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FUNCTIONAL ANALYSIS

Optimal Constants in Khintchine Type Inequalities for Fermions, Rademachers and q-Gaussian Operators

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Summary. For (P_k) being Rademacher, Fermion or q-Gaussian $(-1 \le q \le 0)$ operators, we find the optimal constants C_{2n}, $n \in \mathbb{N}$, in the inequality

$$\Big\| \sum_{k=1}^{N} A_k \otimes P_k \Big\|_{2n} \le [C_{2n}]^{1/2n} \max \Big\{ \Big\| \Big(\sum_{k=1}^{N} A_k^* A_k \Big)^{1/2} \Big\|_{L_{2n}}, \Big\| \Big(\sum_{k=1}^{N} A_k A_k^* \Big)^{1/2} \Big\|_{L_{2n}} \Big\},$$

valid for all finite sequences of operators (A_k) in the non-commutative L_{2n} space related to a semifinite von Neumann algebra with trace. In particular, $C_{2n} = (2n-1)!!$ for the Rademacher and Fermion sequences.

Introduction. The classical Khintchine inequality states that

$$\Big\| \sum \alpha_k r_k \Big\|_{L_p(0,1)} \le \Big(2 \Big\lceil \frac{p}{2} \Big\rceil - 1 \Big)!! \Big(\sum \alpha_k^2 \Big)^{1/2}$$

for $p \ge 2$ and all finite sequences (α_k) of real scalars. (r_k) is the Rademacher sequence, i.e.

$$r_k(x) = \operatorname{sgn}(\sin(2^k \pi x)), \qquad k = 1, 2, \ldots.$$

The inequality appears in many branches of mathematical analysis and it was generalized in many different ways. Also a lot of effort was put into improving the constants in these inequalities.

The first such generalization is due to Orlicz who replaced the sequence of scalars (α_k) by a sequence of vectors in the Banach space L_p.

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A natural generalization of the Orlicz inequality is the result of F. Lust-Piquard [LP] who proved

$$(1)\quad \Big\|\sum_{k=1}^{N} A_k\otimes r_k\Big\|_p \le C_p \max\Big\{\Big\|\Big(\sum_{k=1}^{N} A_k^*A_k\Big)^{1/2}\Big\|_{L_p},\Big\|\Big(\sum_{k=1}^{N} A_kA_k^*\Big)^{1/2}\Big\|_{L_p}\Big\},$$

where A_k are operators from the Schatten trace class L_p. However the constants C_p she obtained are quite large.

In this paper we give the proof of this inequality for $p=2n$ which gives $C_{2n}=(2n-1)!!$. For this we show that in the inequality (1) one can replace the Rademacher sequence by the Fermi sequence (U_k) (see formulas (3) for a definition). This follows from the fact that the Rademacher sequence has the same distribution as the sequence $(U_k\otimes U_k)$ (see Lemma 3 and Corollary 4).

In the last section we study Khintchine type inequalities for q-Gaussian sequences with negative q. We find the optimal constants in this case.

Definitions and notations. Let $H_{\mathbb{R}}$ be a real Hilbert space with $\dim H_{\mathbb{R}}=N$, and let H be its complexification. The inner product on H will be denoted by $\langle\cdot\,|\,\cdot\rangle_H$. We define the vector space $\mathcal{F}^0$ by

$$\mathcal{F}^0=\mathbb{C}\Omega\oplus\bigoplus_{k=1}^{\infty}H^{\otimes k},$$

where $\mathbb{C}\Omega=H^{\otimes 0}$ is the 1-dimensional space spanned by a fixed vector Ω. We choose an orthonormal basis $\{e_1,\dots,e_N\}$ for $H_{\mathbb{R}}$, and we define the linear operators l_k^q, l_k^+ $(k\in\{1,\dots,N\},\ q\in[-1,1])$ acting on $\mathcal{F}^0$ by the formulas

$$\begin{aligned}
l_k^+\Omega &= e_k,\\
l_k^+ e_{i_1}\otimes\cdots\otimes e_{i_m} &= e_k\otimes e_{i_1}\otimes\cdots\otimes e_{i_m},\\
l_k^q\Omega &= 0,\\
l_k^q e_m &= \langle e_m\,|\,e_k\rangle_H\Omega,\\
l_k^q e_{i_1}\otimes\cdots\otimes e_{i_m} &= \sum_{p=1}^{m} q^{p-1}\langle e_{i_p}\,|\,e_k\rangle_H e_{i_1}\otimes\cdots\otimes e_{i_{p-1}}\otimes\hat{e}_{i_p}\otimes e_{i_{p+1}}\otimes e_{i_m},\\
&\qquad m>1,
\end{aligned}$$

