Optimal Constants in Khintchine Type Inequalities for Fermions, Rademachers and $q$-Gaussian Operators

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Summary. For $(P_k)$ being Rademacher, Fermion or $q$-Gaussian ($-1 \leq q \leq 0$) operators, we find the optimal constants $C_{2n}$, $n \in \mathbb{N}$, in the inequality

$$\left\| \sum_{k=1}^{N} A_k \otimes P_k \right\|_{2n} \leq \left[ C_{2n} \right]^{1/2n} \max \left\{ \left\| \left( \sum_{k=1}^{N} A_k^* A_k \right)^{1/2} \right\|_{L_{2n}}, \left\| \left( \sum_{k=1}^{N} A_k A_k^* \right)^{1/2} \right\|_{L_{2n}} \right\},$$

valid for all finite sequences of operators $(A_k)$ in the non-commutative $L_{2n}$ space related to a semifinite von Neumann algebra with trace. In particular, $C_{2n} = (2n - 1)!!$ for the Rademacher and Fermion sequences.

Introduction. The classical Khintchine inequality states that

$$\left\| \sum \alpha_k r_k \right\|_{L_p(0,1)} \leq \left( 2 \left\lceil \frac{p}{2} \right\rceil - 1 \right)!! \left( \sum \alpha_k^2 \right)^{1/2}$$

for $p \geq 2$ and all finite sequences $(\alpha_k)$ of real scalars. $(r_k)$ is the Rademacher sequence, i.e.

$$r_k(x) = \text{sgn}(\sin(2^k \pi x)), \quad k = 1, 2, \ldots.$$ 

The inequality appears in many branches of mathematical analysis and it was generalized in many different ways. Also a lot of effort was put into improving the constants in these inequalities.

The first such generalization is due to Orlicz who replaced the sequence of scalars $(\alpha_k)$ by a sequence of vectors in the Banach space $L_p$. 

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A natural generalization of the Orlicz inequality is the result of F. Lust-Piquard [LP] who proved

\[ \left\| \sum_{k=1}^{N} A_k \otimes r_k \right\|_p \leq C_p \max \left\{ \left\| \left( \sum_{k=1}^{N} A_k^* A_k \right)^{1/2} \right\|_{L_p}, \left\| \left( \sum_{k=1}^{N} A_k^* A_k \right)^{1/2} \right\|_{L_p} \right\}, \]

where \( A_k \) are operators from the Schatten trace class \( L_p \). However the constants \( C_p \) she obtained are quite large.

In this paper we give the proof of this inequality for \( p = 2n \) which gives \( C_{2n} = (2n - 1)!! \). For this we show that in the inequality (1) one can replace the Rademacher sequence by the Fermi sequence \((U_k)\) (see formulas (3) for a definition). This follows from the fact that the Rademacher sequence has the same distribution as the sequence \((U_k \otimes U_k)\) (see Lemma 3 and Corollary 4).

In the last section we study Khintchine type inequalities for \( q \)-Gaussian sequences with negative \( q \). We find the optimal constants in this case.

Definitions and notations. Let \( H_\mathbb{R} \) be a real Hilbert space with \( \dim H_\mathbb{R} = N \), and let \( H \) be its complexification. The inner product on \( H \) will be denoted by \( \langle \cdot \mid \cdot \rangle_H \). We define the vector space \( F^0 \) by

\[ F^0 = \mathbb{C} \Omega \bigoplus \bigoplus_{k=1}^{\infty} H^\otimes k, \]

where \( \mathbb{C} \Omega = H^{\otimes 0} \) is the 1-dimensional space spanned by a fixed vector \( \Omega \). We choose an orthonormal basis \( \{e_1, \ldots, e_N\} \) for \( H_\mathbb{R} \), and we define the linear operators \( l_k^+, l_k^- (k \in \{1, \ldots, N\}, q \in [-1, 1]) \) acting on \( F^0 \) by the formulas

