The Embeddability of \( c_0 \) in Spaces of Operators

by

Ioana GHENCIU and Paul LEWIS

Presented by Stanisław KWAPIÈN

Summary. Results of Emmanuele and Drewnowski are used to study the containment of \( c_0 \) in the space \( K_{w^*}(X^*, Y) \), as well as the complementation of the space \( K_{w^*}(X^*, Y) \) of \( w^*-w \) compact operators in the space \( L_{w^*}(X^*, Y) \) of \( w^*-w \) operators from \( X^* \) to \( Y \).

Definitions and notations. Throughout this paper \( X \) and \( Y \) will denote real Banach spaces and \( X^* \) denotes the continuous linear dual of \( X \). An operator \( T : X \to Y \) will be a continuous and linear function. By \( X \otimes_{\lambda} Y \) we denote the injective tensor product of \( X \) and \( Y \). Notation is consistent with that used in Diestel [5]. Let \( (e_n) \) be the Schauder basis of \( c_0 \), \( (e_n^*) \) be the basis of \( \ell_1 \), and \( (e_n^2) \) the unit vector basis of \( \ell_2 \). The set of all continuous linear transformations from \( X \) to \( Y \) will be denoted by \( L(X, Y) \), and the compact (resp. weakly compact) operators will be denoted by \( K(X, Y) \) (resp. \( W(X, Y) \)). The \( w^*-w \) continuous (resp. \( w^*-w \) continuous compact) maps from \( X^* \) to \( Y \) will be denoted by \( L_{w^*}(X^*, Y) \) (resp. \( K_{w^*}(X^*, Y) \)).

A bounded subset \( A \) of \( X \) is called a limited subset of \( X \) if each \( w^* \)-null sequence in \( X^* \) tends to 0 uniformly on \( A \). If every limited subset of \( X \) is relatively compact, then we say that \( X \) has the Gelfand–Phillips property. If every weakly compact operator defined on \( X \) is completely continuous, then we say that \( X \) has the Dunford–Pettis property (DPP); see [6] and [1] for inventories of classical results related to the DPP.

Introduction. Numerous authors have studied the containment of \( c_0 \) in the spaces of compact operators \( K(X, Y) \) and \( K_{w^*}(X^*, Y) \). This problem has been studied together with the complementation of the space of comp-
pact operators $K_{w^*}(X^*, Y)$ (resp. $K(X, Y)$) in the space $L_{w^*}(X^*, Y)$ (resp. $L(X, Y)$) and the containment of $l_\infty$ in $L_{w^*}(X^*, Y)$ (resp. $L(X, Y)$). See Bartos and Lewis [2], Kalton [23], Emmanuele [13], Emmanuele and John [16], Ghenciu [19], Lewis [25], and Tong and Wilken [31] for an indication of the extensive literature that deals with these problems. The survey paper [29] by Ruess is a valuable resource for the structure of the space of operators $K_{w^*}(X^*, Y)$.

Theorem 4 of Kalton [23] states that $\ell_\infty$ embeds in $K(X, Y)$ if and only if it embeds in $X^*$ or in $Y$. In [8] Drewnowski generalized Theorem 4 of Kalton and proved that $\ell_\infty$ embeds in $K_{w^*}(X^*, Y)$ if and only if it embeds in $X$ or in $Y$. In this paper we use techniques of Emmanuele [11] and Drewnowski’s result [8] to obtain results about the complementation of the space $K_{w^*}(X^*, Y)$ of compact $w^*$-operators in the space $L_{w^*}(X^*, Y)$ of bounded $w^*$-operators. Applications to the complementation of the space $K(X, Y)$ in $W(X, Y)$ are given. We also give sufficient conditions for the containment of $c_0$ in the space $K_{w^*}(X^*, Y)$, resp. $K(X, Y)$. Results in this paper generalize results in [3], [11], [13], [14], [17], [20], [23], and [25].

**Spaces of operators.** We recall the following well-known isometries [29]:

1. $L_{w^*}(X^*, Y) \simeq L_{w^*}(Y^*, X)$ and $K_{w^*}(X^*, Y) \simeq K_{w^*}(Y^*, X)$ ($T \mapsto T^*$),
2. $W(X, Y) \simeq L_{w^*}(X^{**}, Y)$ and $K(X, Y) \simeq K_{w^*}(X^{**}, Y)$ ($T \mapsto T^{**}$).

It is known that if $X$ is infinite-dimensional and $c_0 \hookrightarrow L(X, Y)$, then $\ell_\infty \hookrightarrow L(X, Y)$ (see, e.g., [23] and [25]). Part (i) of the following theorem generalizes this result, as well as Theorem 3 in [3].

**THEOREM 1.**

(i) Suppose that $X$ and $Y$ are infinite-dimensional and $S$ is a closed linear subspace of $L(X, Y)$ which contains all the rank one operators $x^* \otimes y$, $x^* \in X^*$, $y \in Y$. If $c_0 \hookrightarrow S$ and $S$ is complemented in $L(X, Y)$, then $\ell_\infty \hookrightarrow S$.

(ii) Suppose that $X$ and $Y$ fail to have the Schur property, and $S$ is a closed linear subspace of $L_{w^*}(X^*, Y)$ which contains all rank one operators $x \otimes y$, $x \in X$, $y \in Y$. If $c_0 \hookrightarrow S$ and $S$ is complemented in $L_{w^*}(X^*, Y)$, then $\ell_\infty \hookrightarrow S$.

**Proof.** (i) Consider the following two cases.

Suppose first that $c_0 \hookrightarrow Y$ and let $(y_n)$ be a copy of $(e_n)$ in $Y$. Use the Josefson–Nissenzweig theorem and choose a $w^*$-null normalized sequence $(x_n^*)$ in $X^*$. Define $J : \ell_\infty \rightarrow L(X, Y)$ by

$$J(b)(x) = \sum b_n x_n^*(x)y_n, \quad x \in X.$$ 

Then $J$ is an isomorphism, and, if $b$ is finitely supported, $J(b) \in S$. 
Now suppose that \( c_0 \hookrightarrow Y \). Let \( B : c_0 \to S \) be an isomorphic embedding. Note that \( \sum |\langle B(e_n)(x), y^* \rangle| < \infty \) for all \( x \in X \) and \( y^* \in Y^* \). Since \( c_0 \hookrightarrow Y \), \( \sum B(e_n)(x) \) is unconditionally convergent in \( Y \) for all \( x \in X \). Define \( \mu \) by \( \mu(\emptyset) = 0 \) and

\[
\mu(A) = \sum_{n \in A} B(e_n) \quad \text{(strong operator topology)}
\]

for any non-empty subset \( A \) of \( \mathbb{N} \). Note that \( \mu \) is bounded, finitely additive and not strongly additive (\( \|\mu(\{n\})\| \to 0 \)). Apply the Diestel–Faires theorem to obtain \( \ell_\infty \hookrightarrow L(X, Y) \), and observe that if \( A \) is a finite subset of \( \mathbb{N} \), then \( \mu(A) \in S \).