where the hat indicates omission of a vector in the tensor product. Next, we introduce a non-negative Hermitian form $\langle\cdot\,|\,\cdot\rangle_{\mathcal{F}_q}$ on $\mathcal{F}^0$, and we define a Hilbert space $\mathcal{F}_q$. Namely, whenever $k\neq l$, the subspaces $H^{\otimes k}$ and $H^{\otimes l}$ are orthogonal with respect to $\langle\cdot\,|\,\cdot\rangle_{\mathcal{F}_q}$, the vector Ω has norm 1 with respect to this form, and on $H^{\otimes k}$ we put

$$\langle e_{i_1}\otimes\cdots\otimes e_{i_k}\,|\,e_{j_1}\otimes\cdots\otimes e_{j_k}\rangle_{\mathcal{F}_q}=\sum_{\sigma\in S_k} q^{i(\sigma)}\prod_{p=1}^{k}\langle e_{i_p}\,|\,e_{j_{\sigma(p)}}\rangle_H,$$

where S_k is the symmetric group, and $i\,(\sigma)$ is the number of inversions in the permutation σ, i.e.,

$$i(\sigma) = \#\{(k,l) : k < l \text{ and } \sigma(k) > \sigma(l)\}.$$

This form has non-trivial kernel when $q \in \{-1, 1\}$. Note also that $H^{\otimes k} = 0$ in the Hilbert space $\mathcal{F}_{-1}$ whenever $k > N$. The operators l_k^q and l_k^+ preserve the kernel of the form $\langle \cdot \,|\, \cdot \rangle_{\mathcal{F}_q}$. This allows us to consider l_k^q and l_k^+ as operators on the space $\mathcal{F}_q$, defined to be the completion of $\mathcal{F}^0/\ker\langle \cdot \,|\, \cdot \rangle_{\mathcal{F}_q}$ in the norm given by the scalar product $\langle \cdot \,|\, \cdot \rangle_{\mathcal{F}_q}$. From now on l_k^q and l_k^+ stand for these operators on $\mathcal{F}_q$. They are mutually adjoint,

$$(l_k^q)^* = l_k^+.$$

The case $q = 1$ is special since our operators are unbounded.

Definition 1. Let $\{e_1, \ldots, e_N\}$ be an orthonormal basis of the real Hilbert space $H_{\mathbb{R}}$, and let H be the complexification of $H_{\mathbb{R}}$. The sequence $(G_k^q)_{k=1}^N$ of selfadjoint operators defined by

$$G_k^q = l_k^q + l_k^+,$$

where l_k^q and l_k^+ are as above, is called the *q-Gaussian operator sequence*. In the case $q = -1$ those operators will be called *fermions* and denoted by U_k.

The following formula for mixed moments of q-Gaussian operators will be used (see [BSp1]):

$$\langle G_{k_1}^{(q)} \cdots G_{k_{2m}}^{(q)} \Omega \,|\, \Omega \rangle = \sum_{\substack{\nu \in \mathbb{P}_2(2m) \\ \nu = \{\{p_1, q_1\}, \ldots \{p_m, q_m\}\}}} q^{i(\nu)} \prod_{s=1}^{m} \langle e_{k_{p_s}} \,|\, e_{k_{q_s}} \rangle_H, \tag{2}$$

where $\mathbb{P}_2(2m)$ is the set of 2-partitions of the set $\{1, \ldots, 2m\}$, and $i(\nu)$ is the crossing number of the 2-partition ν.