\[
\begin{align*}
l_k^+ \Omega &= e_k, \\
l_k^- e_{i_1} \otimes \cdots \otimes e_{i_m} &= e_k \otimes e_{i_1} \otimes \cdots \otimes e_{i_m}, \\
l_k^q \Omega &= 0, \\
l_k^q e_m &= \langle e_m \mid e_k \rangle_H \Omega, \\
l_k^q e_{i_1} \otimes \cdots \otimes e_{i_m} &= \sum_{p=1}^{m} q^{p-1} \langle e_{i_p} \mid e_k \rangle_H e_{i_1} \otimes \cdots \otimes \hat{e}_{i_p} \otimes e_{i_{p+1}} \otimes e_{i_m},
\end{align*}
\]

where the hat indicates omission of a vector in the tensor product. Next, we introduce a non-negative Hermitian form \( \langle \cdot \mid \cdot \rangle_{F^q} \) on \( F^0 \), and we define a Hilbert space \( F^q \). Namely, whenever \( k \neq l \), the subspaces \( H^{\otimes k} \) and \( H^{\otimes l} \) are orthogonal with respect to \( \langle \cdot \mid \cdot \rangle_{F^q} \), the vector \( \Omega \) has norm 1 with respect to this form, and on \( H^{\otimes k} \) we put

\[
\langle e_{i_1} \otimes \cdots \otimes e_{i_k} \mid e_{j_1} \otimes \cdots \otimes e_{j_k} \rangle_{F^q} = \sum_{\sigma \in S_k} q^{|i(\sigma)|} \prod_{p=1}^{k} \langle e_{i_p} \mid e_{j_{\sigma(p)}} \rangle_H,
\]
where $S_k$ is the symmetric group, and $i(\sigma)$ is the number of inversions in the permutation $\sigma$, i.e.,

$$i(\sigma) = \#\{(k, l) : k < l \text{ and } \sigma(k) > \sigma(l)\}.$$ 

This form has non-trivial kernel when $q \in \{-1, 1\}$. Note also that $H^\otimes k = 0$ in the Hilbert space $F_1$ whenever $k > N$. The operators $l_k^q$ and $l_k^+$ preserve the kernel of the form $\langle \cdot | \cdot \rangle_{F_q}$. This allows us to consider $l_k^q$ and $l_k^+$ as operators on the space $F_q$, defined to be the completion of $F^0 / \ker \langle \cdot | \cdot \rangle_{F_q}$ in the norm given by the scalar product $\langle \cdot | \cdot \rangle_{F_q}$. From now on $l_k^q$ and $l_k^+$ stand for these operators on $F_q$. They are mutually adjoint,

$$\left(l_k^q\right)^* = l_k^+.$$ 

The case $q = 1$ is special since our operators are unbounded.

**Definition 1.** Let $\{e_1, \ldots, e_N\}$ be an orthonormal basis of the real Hilbert space $H_{\mathbb{R}}$, and let $H$ be the complexification of $H_{\mathbb{R}}$. The sequence $(G^q_k)_{k=1}^N$ of selfadjoint operators defined by

$$G^q_k = l_k^q + l_k^+,$$

where $l_k^q$ and $l_k^+$ are as above, is called the $q$-Gaussian operator sequence. In the case $q = -1$ those operators will be called fermions and denoted by $U_k$.

The following formula for mixed moments of $q$-Gaussian operators will be used (see [BSp1]):

$$(2) \quad \langle G^{(q)}_{k_1} \cdots G^{(q)}_{k_{2m}} \Omega | \Omega \rangle = \sum_{\nu \in \mathbb{P}_2(2m)} q^i(\nu) \prod_{s=1}^m \langle e_{k_{ps}} | e_{k_{qs}} \rangle_H,$$

where $\mathbb{P}_2(2m)$ is the set of 2-partitions of the set $\{1, \ldots, 2m\}$, and $i(\nu)$ is the crossing number of the 2-partition $\nu$.

In [Bu] a Khintchine type inequality for $q$-Gaussian operator sequences was proved. In the last section we will show the optimality for $q < 0$ in the following theorem (see Theorem 5 of [Bu]):