Now suppose that \( S \) is complemented in \( L(X, Y) \), and let \( P : L(X, Y) \to S \) be a projection. Let \( \nu(A) = P(\chi_A) \) for \( A \subseteq \mathbb{N} \). The first part of the proof shows that \( \ell_\infty \hookrightarrow L(X, Y) \), thus \( \nu \) is well-defined. Then \( \nu : \mathcal{P}(\mathbb{N}) \to S \) is bounded and finitely additive. Moreover, \( \|\nu(\{n\})\| \to 0 \). Therefore another application of the Diestel–Faires theorem tells us that \( \ell_\infty \hookrightarrow S \).

(ii) Assume first that \( c_0 \hookrightarrow Y \). Let \((x_n)\) be a \( w \)-null normalized sequence in \( X \) and \((y_n)\) be a copy of \((e_n)\) in \( Y \). Define \( \phi : \ell_\infty \to L_{w^*}(X^*, Y) \) by

\[
\phi(b)(x^*) = \sum b_n x_n^*(x_n)y_n, \quad x^* \in X^*.
\]

We note that the series converges unconditionally. To show that \( \phi(b) \) is a \( w^*-w \) operator, we need to prove that \( (\phi(x_n^*)\) is \( w \)-null for each \( w^*-\)null net \((x_n^*)\) in \( X^* \). We can suppose that \((x_n^*)\) is a \( w\)-null net in \( B_{X^*} \) by results about the bounded \( X \) topology (or \( BX \) topology) for \( X^* \) ([10, Chapter V]). Let \( \varepsilon > 0 \) and \( y^* \in B_{Y^*} \). Since \( \sum y_n \) is \( wuc \), there is an \( n \in \mathbb{N} \) such that \( \sum_{i=n+1}^\infty |y^*(y_i)| < \varepsilon/(2\|b\|_\infty) \). Then

\[
\left| \sum_{i=n+1}^\infty b_i x_n^*(x_i)y^*(y_i) \right| \leq \|b\|_\infty \sum_{i=n+1}^\infty |y^*(y_i)| < \frac{\varepsilon}{2}.
\]

On the other hand, \( \lim_{\alpha} \sum_{i=1}^n |b_i x_n^*(x_i)y^*(y_i)| = 0 \) since \((x_n^*)\) is a \( w^* \)-null net. Therefore, for \( \alpha \) large,

\[
|\langle \phi(b)(x_n^*), y^* \rangle| \leq \left| \sum_{i=1}^n b_i x_n^*(x_i)y^*(y_i) \right| + \left| \sum_{i=n+1}^\infty b_i x_n^*(x_i)y^*(y_i) \right| < \varepsilon.
\]

Hence \( \phi(b) \) is a \( w^*-w \) operator. Further, if \( b \in \ell_\infty \) is finitely supported, \( \phi(b) \in S \). A result in [28] implies that \( \ell_\infty \hookrightarrow L_{w^*}(X^*, Y) \) since \( \|\phi(e_n)\| \to 0 \). Similarly, if \( c_0 \hookrightarrow X \) (and \( Y \) does not have the Schur property), then \( \ell_\infty \hookrightarrow L_{w^*}(X^*, Y) \).

Without loss of generality assume that \( c_0 \hookrightarrow X, Y \) and let \( B : c_0 \to S \) be an isomorphic embedding. Note that \( \sum B(e_n)(x^*) \) is \( wuc \), hence unconditionally convergent for each \( x^* \in X^* \) (since \( c_0 \hookrightarrow Y \)). Similarly, \( \sum B(e_n)^*(y^*) \) is
unconditionally convergent in $X$ for each $y^* \in Y^*$. Then
\[ \sum B(e_n) \quad \text{(strong operator topology)} \]
is a $w^*-w$ operator from $X^*$ to $Y$. Define $\mu : \mathcal{P}(\mathbb{N}) \to L_{w^*}(X^*, Y)$ by $\mu(0) = 0$ and
\[ \mu(A) = \sum_{n \in A} B(e_n) \quad \text{(strong operator topology)} \]
if $A$ is a non-empty subset of $\mathbb{N}$. Then $\mu$ is bounded (by the Uniform Boundedness Principle) and finitely additive, but $\mu(\{n\}) \to 0$. The $\sigma$-algebra version of the Diestel–Faires theorem [7] implies that $\ell_\infty \hookrightarrow L_{w^*}(X^*, Y)$. Observe that if $A$ is a finite subset of $\mathbb{N}$, then $\mu(A) \in S$.

Now suppose that $S$ is complemented in $L_{w^*}(X^*, Y)$, and let $P : L_{w^*}(X^*, Y) \to S$ be a projection. Let $\nu(A) = P(\chi_A)$ for $A \subseteq \mathbb{N}$. Then $\nu : \mathcal{P}(\mathbb{N}) \to S$ is bounded and finitely additive. Moreover, $\|\nu(\{n\})\| \to 0$. By another application of the Diestel–Faires theorem we conclude that $\ell_\infty \hookrightarrow S$.

If $X$ is infinite-dimensional and $c_0 \hookrightarrow L_{w^*}(X^*, Y)$, then $L_{w^*}(X^*, Y)$ may fail to contain $\ell_\infty$. It is not difficult to check that $c_0 \hookrightarrow K_{w^*}(\ell_1, \ell_1)$. In fact, $c_0 \hookrightarrow K_{w^*}(\ell_1, \ell_1)$; see the closing remarks in this paper. However, since $K_{w^*}(\ell_1, \ell_1) = L_{w^*}(\ell_1, \ell_1)$, Drewnowski’s theorem makes it clear that $\ell_\infty \hookrightarrow L_{w^*}(\ell_1, \ell_1)$.

Our first corollary points out that the exclusion of $\ell_\infty$ is not possible if $X$ and $Y$ do not have the Schur property.

**Corollary 2.** Suppose that $c_0 \hookrightarrow L_{w^*}(X^*, Y)$ and $X$ and $Y$ do not have the Schur property. Then $\ell_\infty \hookrightarrow L_{w^*}(X^*, Y)$.

**Corollary 3** (Ghenciu and Lewis, [20]).

(i) If $X$ does not have the Schur property and $c_0 \hookrightarrow Y$, then $\ell_\infty \hookrightarrow L_{w^*}(X^*, Y)$.

(ii) If $c_0$ does not embed in $X$ or $Y$ and $c_0 \hookrightarrow K_{w^*}(X^*, Y)$, then $\ell_\infty \hookrightarrow L_{w^*}(X^*, Y)$ provided that $X$ and $Y$ do not have the Schur property.

**Proof.** Part (i) follows from the proof of Theorem 1, and (ii) is an immediate corollary of the statement of the theorem.

The next theorem is motivated by results in [13].