In [Bu] a Khintchine type inequality for q-Gaussian operator sequences was proved. In the last section we will show the optimality for $q < 0$ in the following theorem (see Theorem 5 of [Bu]):

Theorem 2. *Let $(G_k^q)_{k=1}^N$ be the q-Gaussian operator sequence, and let $(A_k)_{k=1}^N$ be a sequence of operators from the non-commutative L_{2n}-space related to a semifinite von Neumann algebra with trace* Tr. *Then*

$$\Big\| \sum_{k=1}^N A_k \otimes G_k^q \Big\|_{2n} \le [C_{2n}(q)]^{1/2n} \max\Big\{ \Big\| \Big(\sum_{k=1}^N A_k^* A_k \Big)^{1/2} \Big\|_{L_{2n}}, \Big\| \Big(\sum_{k=1}^N A_k A_k^* \Big)^{1/2} \Big\|_{L_{2n}} \Big\},$$

where $\|T\|_{2n}^{2n} = \mathrm{Tr} \otimes \mathbb{E}_\Omega((TT^*)^n)$ *for the state* $\mathbb{E}_\Omega(S) = \langle S\Omega \mid \Omega\rangle_{\mathcal{F}_q}$ *and*

$$C_{2n}(q) = \langle (G_1^{|q|})^{2n}\Omega \mid \Omega\rangle.$$

Moreover the constant $C_{2n}(q)$ *in the above inequality is optimal for non-negative* q.

Fermions. In this section we consider the case $q = -1$. We will write U_k instead of G_k^{-1}. The main properties of the operators U_k are their unitarity and commutation properties, i.e.

$$\begin{aligned} U_k &= U_k^*, \\ U_k U_k &= \mathbb{1}, \\ U_k U_l &= -U_l U_k, \quad k \neq l, \end{aligned} \tag{3}$$

where $\mathbb{1}$ is the identity operator on $\mathcal{F}_{-1}$. The next lemma establishes a relationship between the Fermion sequence and the Rademacher sequence.

LEMMA 3. *Let* $(U_k)_{k=1}^N$ *and* $(r_k)_{k=1}^N$ *be the Fermion and Rademacher sequences respectively. Moreover let* G_R *be the group generated by the operators* $R_k = U_k \otimes U_k$ *acting on the Hilbert space* $\mathcal{F}_{-1} \otimes_2 \mathcal{F}_{-1}$, *and let* G_r *be the group under pointwise multiplication generated by the rademachers* r_k. *Then the mapping*

$$R_k \mapsto r_k$$

extends to a group isomorphism between G_R *and* G_r.

Proof. By (3), R_k have order two and form a commuting family. We have to check that for any $m \leq N$,

$$R_{k_1} \cdots R_{k_m} \neq \mathbb{1},$$

where $k_1, \ldots, k_m$ are different positive integers. This follows from

$$R_{k_1} \cdots R_{k_m}[\Omega] \otimes [\Omega] = [e_{k_1} \otimes \cdots \otimes e_{k_m}] \otimes [e_{k_1} \otimes \cdots \otimes e_{k_m}] \neq [\Omega] \otimes [\Omega]. \ \blacksquare$$

The above lemma can be equivalently stated as follows:

COROLLARY 4. *The joint distribution of the Rademacher sequence is the same as the distribution of the sequence* (R_k), *i.e.*

$$\langle R_{k_1} \cdots R_{k_m}[\Omega] \otimes [\Omega] \mid [\Omega] \otimes [\Omega]\rangle_{\mathcal{F} \otimes \mathcal{F}} = \int_0^1 r_{k_1}(t) \cdots r_{k_m}(t)\, dt.$$

From now on, the symbol C_{2n} will denote the number $(2n-1)!!$. Theorem 2 and Corollary 4 imply

THEOREM 5. *Let* $(r_k)_{k=1}^N$ *be the Rademacher sequence. Moreover let* $(A_k)_{k=1}^N$ *be as in the preceding theorem. Then*

$$\Big\|\sum_{k=1}^N A_k\otimes r_k\Big\|_{2n} \le (C_{2n})^{1/2n}\max\Big\{\Big\|\Big(\sum_{k=1}^N A_k^*A_k\Big)^{1/2}\Big\|_{L_{2n}},\Big\|\Big(\sum_{k=1}^N A_kA_k^*\Big)^{1/2}\Big\|_{L_{2n}}\Big\}. \tag{4}$$

Moreover the constant $C_{2n}=(2n-1)!!$ *is optimal as* N *runs over the natural numbers and* (A_k) *runs over the non-commutative* L_{2n}*-spaces.*

Proof. By Corollary 4 we have

$$\Big\|\sum_{k=1}^N A_k\otimes r_k\Big\|_{2n}^{2n} = \mathrm{Tr}\int_0^1\Big(\Big(\sum_{k=1}^N A_kr_k(t)\Big)\Big(\sum_{k=1}^N A_k^*r_k(t)\Big)\Big)^n\,dt = \Big\|\sum_{k=1}^N A_k\otimes U_k\otimes U_k\Big\|_{2n}^{2n}. \tag{5}$$

The equality above, Theorem 2 applied to A_k replaced by $A_k\otimes U_k$, and the first equality in (3) complete the proof. ∎

Also, as a consequence of the above proof we obtain equality between the optimal constants in the operator Khintchine inequality for fermions and rademachers as well as the optimality of the constants $C_{2n}(-1)=(2n-1)!!$ in Theorem 2.