**Theorem 2.** Let $(G^q_k)_{k=1}^N$ be the $q$-Gaussian operator sequence, and let $(A_k)_{k=1}^N$ be a sequence of operators from the non-commutative $L_{2n}$-space related to a semifinite von Neumann algebra with trace $\text{Tr}$. Then

$$\left\| \sum_{k=1}^N A_k \otimes G^q_k \right\|_{2n} \leq \left| C_{2n}(q) \right|^{1/2n} \max \left\{ \left\| \left( \sum_{k=1}^N A^*_k A_k \right)^{1/2} \right\|_{L_{2n}}, \left\| \left( \sum_{k=1}^N A_k A^*_k \right)^{1/2} \right\|_{L_{2n}} \right\},$$
where \( \|T\|_{2n}^2 = \text{Tr} \otimes \mathbb{E}_{\Omega}((TT^*)^n) \) for the state \( \mathbb{E}_{\Omega}(S) = \langle S\Omega | \Omega \rangle_{\mathcal{F}_q} \) and
\[
C_{2n}(q) = \langle (G_1^{[q]} )^{2n} \Omega | \Omega \rangle.
\]
Moreover the constant \( C_{2n}(q) \) in the above inequality is optimal for non-negative \( q \).

**Fermions.** In this section we consider the case \( q = -1 \). We will write \( U_k \) instead of \( G_{k}^{-1} \). The main properties of the operators \( U_k \) are their unitarity and commutation properties, i.e.
\[
\begin{align*}
U_k &= U_k^*, \\
U_k U_k &= 1, \\
U_k U_l &= -U_l U_k, \quad k \neq l,
\end{align*}
\]
where \( 1 \) is the identity operator on \( \mathcal{F}_{-1} \). The next lemma establishes a relationship between the Fermion sequence and the Rademacher sequence.

**Lemma 3.** Let \( (U_k)_{k=1}^N \) and \( (r_k)_{k=1}^N \) be the Fermion and Rademacher sequences respectively. Moreover let \( G_R \) be the group generated by the operators \( R_k = U_k \otimes U_k \) acting on the Hilbert space \( \mathcal{F}_{-1} \otimes_2 \mathcal{F}_{-1} \), and let \( G_r \) be the group under pointwise multiplication generated by the rademachers \( r_k \). Then the mapping
\[
R_k \mapsto r_k
\]
extends to a group isomorphism between \( G_R \) and \( G_r \).

**Proof.** By (3), \( R_k \) have order two and form a commuting family. We have to check that for any \( m \leq N \),
\[
R_{k_1} \cdots R_{k_m} \neq 1,
\]
where \( k_1, \ldots, k_m \) are different positive integers. This follows from
\[
R_{k_1} \cdots R_{k_m} [\Omega] \otimes [\Omega] = [e_{k_1} \otimes \cdots \otimes e_{k_m}] \otimes [e_{k_1} \otimes \cdots \otimes e_{k_m}] \neq [\Omega] \otimes [\Omega]. \]

The above lemma can be equivalently stated as follows:

**Corollary 4.** The joint distribution of the Rademacher sequence is the same as the distribution of the sequence \( (R_k) \), i.e.
\[
\langle R_{k_1} \cdots R_{k_m} [\Omega] \otimes [\Omega] | [\Omega] \otimes [\Omega] \rangle_{\mathcal{F} \otimes \mathcal{F}} = \int_0^1 r_{k_1}(t) \cdots r_{k_m}(t) \, dt.
\]

From now on, the symbol \( C_{2n} \) will denote the number \( (2n - 1)!! \). Theorem 2 and Corollary 4 imply
Theorem 5. Let \((r_k)_{k=1}^N\) be the Rademacher sequence. Moreover let \((A_k)_{k=1}^N\) be as in the preceding theorem. Then

\[
\left\| \sum_{k=1}^N A_k \otimes r_k \right\|_{2n} \leq (C_{2n})^{1/2n} \max \left\{ \left\| \left( \sum_{k=1}^N A_k^* A_k \right)^{1/2} \right\|_{L_2}, \left\| \left( \sum_{k=1}^N A_k^* A_k \right)^{1/2} \right\|_{L_2} \right\}.
\]

Moreover the constant \(C_{2n} = (2n-1)!!\) is optimal as \(N\) runs over the natural numbers and \((A_k)\) runs over the non-commutative \(L_2\)-spaces.

Proof. By Corollary 4 we have

\[
\left\| \sum_{k=1}^N A_k \otimes r_k \right\|_{2n}^{2n} = \text{Tr} \left[ \left( \sum_{k=1}^N A_k r_k(t) \right)^n dt \right]
\]

\[
= \left\| \sum_{k=1}^N A_k \otimes U_k \otimes U_k \right\|_{2n}^{2n}.
\]

The equality above, Theorem 2 applied to \(A_k\) replaced by \(A_k \otimes U_k\), and the first equality in (3) complete the proof.