**Theorem 4.** Suppose that $X$ has an unconditional and seminormalized basis $(x_i)$ with biorthogonal coefficients $(x_i^*)$, and $T : X \to Y$ is an operator such that $(T(x_i))$ is a weakly null seminormalized basic sequence in $Y$. Let $S(X, Y)$ be a closed linear subspace of $L(X, Y)$ which properly contains $K(X, Y)$ such that $\phi(b) \in S(X, Y)$ for all $b \in \ell_\infty$, where $\phi(b)(x) = \sum b_ix_i^*(x)T(x_i)$, $x \in X$. Then $K(X, Y)$ is not complemented in $S(X, Y)$. 
Therefore $A \tau \in \ell_\infty$ is a contradiction since $\tau$ is continuous linear and compact. Then $A \tau : \ell_\infty \to L(X,Y)$ by
\[
\tau(b)(x) = \sum_j b_j u_j^*(x) T(u_j), \quad x \in X.
\]
Note that $\tau(\ell_\infty) \subseteq S(X,Y)$. Consider the operators $A P \tau : \ell_\infty \to K(X,\ell_\infty)$ and $A \tau : \ell_\infty \to S(X,\ell_\infty)$. Since $\tau(e_j) = u_j^* \otimes T(u_j)$, $\tau(e_j)$ is a rank one operator, thus compact. Then $A P \tau(e_j) = A \tau(e_j)$ for each $j \in \mathbb{N}$. Proposition 5 of Kalton [23] produces an infinite subset $M$ of $\mathbb{N}$ such that
\[
A P \tau(b) = A \tau(b), \quad b \in l_\infty(M).
\]
Therefore $A \tau(\chi_M)$ is compact. But $\tau(\chi_M)(u_j) = T(u_j)$, $j \in M$, and $\{T(u_j) : j \in M\}$ is not relatively compact. Therefore $\tau(\chi_M)$ is not compact. However, this is a contradiction since $A|_{\tau(u_j)}$ is an isometry. □

**Corollary 5 (Emmanuele, [13]).** Let $Y$ be a Banach space without the Schur property. Then $K(\ell_1,Y)$ is not complemented in $W(\ell_1,Y)$.

**Proof.** Let $(y_n)$ be a $w$-null normalized basic sequence in $Y$, $X = \ell_1$, and $S(\ell_1,Y) = W(\ell_1,Y)$. Define $T : \ell_1 \to Y$ by $T(x) = \sum x_n y_n$, $x = (x_n) \in \ell_1$. If $\phi : \ell_\infty \to L(\ell_1,Y)$ is defined as in the previous theorem, then $\phi(b)(x) = \sum_j b_j x_j y_j$ for $x = (x_n) \in \ell_1$. Since $\phi(b)(e_n^*) = (b_n y_n)$ is $w$-null, $\phi(b)$ is weakly compact for all $b \in \ell_\infty$. By Theorem 4, $K(\ell_1,Y) \hookrightarrow W(\ell_1,Y)$. □

The next corollary contains principal results of [11], [13] and [16].

**Corollary 6.**

(i) If $\ell_\infty \hookrightarrow Y$ and $X$ does not have the Schur property (or $\ell_\infty \hookrightarrow X$ and $Y$ does not have the Schur property), then $K_{w^*}(X^*,Y)$ is not complemented in $L_{w^*}(X^*,Y)$.

(ii) If $c_0 \hookrightarrow K(X,Y)$ and $K(X,Y) \neq L(X,Y)$, then $K(X,Y)$ is not complemented in $L(X,Y)$.

(iii) If $c_0 \hookrightarrow Y$ and $X$ does not have the Schur property (or $c_0 \hookrightarrow X$ and $Y$ does not have the Schur property), then $K_{w^*}(X^*,Y)$ is not complemented in $L_{w^*}(X^*,Y)$.

(iv) If $c_0 \hookrightarrow K_{w^*}(X^*,Y)$ and $X$ and $Y$ do not have the Schur property, then $K_{w^*}(X^*,Y)$ is not complemented in $L_{w^*}(X^*,Y)$. 


Proof. (i) Since \( \ell_\infty \hookrightarrow Y \) and \( \ell_\infty \) is injective, \( \ell_\infty \) is complemented in \( Y \). Suppose that \( K_{w^*}(X^*, Y) \overset{c}{\hookrightarrow} L_{w^*}(X^*, Y) \). Then \( K_{w^*}(X^*, \ell_\infty) \overset{c}{\hookrightarrow} L_{w^*}(X^*, \ell_\infty) \). Let \( P \) be a projection of \( L_{w^*}(X^*, \ell_\infty) \) onto \( K_{w^*}(X^*, \ell_\infty) \). Note that \( W(\ell_1, X) \sim L_{w^*}(X^*, \ell_\infty) \) and \( K(\ell_1, X) \sim K_{w^*}(X^*, \ell_\infty) \). Hence the projection \( P \) may be viewed as an operator from \( W(\ell_1, X) \) onto \( K(\ell_1, X) \). Apply Corollary 5 now.

(ii) Supposethat \( K(X, Y) \overset{c}{\hookrightarrow} L(X,Y) \). By Theorem 1, \( \ell_\infty \hookrightarrow K(X,Y) \). Apply Theorem 4 of Kalton [23] to conclude that \( \ell_\infty \hookrightarrow X^* \) or \( \ell_\infty \hookrightarrow Y \). The first case produces a contradiction in view of Lemma 3 of Kalton [23]. If \( \ell_\infty \hookrightarrow Y \), then \( c_0 \hookrightarrow Y \), and the conclusion follows from Corollary 1 of Feder [17].

(iii) Suppose that \( c_0 \hookrightarrow Y \) and \( X \) does not have the Schur property. Assume that \( K_{w^*}(X^*, Y) \overset{c}{\hookrightarrow} L_{w^*}(X^*, Y) \). Theorem 1 implies that \( \ell_\infty \hookrightarrow K_{w^*}(X^*, Y) \). Drewnowski’s result [8] implies that \( \ell_\infty \hookrightarrow X \) or \( \ell_\infty \hookrightarrow Y \). However, this is not possible by part (i).

(iv) The same proof as for (iii). ■

Our proof of Corollary 6 made use of the following result in [17]:

**Theorem 7** (Feder, [17]). Suppose \( T \) is an operator in \( L(X,Y) \) which is not compact and \( (T_n) \) is a sequence in \( K(X,Y) \) such that for each \( x \in X \), the series \( \sum T_n(x) \) converges unconditionally to \( T(x) \). Then \( K(X,Y) \) is not complemented in \( L(X,Y) \).

In [11] Emmanuele proved that the containment of \( c_0 \) in \( K(X,Y) \) is equivalent to the hypothesis of Feder’s theorem. He used this to obtain (ii) of Corollary 6 above. In the next theorem we obtain an analogue of Feder’s theorem in \( L_{w^*}(X^*, Y) \).

**Theorem 8.** Suppose \( T \) is an operator in \( L_{w^*}(X^*, Y) \) which is not compact and \( (T_n) \) is a sequence in \( K_{w^*}(X^*, Y) \) such that for each \( x^* \in X^* \), the series \( \sum T_n(x^*) \) converges unconditionally to \( T(x^*) \). Then \( K_{w^*}(X^*, Y) \) is not complemented in \( L_{w^*}(X^*, Y) \). Furthermore, \( \ell_\infty \hookrightarrow L_{w^*}(X^*, Y) \).