THEOREM 6. *For any* $p\ge 2$ *the optimal constants in the operator Khintchine inequality for the Fermion and Rademacher sequences are identical.*

Proof. As was mentioned above, the equality (5) implies that for any even polynomial w and any bounded sequence (B_k),

$$\mathrm{Tr}\otimes\int\Big(w\Big(\sum B_k\otimes r_k\Big)\Big) = \mathrm{Tr}\otimes\mathbb{E}_\Omega\otimes\mathbb{E}_\Omega\Big(w\Big(\sum B_k\otimes U_k\otimes U_k\Big)\Big).$$

Since the function $|\cdot|^p$ can be uniformly approximated on compact sets by even polynomials and since L_∞ is dense in L_p we get the assertion. ∎

REMARK 7. The constant $C_{2n}(-1)$ is optimal in Theorem 2.

q-Gaussian. We will make use of the following non-commutative Central Limit Theorem (see Theorem 0 in [BSp1]).

THEOREM 8. *Let* $\mathcal{B}$ *be a unital* $*$*-algebra with a state* ϕ. *Consider self-adjoint elements* $b_i=b_i^*\in\mathcal{B}$ $i\in\mathbb{N}$ *normalized by* $\phi(b_i^2)=1$, *which satisfy the following assumptions*:

$$\phi(b_{i_1}\cdots b_{i_n})=0 \quad \textit{whenever} \quad \#\{m : i_k=i_m\}=1 \textit{ for some } k,$$
$$\phi(b_{i_1}\cdots b_{i_n})=\phi(b_{\pi(i_1)}\cdots b_{\pi(i_n)}) \quad \textit{for any injection } \pi:\mathbb{N}\to\mathbb{N}.$$

Then for the operators

$$S_N(k)=\frac{1}{\sqrt{N}}\sum_{i\in A_{N,k}} b_i$$

where for each N the sets $A_{N,k}$ are disjoint and of cardinality N each, the following equalities hold:

$$\lim_{N\to\infty}\phi(S_N(k_1)\cdots S_N(k_n))=0 \quad \textit{whenever} \quad n\in 2\mathbb{N}+1,$$
$$\lim_{N\to\infty}\phi(S_N(k_1)\cdots S_N(k_{2n}))=\sum_{\nu\in\mathbb{P}_2(\{1,\ldots,2n\})}\delta_{k_{i_1},k_{j_1}}\cdots\delta_{k_{i_n},k_{j_n}}t(\nu),$$

where $\nu=\{\{i_1,j_1\},\ldots,\{i_n,j_n\}\}$ and t is some positive definite function on $\mathbb{P}_2$.

To show the optimality of the constants $C_{2n}(q)=\langle (G_1^{|q|})^{2n}\Omega \mid \Omega\rangle^{1/2n}$ in Theorem 2 we will follow the method used for fermions.

Consider the operators

$$R_k^q=U_k\otimes G_k^q.$$

Since the sequence $(R_k^q)_{k=1}^{\infty}$ satisfies the assumptions of Theorem 8, the optimal constants in Theorem 2 cannot be smaller than the moments of the central measure associated with the sequence $(R_k^q)_{k=1}^{\infty}$. By (3) and (2) this measure has the $2n$-moments equal to the corresponding moments for the operator $G_1^{|q|}$,

$$\left\langle \left(\lim_{N\to\infty}\frac{1}{\sqrt{N}}\sum_{k=1}^{N}R_k^q\right)^{2n}[\Omega]\otimes\Omega \,\middle|\, [\Omega]\otimes\Omega\right\rangle=\langle (G_1^{|q|})^{2n}\Omega\mid\Omega\rangle.$$

The above considerations prove the following theorem:

THEOREM 9. *The constants in Theorem 2 remain optimal when $q\in[-1,1]$.*

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