = (u, 1/u)$, $V = (1/v, v)$, respectively. The remaining two straight lines through U or V are described by the equations $x = 1/v$, $y = 1/u$ and they intersect at the point $Q = (1/v, 1/u)$. In the same way we consider the case $u = 0$ or $v = 0$. Therefore, the proper line a determines the following bijection S_a of the point set:

$$S_a(x,y) = \begin{cases} (1/y, 1/x) & \text{for } x \neq 0 \neq y, \\ (0, 1/x) & \text{for } x \neq 0 = y, \\ (1/y, 0) & \text{for } x = 0 \neq y, \\ (0,0) & \text{for } x = 0 = y. \end{cases}$$

Clearly, $S_a(P) = P \Leftrightarrow P \in a$. Note that S_a is never an automorphism of the nearaffine plane, since the line $\{(p,q)\} \cup \{(x,y);\ (x-p)(y-q) = pq\}$ with $p \neq 0 \neq q$ is mapped onto the set $\{(1/q, 1/p)\} \cup \{(x,y);\ qx + py = 1\}$ which is not a line. But any proper line c with base point on the straight line g described by $x = 0$ is given by $x(y-q) = r$. We find that the image of c is given by $x(y + q/r) = 1/r$. Note that the base point $(0,q)$ with $q \neq 0$ of c is mapped to $(1/q, 0)$ which is not the base point of $S_a(c)$ but it is on $S_a(c)$. The line a is the axis of the pseudo-automorphism S_a and the class of proper lines described by $x(y-q) = -1$ is the center of S_a. In the same way we conclude that every proper line $b = \{(p,q)\} \cup \{(x,y)\};\ (x-p)(y-q) = v\}$ with $v \neq 0$ and arbitrary p, q determines a pseudo-automorphism S_b.

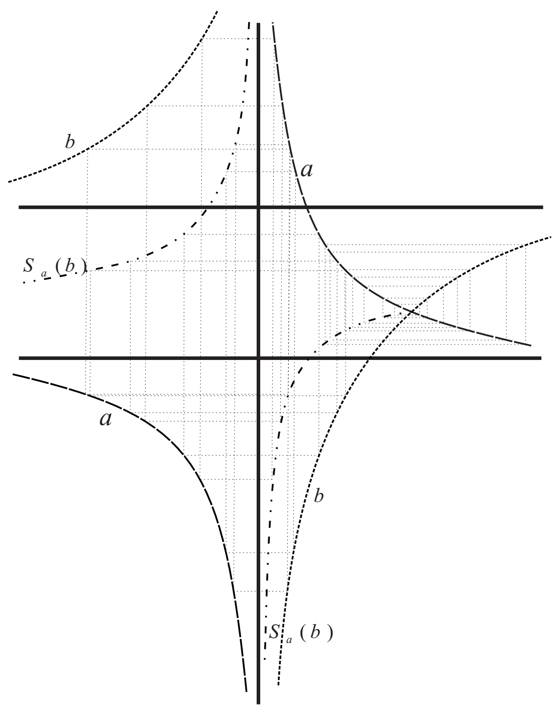

Fig. 5

EXAMPLE 4.2. Let F and the nearaffine plane **NA** be as described in Example 3.3. For the line $a = \{(0,0)\} \cup \{(x,y)\};\ xy = v\}$, where $v \neq 0$, we use the same arguments which work for Example 4.1 and obtain:

$$S_a(x,y) = \begin{cases} (v/y, v/x) & \text{for } x \neq 0 \neq y, \\ (0, v/x) & \text{for } x \neq 0 = y, \\ (v/y, 0) & \text{for } x = 0 \neq y, \\ (0,0) & \text{for } x = 0 = y. \end{cases}$$

The line $b = \{(p,q)\} \cup \{(x,y);\ (x-p)(y-q) = pq\}$ with $p \neq 0 \neq q$, $pq < 0$, is not mapped onto a line again. Clearly, S_a maps any line

$$c = \{(0,q)\} \cup \{(x,y);\, x(y-q) = r\},$$

where $r < 0$, onto the line

$$S_a(c) = \left\{ \left(0, -\frac{vq}{r}\right) \right\} \cup \left\{ (x,y);\, x\left(y + \frac{vq}{r}\right) = \frac{v^2}{r} \right\}.$$

Consider any line

$$d = \{(0,q)\} \cup \{(x,y);\ x^3(y^3 - q^3) = r\}$$

with $r > 0$ and with its base point $(0,q)$ on the straight line $x = 0$. We have

$$S_a(d) = \left\{ \left(0, -\frac{vq}{\sqrt[3]{r}}\right) \right\} \cup \left\{ (x,y);\, x^3\left(y^3 + \frac{v^3q^3}{r}\right) = \frac{v^6}{r} \right\}.$$

Note that v^2/r and v^6/r are of the same sign as r. Therefore S_a is a pseudo-automorphism. It is central since a is its axis. A simple calculation shows that the class $\{\{(0,q)\} \cup \{(x,y);\, x^3(y^3 - q^3) = -v^3\};\, q \in F\}$ (resp. $\{\{(0,q)\} \cup \{(x,y);\, x(y-q) = -v\};\, q \in F\}$) of parallel lines is the center of S_a if $v < 0$ (resp. $v > 0$).

Now if $b = \{(p,0)\} \cup \{(x,y);\ (x-p)y = r\}$, $r < 0$ and $p \neq 0$ then set

$$S_b(x,y) = \begin{cases} \left(\dfrac{r}{y} + p, \dfrac{r}{x-p}\right) & \text{for } x \neq p,\ y \neq 0, \\ \left(p, \dfrac{r}{x-p}\right) & \text{for } x \neq p,\ y = 0, \\ \left(\dfrac{r}{y} + p, 0\right) & \text{for } x = p,\ y \neq 0, \\ (p,0) & \text{for } x = p,\ y = 0. \end{cases}$$

This time S_b is not a pseudo-automorphism. The line

$$\{(p,q)\} \cup \{(x,y);\ (x^3 - p^3)(y^3 - q^3) = s\}$$

with $s > 0$ and the base point on the straight line $x = p$ is mapped onto the set

$$\left\{ \left(\frac{r}{q} + p, 0\right) \right\} \cup \left\{ (x,y);\ \left(\left(\frac{r}{y} + p\right)^3 - p^3\right)\left(\left(\frac{r}{x-p}\right)^3 - q^3\right) = s \right\}$$

which is not a line. Using the same arguments, we conclude that in this plane S_b is a pseudo-automorphism if and only if $(0, 0)$ is the base point of a.

References

[1] J. André, *On finite non-commutative affine spaces*, in M. Hall and J. H. von Lint (eds.), Combinatorics, Part I, Math. Centre Tracts 55, 1974, 60–107.
[2] P. Dembowski, *Finite Geometries*, Springer, Berlin, 1968.
[3] J. Jakóbowski, *Nearaffine planes related to pseudo-ordered fields*, Bull. Polish Acad. Sci. Math. 50 (2002), 345–360.
[4] G. Pickert, *Projektive Ebenen*, Springer, Berlin, 1955.
[5] H. A. Wilbrink, *Nearaffine planes*, Geom. Dedicata 12 (1982), 53–62.

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