Proof. Since \( L_{w^*}(X^*, Y) \neq K_{w^*}(X^*, Y) \), \( X \) and \( Y \) do not have the Schur property (if \( X \) or \( Y \) has the Schur property, \( K_{w^*}(X^*, Y) = L_{w^*}(X^*, Y) \)). Without loss of generality assume \( c_0 \hookrightarrow X,Y \) (by Corollary 6(iii)), hence \( \ell_\infty \hookrightarrow X,Y \). Suppose the operator \( T \) and the sequence \( (T_n) \) are as in the hypothesis. Since \( T \) is not compact, \( \sum T_n \) diverges in the norm topology of \( K_{w^*}(X^*, Y) \). This divergence and the pointwise unconditional convergence of the series \( \sum T_n(x^*) \) allow us to reblock the sum and to assume that \( \|T_n\| \to 0 \).

Now use the Uniform Boundedness Principle, the finite-cofinite algebra of the subsets of \( N \), and the Diestel–Faires theorem to conclude that \( c_0 \hookrightarrow K_{w^*}(X^*, Y) \); see the proof of Theorem 1 for details. (Alternatively,
note that $\sum T_n$ is weakly unconditionally convergent and not unconditionally convergent.) If $K_{w^*}(X^*, Y)$ were complemented in $L_{w^*}(X^*, Y)$, then Theorem 1 would place $\ell_\infty$ in $K_{w^*}(X^*, Y)$. Another application of Drewnowski’s result [8] would provide the contradiction that $\ell_\infty$ would embed in either $X$ or $Y$. To see that $\ell_\infty$ embeds in $L_{w^*}(X^*, Y)$ simply apply Theorem 1 again. ■

**Remark.** The hypothesis of the previous theorem implies that the series $\sum T_n$ is $wuc$ (by the Uniform Boundedness Principle) and not unconditionally convergent in $K_{w^*}(X^*, Y)$, hence $c_0$ embeds in $K_{w^*}(X^*, Y)$. Conversely, if $c_0$ embeds in $K_{w^*}(X^*, Y)$, but neither in $X$ nor in $Y$, then there is a sequence $(T_n)$ which satisfies the hypothesis of Theorem 8. In fact, if $c_0 \hookrightarrow X, Y$, then $l_\infty \hookrightarrow X, Y$ and thus $l_\infty \hookrightarrow K_{w^*}(X^*, Y)$ [8]. Let $(T_n)$ be a copy of $(e_n)$ in $K_{w^*}(X^*, Y)$. Define $\phi : \ell_\infty \rightarrow L(X^*, Y)$ by

$$\phi(b)(x^*) = \sum b_n T_n(x^*), \quad x^* \in X^*.$$  

This series is unconditionally convergent and $\phi(b)$ is a $w^*$-$w$ operator. If $\phi(b)$ is compact for each $b \in \ell_\infty$, then $\phi : \ell_\infty \rightarrow K_{w^*}(X^*, Y)$ is weakly compact (since $\ell_\infty \hookrightarrow K_{w^*}(X^*, Y)$, [28]). Then $\|\phi(e_n)\| = \|T_n\| \rightarrow 0$. This is a contradiction. Therefore there is a $b_0 \in \ell_\infty$ such that $\phi(b_0)$ is not compact. The series $\sum b_0 n T_n$ and the operator $\phi(b_0)$ satisfy the hypothesis of Theorem 8.

We are now in a position to present a concise and straightforward proof of the main result in [13] and to obtain several corollaries concerning the structure of $K(X, Y)$ and $W(X, Y)$.

**Theorem 9 ([13, Theorem 4]).** Suppose $c_0 \hookrightarrow K_{w^*}(X^*, Y)$. Then either $K_{w^*}(X^*, Y) = L_{w^*}(X^*, Y)$, or $K_{w^*}(X^*, Y)$ is not complemented in $L_{w^*}(X^*, Y)$.

Furthermore, $K_{w^*}(X^*, Y) = L_{w^*}(X^*, Y)$ if and only if only one of the following is true:

(i) $c_0 \hookrightarrow Y$ and $X$ has the Schur property,
(ii) $c_0 \hookrightarrow X$ and $Y$ has the Schur property.

**Proof.** If $c_0 \hookrightarrow K_{w^*}(X^*, Y)$ and $K_{w^*}(X^*, Y) \neq L_{w^*}(X^*, Y)$, then Corollary 6(iv) implies that $K_{w^*}(X^*, Y)$ is not complemented in $L_{w^*}(X^*, Y)$.

Now assume that $K_{w^*}(X^*, Y) = L_{w^*}(X^*, Y)$ and $c_0$ embeds neither in $X$ nor in $Y$. The proof of Theorem 1 shows that if $c_0 \hookrightarrow K_{w^*}(X^*, Y)$, but $c_0 \hookrightarrow X, Y$, then $\ell_\infty \hookrightarrow K_{w^*}(X^*, Y)$. Therefore $\ell_\infty \hookrightarrow X$ or $\ell_\infty \hookrightarrow Y$ [8]. This contradiction shows that either $c_0 \hookrightarrow X$ or $c_0 \hookrightarrow Y$.

If $c_0 \hookrightarrow Y$ and $X$ does not have the Schur property, then $K_{w^*}(X^*, Y)$ is not complemented in $L_{w^*}(X^*, Y)$ by Corollary 6(iii). Hence $X$ has the Schur property and (i) must hold. ■
Corollaries 10–12 make use of the following isometries:

\[ W(X,Y) \simeq L^*_w(X^{**},Y), \quad K(X,Y) \simeq K^*_w(X^{**},Y). \]

**Corollary 10.** Suppose \( Y \) is the second Bourgain–Delbaen space which is an \( L_{\infty} \)-space which has the RNP and \( Y^* \) is isomorphic to \( \ell_1 \). Then \( c_0 \hookrightarrow K(Y,Y) \).

**Proof.** Since \( Y^* \) is a Schur space, it follows that \( K(Y,Y) = W(Y,Y) \) and \( c_0 \hookrightarrow Y^* \). Further, \( c_0 \hookrightarrow Y \) since \( Y \) has the RNP. By Theorem 9, \( c_0 \hookrightarrow K(Y,Y) \). \( \blacksquare \)

**Corollary 11.** Suppose \( T : X \to Y \) is a weakly compact operator which is not compact and \( (T_n) \) is a sequence in \( K(X,Y) \) such that for each \( x \in X \), the series \( \sum T_n(x) \) converges unconditionally to \( T(x) \). Then \( K(X,Y) \) is not complemented in \( W(X,Y) \). Furthermore, \( \ell_\infty \hookrightarrow W(X,Y) \).

**Proof.** Apply Theorem 8. \( \blacksquare \)

**Corollary 12.**

(i) If \( c_0 \hookrightarrow Y \) and \( X^* \) does not have the Schur property, then \( K(X,Y) \) is not complemented in \( W(X,Y) \) and \( \ell_\infty \hookrightarrow W(X,Y) \).

(ii) If \( c_0 \hookrightarrow K(X,Y) \) and \( K(X,Y) \neq W(X,Y) \), then \( K(X,Y) \) is not complemented in \( W(X,Y) \) and \( \ell_\infty \hookrightarrow W(X,Y) \).

**Proof.** (i) Apply Corollary 6(iii) to deduce that \( K(X,Y) \hookrightarrow W(X,Y) \). An application of Corollary 2 concludes the proof.