Also, as a consequence of the above proof we obtain equality between the optimal constants in the operator Khintchine inequality for fermions and rademachers as well as the optimality of the constants \(C_{2n}(-1) = (2n - 1)!!\) in Theorem 2.

Theorem 6. For any \(p \geq 2\) the optimal constants in the operator Khintchine inequality for the Fermion and Rademacher sequences are identical.

Proof. As was mentioned above, the equality (5) implies that for any even polynomial \(w\) and any bounded sequence \((B_k)\),

\[
\text{Tr} \otimes \int \left( w \left( \sum B_k \otimes r_k \right) \right) = \text{Tr} \otimes \mathbb{E}_\Omega \otimes \mathbb{E}_\Omega \left( w \left( \sum B_k \otimes U_k \otimes U_k \right) \right).
\]

Since the function \(|\cdot|^p\) can be uniformly approximated on compact sets by even polynomials and since \(L_\infty\) is dense in \(L_p\) we get the assertion.

Remark 7. The constant \(C_{2n}(-1)\) is optimal in Theorem 2.

q-Gaussian. We will make use of the following non-commutative Central Limit Theorem (see Theorem 0 in [BSp1]).

Theorem 8. Let \(\mathcal{B}\) be a unital \(*\)-algebra with a state \(\phi\). Consider self-adjoint elements \(b_i = b_i^* \in \mathcal{B} \ i \in \mathbb{N}\) normalized by \(\phi(b_i^2) = 1\), which satisfy the following assumptions:
\[ \phi(b_{i_1} \cdots b_{i_n}) = 0 \quad \text{whenever} \quad \# \{ m : i_k = i_m \} = 1 \text{ for some } k, \]
\[ \phi(b_{i_1} \cdots b_{i_n}) = \phi(b_{\pi(i_1)} \cdots b_{\pi(i_n)}) \quad \text{for any injection } \pi : \mathbb{N} \to \mathbb{N}. \]

Then for the operators
\[ S_N(k) = \frac{1}{\sqrt{N}} \sum_{i \in A_{N,k}} b_i \]
where for each \( N \) the sets \( A_{N,k} \) are disjoint and of cardinality \( N \) each, the following equalities hold:
\[ \lim_{N \to \infty} \phi(S_N(k_1) \cdots S_N(k_n)) = 0 \quad \text{whenever} \quad n \in 2\mathbb{N} + 1, \]
\[ \lim_{N \to \infty} \phi(S_N(k_1) \cdots S_N(k_{2n})) = \sum_{\nu \in \mathbb{P}_2(\{1, \ldots, 2n\})} \delta_{k_{i_1},k_{j_1}} \cdots \delta_{k_{i_n},k_{j_n}} t(\nu), \]
where \( \nu = \{ \{i_1,j_1\}, \ldots, \{i_n,j_n\} \} \) and \( t \) is some positive definite function on \( \mathbb{P}_2 \).

To show the optimality of the constants \( C_{2n}(q) = \langle (G_{1}^{\lfloor q \rfloor})^{2n} \Omega \mid \Omega \rangle^{1/2n} \) in Theorem 2 we will follow the method used for fermions.

Consider the operators
\[ R^q_k = U_k \otimes G^q_k. \]
Since the sequence \( (R^q_k)_{k=1}^{\infty} \) satisfies the assumptions of Theorem 8, the optimal constants in Theorem 2 cannot be smaller than the moments of the central measure associated with the sequence \( (R^q_k)_{k=1}^{\infty} \). By (3) and (2) this measure has the \( 2n \)-moments equal to the corresponding moments for the operator \( G_{1}^{\lfloor q \rfloor} \),
\[ \left\langle \left( \lim_{N \to \infty} \frac{1}{\sqrt{N}} \sum_{k=1}^{N} R^q_k \right)^{2n} \Omega \otimes \Omega \right\rangle = \langle (G_{1}^{\lfloor q \rfloor})^{2n} \Omega \mid \Omega \rangle. \]

The above considerations prove the following theorem:

**Theorem 9.** The constants in Theorem 2 remain optimal when \( q \in [-1,1] \).

**References**


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