(ii) Apply Theorem 9 to find that \( K(X,Y) \hookrightarrow W(X,Y) \). An application of Corollary 2 concludes the proof. \( \blacksquare \)

The next theorem, as well as several subsequent corollaries, show that many familiar spaces of operators contain complemented copies of \( c_0 \).

**Theorem 13.** Suppose that \( (x_i) \) is an unconditional and seminormalized shrinking basis for \( X \) and \( (x^*_i) \) is the associated sequence of coefficient functionals. Let \( T \) be an operator in \( L^*_w(X^*,Y) \) such that \( (T(x^*_i)) \) is seminormalized. Then \( c_0 \hookrightarrow K_{w^*}(X^*,Y) \), \( K_{w^*}(X^*,Y) \hookrightarrow L^*_w(X^*,Y) \), and \( \ell_\infty \hookrightarrow L^*_w(X^*,Y) \). Moreover, \( c_0 \hookrightarrow K_{w^*}(X^*,Y) \).

**Proof.** Since \( (x_n) \) is an unconditional shrinking basis for \( X \), \( (x^*_n) \) is an unconditional basis for \( X^* \), and the series \( \sum x^*(x_n)x^*_n \) converges unconditionally to \( x^* \) for all \( x^* \in X^* \) ([32, Thm. 17.7]). Note that \( (T(x^*_i)) \) is \( w \)-null since \( (x^*_i) \) is \( w^* \)-null. Bessaga–Pełczyński’s selection principle allows us to assume that \( (T(x^*_i)) \) is a \( w \)-null basic sequence in \( Y \). If \( T_i : X^* \to Y \), \( T_i(x^*) = x^*(x_i)T(x^*_i) \), then \( T_i \in K_{w^*}(X^*,Y) \) and the series \( \sum T_i(x^*) \) converges unconditionally to \( T(x^*) \) for all \( x^* \in X^* \). Since \( T \) is not compact,
\[ \sum T_n \] is weakly unconditionally convergent and not unconditionally convergent, and thus \( c_0 \hookrightarrow K_w^*(X^*, Y) \). By Theorem 8, \( K_w^*(X^*, Y) \) is not complemented in \( L_{w^*}(X^*, Y) \) and \( \ell_\infty \hookrightarrow L_{w^*}(X^*, Y) \).

Choose \( \varepsilon > 0 \) and intertwining sequences \( (m_k), (n_k) \) of positive integers so that \( \| \sum_{i=m_k}^{n_k} T_i \| > \varepsilon \) for each \( k \). Let \( L_k = \sum_{i=m_k}^{n_k} T_i, k \in \mathbb{N} \). Note that \( \sum L_k(x^*) \) converges unconditionally for each \( x^* \in X^* \) since \( \sum T_i(x^*) \) is unconditionally convergent. Hence \( \sum L_k \) is weakly unconditionally convergent in \( K_w^*(X^*, Y) \). Moreover, \( \inf \| L_k \| > 0 \). By Lemma 3 on p. 160 of [3], \( (L_k) \sim (e_k) \).

Let \( (y_i^*) \) in \( Y^* \) be a biorthogonal sequence of coefficients of \( (T(x_i^*)) \). We may suppose that \( \| y_i^* \| \leq 1 \). If \( L \in K_w^*(X^*, Y) \), then \( \langle x_i^* \otimes y_i^*, L \rangle \leq \| L(x_i^*) \| \to 0 \). Hence \( (x_i^* \otimes y_i^*) \) is \( w^* \)-null in \( (K_w^*(X^*, Y))^* \). For each \( m_k \leq i \leq n_k \), \( \langle x_i^* \otimes y_i^*, L_k \rangle = \langle x_i^* \otimes y_i^*, T_i \rangle = 1 \). Then \( (L_k) \) is not limited. By a result on p. 36 of Schlumprecht [30], \( c_0 \hookrightarrow K_w^*(X^*, Y) \).

**Theorem 14.** Let \( X \) and \( Y \) be infinite-dimensional Banach spaces satisfying the following assumption: if \( T \) is an operator in \( L_{w^*}(X^*, Y) \), then there is a sequence of operators \( (T_n) \) in \( K_w^*(X^*, Y) \) such that for each \( x^* \in X^* \), the series \( \sum T_n(x^*) \) converges unconditionally to \( T(x^*) \). Then the following are equivalent:

(i) \( K_w^*(X^*, Y) \neq L_{w^*}(X^*, Y) \).
(ii) \( X \) and \( Y \) do not have the Schur property and \( c_0 \hookrightarrow K_w^*(X^*, Y) \).
(iii) \( X \) and \( Y \) do not have the Schur property and \( \ell_\infty \hookrightarrow L_{w^*}(X^*, Y) \).
(iv) \( K_w^*(X^*, Y) \) is not complemented in \( L_{w^*}(X^*, Y) \).

**Proof.** (i)\( \Rightarrow \) (ii). Let \( T \in L_{w^*}(X^*, Y) \) be noncompact. Then \( X \) and \( Y \) do not have the Schur property. Let \( (T_n) \) be a sequence as in the hypothesis. By the remark after Theorem 8, \( c_0 \hookrightarrow K_w^*(X^*, Y) \).

(ii)\( \Rightarrow \) (iii) by Corollary 3 (or Corollary 2).

(iii)\( \Rightarrow \) (i). If \( K_w^*(X^*, Y) = L_{w^*}(X^*, Y) \), then \( \ell_\infty \hookrightarrow K_w^*(X^*, Y) \). By Drewnowski’s result [8], \( \ell_\infty \hookrightarrow X \) or \( \ell_\infty \hookrightarrow Y \). By Corollary 6(i), \( K_w^*(X^*, Y) \hookrightarrow L_{w^*}(X^*, Y) \), a contradiction.

(iv)\( \Rightarrow \) (i) is trivial, and (ii)\( \Rightarrow \) (iv) by Corollary 6(iv).

A separable Banach space \( X \) has an unconditional finite-dimensional expansion of the identity (u.f.d.e.i.) if there is a sequence \( (A_n) \) of finite rank operators from \( X \) to \( X \) such that \( \sum A_n(x) \) converges unconditionally to \( x \) for all \( x \in X \). In this case, \( (A_n) \) is called an u.f.d.e.i. of \( X \) [18].

**Corollary 15.** If either \( Y \) or \( X \) has an u.f.d.e.i., then the following are equivalent:

(i) \( K_w^*(X^*, Y) \neq L_{w^*}(X^*, Y) \).
(ii) \( X \) and \( Y \) do not have the Schur property and \( c_0 \hookrightarrow K_w^*(X^*, Y) \).
(iii) $X$ and $Y$ do not have the Schur property and $\ell_\infty \hookrightarrow L_{w^*}(X^*, Y)$.
(iv) $K_{w^*}(X^*, Y)$ is not complemented in $L_{w^*}(X^*, Y)$.
(v) $X$ and $Y$ do not have the Schur property and $c_0 \hookrightarrow c c_{w^*}(X^*, Y)$.

Proof. Suppose $Y$ has an u.f.d.e.i. $(A_n)$. Then $A_n : Y \to Y$ is compact for each $n$ and $y = \sum A_n(y)$ unconditionally for each $y \in Y$. Let $T \in L_{w^*}(X^*, Y)$. Hence $T(x^*) = \sum A_n T(x^*)$ unconditionally for each $x^* \in X^*$ and $A_n T \in K_{w^*}(X^*, Y)$. Apply Theorem 14 to find that the first four statements are equivalent.

Now, if $Y$ has an u.f.d.e.i. then $Y$ must be separable, hence it has the Gelfand–Phillips property [4]. By Theorem 18 in [13], if $c_0 \hookrightarrow K_{w^*}(X^*, Y)$, then $c_0 \hookrightarrow c c_{w^*}(X^*, Y)$. Hence (ii) $\Rightarrow$ (v). (v) $\Rightarrow$ (ii) is trivial.

Assume that $X$ has an u.f.d.e.i. $(A_n)$. Then $A_n : X \to X$ is compact and $x = \sum A_n(x)$ unconditionally for each $x \in X$. Let $T \in L_{w^*}(X^*, Y)$. Then $T^*(y^*) = \sum A_n T^*(y^*)$ unconditionally for each $y^* \in Y^*$ and $T_n = A_n T^* \in K_{w^*}(Y^*, X)$. Now apply Theorem 14 and use the isometry $K_{w^*}(X^*, Y) \simeq K_{w^*}(Y^*, X)$. ■

**Corollary 16** ([13, Corollary 9]). Let $X$ and $Y$ be infinite-dimensional Banach spaces such that $X^*$ or $Y$ has an u.f.d.e.i. Then the following are equivalent:

1. $K(X, Y) \neq W(X, Y)$.
2. $X^*$ and $Y$ do not have the Schur property and $c_0 \hookrightarrow K(X, Y)$.
3. $X^*$ and $Y$ do not have the Schur property and $\ell_\infty \hookrightarrow W(X, Y)$.
4. $K(X, Y)$ is not complemented in $W(X, Y)$.
5. $X^*$ and $Y$ do not have the Schur property and $c_0 \hookrightarrow c K(X, Y)$.

Proof. Apply the isometries at the beginning of this section and Corollary 15. ■

**Corollary 17.** Suppose that $X^*$ has an u.f.d.e.i. $(A_n)$ consisting of $w^*$-$w$ operators. Then the conclusion of Corollary 15 is true.

Proof. Let $(A_n)$ be an u.f.d.e.i. for $X^*$ consisting of $w^*$-$w$ operators. Let $T \in L_{w^*}(X^*, Y)$ and $T_n = TA_n$. Then $x^* = \sum A_n(x^*)$ unconditionally for each $x^* \in X^*$, $T^*(y^*) \subseteq X$, $A_n(X^{**}) \subseteq X$, and $T_n$ is compact for each $n$.

We will show that $T_n$ is $w^*$-$w$ continuous. Let $(x^*_\alpha)$ be a $w^*$-null net in $B_{X^*}$ and $y^* \in Y^*$. For each $n \in \mathbb{N}$,

$$\langle y^*, T_n(x^*_\alpha) \rangle = \langle A_n T^*(y^*), x^*_\alpha \rangle \to 0.$$ 

Then $T_n \in L_{w^*}(X^*, Y)$, and thus $T_n \in K_{w^*}(X^*, Y)$. Since the series $\sum T_n(x^*)$ converges unconditionally to $T(x^*)$ for each $x^* \in X^*$, an application of Theorem 14, Theorem 18 in [13], and the isometry $K_{w^*}(Y^*, X) \simeq K_{w^*}(X^*, Y)$ concludes the proof. ■
The following result is motivated by Theorem 1 in [14].

A sequence \((X_n)\) of closed subspaces of a Banach space \(X\) is called an **unconditional Schauder decomposition** of \(X\) if every \(x \in X\) has a unique representation of the form \(x = \sum x_n\) with \(x_n \in X_n\) for every \(n\), and the series converges unconditionally [26].

**Corollary 18.** Let \(X\) and \(Y\) be infinite-dimensional Banach spaces satisfying the following assumptions:

(a) \(Y\) is complemented in a Banach space \(Z\) which has an unconditional Schauder decomposition \((Z_n)\).
(b) \(L(X^*, Z_n) = K(X^*, Z_n)\) for each \(n\). Then the conclusion of Theorem 14 is true.

**Proof.** Let \(T \in L_{w^*}(X^*, Y)\), \(A_n : Z \to Z_n\), \(A_n(\sum z_i) = z_n\), and \(P\) the projection of \(Z\) onto \(Y\). Define \(T_n : X^* \to Y\) by \(T_n(x^*) = PA_nT(x^*)\), \(x^* \in X^*, n \in \mathbb{N}\). Note that \(T_n\) is compact since \(L(X^*, Z_n) = K(X^*, Z_n)\), and \(T_n\) is \(w^*\)-\(w\) continuous for each \(n\). Since for each \(z \in Z\), \(z = \sum A_n(z)\) and the convergence is unconditional, \(\sum T_n(x^*)\) converges unconditionally to \(T(x^*)\) for each \(x^* \in X^*\). An application of Theorem 14 gives the conclusion. \(\blacksquare\)

The hypothesis (b) of the previous theorem is satisfied, for instance, in the following cases:

1. \(X\) is arbitrary and each \(Z_n\) is finite-dimensional;
2. \(\ell_1 \preccurlyeq X^*\) and each \(Z_n\) has the Schur property;
3. \(X = \ell_1\) and each \(Z_n\) has the Schur property;
4. \(X^{**}\) has the Schur property and each \(Z_n\) has \((RDP)^*\).

**Corollary 19.** If \(\ell_1 \preccurlyeq X^*, Y\) is complemented in a Banach space \(Z\) which has an unconditional Schauder decomposition \((Z_n)\), and each \(Z_n\) has the Schur property, then the conclusion of Corollary 15 is true.

**Proof.** Since \(Z\) has an unconditional Schauder decomposition \((Z_n)\) and each \(Z_n\) has the Schur property, \(Z\), hence \(Y\), has the Gelfand–Phillips property [9]. Apply Corollary 18 and Theorem 18 in [13] to get the conclusion. \(\blacksquare\)

The following theorem continues a theme of Theorem 13 and gives sufficient conditions for \(K_{w^*}(X^*, Y)\) to contain isomorphic (complemented) copies of \(c_0\).

**Theorem 20.** Let \(X\) and \(Y\) be Banach spaces satisfying the following assumption: there exists a Banach space \(G\) with an unconditional basis \((g_n)\) and biorthogonal coefficients \((g^*_n)\) and two operators \(R : G \to Y\) and \(S : G^* \to X\) such that \((R(g_i))\) and \((S(g^*_i))\) are seminormalized sequences and either \((R(g_i))\) or \((S(g^*_i))\) is a basic sequence. Then \(c_0\) embeds in \(K_{w^*}(X^*, Y)\) (indeed, in any subspace \(H\) of \(L(X^*, Y)\) which contains \(X \otimes \chi Y\)).
Moreover, if \((R(g_i))\) and \((S(g_i^*))\) are basic and \(Y\) (or \(X\)) has the Gelfand–Phillips property, then \(K_{w^*}(X^*, Y)\) contains a complemented copy of \(c_0\).

**Proof.** Suppose that \(p \leq \|R(g_i)\| \leq q\) and \(p \leq \|S(g_i^*)\| \leq q\) for all \(i\). Let \(S(g_i^*) \otimes R(g_i) \in K_{w^*}(X^*, Y)\), \(S(g_i^*) \otimes R(g_i), x^* = x^*(S(g_i^*))R(g_i), x^* \in X^*\).

Assume without loss of generality that \((R(g_i))\) is a basic sequence. Choose \(C_1 > 0\) so that for all real numbers \((b_i)\) and all positive integers \(m \leq n\),

\[
\left\| \sum_{i=1}^{m} b_i R(g_i) \right\| \leq C_1 \left\| \sum_{i=1}^{n} b_i R(g_i) \right\|.
\]

Then \(\|b_i R(g_i)\| \leq 2C_1 \left\| \sum_{j=1}^{n} b_i R(g_i) \right\|\) for each \(1 \leq i \leq n\).

We have, for any sequence \((a_n)\) of real numbers,

\[
\left\| \sum_{i=1}^{n} a_i [S(g_i^*) \otimes R(g_i)] \right\|_{\lambda} = \sup \left\{ \left\| \sum_{i=1}^{n} a_i x^*(S(g_i^*))R(g_i) \right\| : x^* \in B_{X^*} \right\}
\]

\[
\geq \sup \left\{ \frac{1}{2C_1} \left\| a_i x^*(S(g_i^*))R(g_i) \right\| : x^* \in B_{X^*} \right\}
\]

\[
\geq \frac{1}{2C_1} p |a_i| \|S(g_i^*)\| \geq \frac{1}{2C_1} p^2 |a_i|
\]

for each \(1 \leq i \leq n\). Hence

\[
\left\| \sum_{i=1}^{n} a_i [S(g_i^*) \otimes R(g_i)] \right\|_{\lambda} \geq \frac{1}{2C_1} p^2 (\max_{i=1}^{n} |a_i|).
\]

On the other hand, \(S\) and \(R\) induce an operator \(S \otimes_\lambda R : G^* \otimes_\lambda G \to X \otimes_\lambda Y\), which maps \((g_n^* \otimes g_n)\) into \((S(g_n^*) \otimes R(g_n))\) ([7, Chapter VIII]). So we have

\[
\left\| \sum_{i=1}^{n} a_i [S(g_i^*) \otimes R(g_i)] \right\|_{\lambda} \leq \|S \otimes_\lambda R\| \left\| \sum_{i=1}^{n} a_i (g_i^* \otimes g_i) \right\|_{\lambda}.
\]

Let \(\varepsilon(\{g_n^*(g)g_n\}) = \sup \{ \sum |g^*(g_n^*(g)g_n)| : g_n^* \in B_{G^*} \}\) for \(g \in G\) and let \(M\) be the uncondition basis constant of the uncondition basis \((g_n)\).

If \(g \in G\) and \(g^* \in B_{G^*}\), then \(g = \sum g_i^*(g)g_i\) unconditionally, \(\sum |g^*(g_i^*(g)g_i)| \leq 2M \|g\|\), and \(\varepsilon(\{g_i^*(g)g_i\}) : g \in B_G\) \(\leq 2M\). Consequently,

\[
\left\| \sum_{i=1}^{n} a_i (g_i^* \otimes g_i) \right\|_{\lambda} \leq \sup \left\{ \sum_{i=1}^{n} |a_i g_i^*(g)g^*(g_i)| : g \in B_G, g^* \in B_{G^*} \right\}
\]

\[
\leq 2M (\max_{i=1}^{n} |a_i|),
\]
and therefore
\[ \left\| \sum_{i=1}^{n} a_i [S(g_i^*) \otimes R(g_i)] \right\|_{\lambda} \leq 2M \| S \otimes \lambda R \| \max_{i=1}^{n} |a_i| . \]

Hence \((S(g_n^*) \otimes R(g_n)) \sim (e_n)\) and thus \(c_0 \hookrightarrow K_{w^*}(X^*, Y)\).

To prove the last part of the theorem, suppose that \(Y\) has the Gelfand–Phillips property and both \((R(g_n))\) and \((S(g_n^*))\) are basic. If \((R(g_n))\) is limited, then \(R(g_n) \to 0\) since \((R(g_n))\) is relatively compact and the only weak limit of a basic sequence is zero [5, p. 42]. Therefore \((R(g_n))\) is not limited. By a result of Schlumprecht [30], we can choose a \(w^*\)-null sequence \((y_n^*)\) in \(Y^*\) such that \(\langle y_n^*, R(g_n) \rangle = \delta_{nm}\). Let \((x_n^*)\) be a bounded sequence in \(X^*\) such that \(\langle x_n^*, S(g_m^*) \rangle = \delta_{nm}\). We may assume that \(\|x_n^*\| \leq 1\). Then \((x_n^* \otimes y_n^*)\) is a \(w^*\)-null sequence in \((K_{w^*}(X^*, Y))^*\) since for each \(T \in K_{w^*}(X^*, Y)\),
\[ \langle x_n^* \otimes y_n^*, T \rangle = \langle T(x_n^*), y_n^* \rangle \leq \|T^*(y_n^*)\| \to 0 . \]

Also, \(\langle x_n^* \otimes y_n^*, S(g_m^*) \otimes R(g_m) \rangle = \delta_{nm}\), thus \((S(g_m^*) \otimes R(g_m))\) is not limited. By Theorem 1.3.2 in [30], \(c_0 \hookrightarrow K_{w^*}(X^*, Y)\).

Remark. From Theorem 20 and the first example at the end of the paper it follows that \(c_0\) embeds in \(K_{w^*}(X^*, Y)\) when \(\ell_2\) embeds in both \(X\) and \(Y\). In fact, \(c_0 \hookrightarrow K_{w^*}(X^*, Y)\).

Corollary 21 ([11, Theorem 3]). Let \(X\) and \(Y\) be Banach spaces satisfying the following assumption: there exists a Banach space \(G\) with an unconditional basis \((g_n)\) and biorthogonal coefficients \((g_n^*)\) and two operators \(R : G \to Y\) and \(S : G^* \to X^*\) such that \((R(g_i))\) and \((S(g_i^*))\) are normalized basic sequences. Then \(c_0 \hookrightarrow K(X, Y)\).

Moreover, if \(Y\) (or \(X^*\)) has the Gelfand–Phillips property, then \(K(X, Y)\) contains a complemented copy of \(c_0\).

Proof. Apply Theorem 20 and the isometry \(K_{w^*}(X^{**}, Y) \simeq K(X, Y)\).

Recall that a basis \((x_n)\) for \(X\) is said to be perfectly homogeneous if it is semi-normalized and every semi-normalized block basic sequence with respect to \((x_n)\) is equivalent to \((x_n)\) [32]. A perfectly homogeneous basis is unconditional. The unit vector bases of \(c_0\) and \(\ell_p\), \(1 \leq p < \infty\), are, up to equivalence, the only perfectly homogeneous bases (Zippin) [32, p. 609].

Theorem 22. Suppose that \((x_n^*)\) is a perfectly homogeneous basic sequence in \(X^*\), \([x_n^*]^* \hookrightarrow X\) and \(T : [x_n^*] \to Y\) is a non-completely continuous operator. Then \(c_0 \hookrightarrow K_{w^*}(X^*, Y)\), \(\ell_\infty \hookrightarrow L_{w^*}(X^*, Y)\), and \(K_{w^*}(X^*, Y)\) is not complemented in \(L_{w^*}(X^*, Y)\).

Proof. Suppose that \((x_n^*)\) is a perfectly homogeneous basic sequence, \([x_n^*]^* \hookrightarrow X\), and \(T : [x_n^*] \to Y\) is an operator which is not completely continuous. Let \(U = [x_n]\), and let \((u_n^*)\) be a weakly null sequence in \(U\) so that
\( (T(u_n^*)) \to 0 \). Without loss of generality, suppose that \( \varepsilon > 0 \) and \( \|T(u_n^*)\| > \varepsilon \) for each \( n \). Apply the Bessaga–Pełczyński selection principle [5] and let \( (v_n^*) \) be a subsequence of \( (u_n^*) \) so that \( (v_n^*) \) is equivalent to a block basic sequence of \( (x_n^*) \). In fact, an inspection of the Bessaga–Pełczyński theorem shows that we may assume that \( (v_n^*) \) is seminormalized. Therefore \( (v_n^*) \sim (x_n^*) \). Since \( (x_n^*) \) is unconditional, \( c_0 \hookrightarrow K_{w^*}(X^*, Y) \) by Theorem 20. Apply Corollary 2 and Corollary 6(iv) to conclude the argument.

**Corollary 23.**

(a) Assume that \( \ell_2 \hookrightarrow X \) and there is an operator \( T : \ell_2 \to Y \) such that the sequence \( (T(e_n^2)) \) is seminormalized. Then the four statements in the conclusion of Theorem 14 hold.

(b) Assume that \( \ell_2 \hookrightarrow Y \) and there is an operator \( T : \ell_2 \to X \) such that the sequence \( (T(e_n^2)) \) is seminormalized. Then the four statements in the conclusion of Theorem 14 hold.

**Proof.** We prove (a); the case (b) is similar. An application of Theorem 22 (or 20) gives \( c_0 \hookrightarrow K_{w^*}(X^*, Y) \). We note that \( X \) and \( Y \) are not Schur spaces by hypothesis. Thus, (ii) holds. The proof of Theorem 14 shows that (ii) \( \Rightarrow \) (i), (ii) \( \Rightarrow \) (iii), and (ii) \( \Rightarrow \) (iv).

**Remark.** A similar proof shows that if \( \ell_2 \hookrightarrow X \) and \( \ell_p \hookrightarrow Y \) for some \( p \geq 2 \), then the four statements in the conclusion of Theorem 14 hold.

**Corollary 24.** Assume that \( \ell_2 \hookrightarrow X^* \) and there is an operator \( T : \ell_2 \to Y \) such that the sequence \( (T(e_n^2)) \) is seminormalized. Then the first four statements in the conclusion of Corollary 16 are true.

**Proof.** Apply Corollary 23.

In [17] Feder proved that \( K(C(S), L^1) \) is not complemented in \( L(C(S), L^1) \) when \( S \) is not dispersed. See also [12]. The following corollary improves Feder’s result.

**Corollary 25.** Assume that \( S \) is a Hausdorff compact space which is not dispersed. Then \( K(C(S), L^1) \) is not complemented in \( W(C(S), L^1) \).

**Proof.** Since \( S \) is not dispersed, \( \ell_1 \hookrightarrow C(S) \) [24]. Then \( L^1 \hookrightarrow C(S)^* \) [27]. Also, the Rademacher functions span \( \ell_2 \) inside of \( L^1 \), and thus \( \ell_2 \hookrightarrow C(S)^* \). Corollary 21 implies that \( c_0 \hookrightarrow K(C(S), L^1) \). By Corollary 24, \( K(C(S), L^1) \) is not complemented in \( W(C(S), L^1) \).

See the last section of this paper for a generalization of Corollary 25.

**Corollary 26 ([13, Corollary 12]).** Assume that \( X \) has the DPP and there is an operator \( T : \ell_2 \to Y \) such that the sequence \( (T(e_n^2)) \) is seminormalized. Then the first four statements in the conclusion of Corollary 16 are equivalent.
Thus the hypotheses of Theorem 13. (Theorems 14 and 20 also apply to this set-
contains an infinite-dimensional subspace \(c_0\). Further, Theorem 13, as well as the Emmanuele result just cited, show that \(c_0 \hookrightarrow K(X,Y)\). The rest follows from Corollary 24.

In [22] the authors proved that if \(X\) and \(Y\) are weakly sequentially complete and \(K_w^*(X^*,Y) = L_w^*(X^*,Y)\), then \(K_w^*(X^*,Y)\) is weakly sequentially complete. Now we give a partial converse.

**Corollary 27.** If \(Y\) (or \(X\)) has an u.f.d.e.i. and \(K_w^*(X^*,Y)\) is weakly sequentially complete, then \(K_w^*(X^*,Y) = L_w^*(X^*,Y)\).

**Proof.** By Corollary 15, if \(K_w^*(X^*,Y) \neq L_w^*(X^*,Y)\), then \(c_0 \hookrightarrow K_w^*(X^*,Y)\), a contradiction.

**Closing remarks.** Emmanuele made the following two observations on p. 334 of [11]:

(a) If \(\ell_1 \hookrightarrow X\) and \(\ell_p \hookrightarrow Y\) for some \(p \geq 2\), then \(c_0 \hookrightarrow K(X,Y)\) and \(K(X,Y) \overset{c}{\hookrightarrow} L(X,Y)\).

(b) If \(1/p + 1/p' = 1\) and \(1 < p' \leq q < \infty\), then \(c_0 \overset{c}{\hookrightarrow} \ell_p \otimes_{\varepsilon} \ell_q\).

In case (a) we can actually show that \(K(X,Y) \overset{c}{\hookrightarrow} W(X,Y)\). Suppose that \(\ell_1 \hookrightarrow X\) and \(\ell_p \hookrightarrow Y\), \(p \geq 2\). Then \(L_1 \hookrightarrow X^*\), and thus \(\ell_2 \hookrightarrow X^*\). By Theorem 20, \(c_0 \hookrightarrow K_w^*(X^{**},Y)\). By Corollary 6, \(K_w^*(X^{**},Y)\) is not complemented in \(L_w^*(X^{**},Y)\). Now use the natural isometries at the beginning of the previous section to conclude that \(K(X,Y)\) is not complemented in \(W(X,Y)\).

Since \(K(\ell_p,\ell_q) = K_w^*(\ell_p,\ell_q) \neq L(\ell_p,\ell_q) = L_w^*(\ell_p,\ell_q)\), Theorem 13 allows us to see that \(c_0 \overset{c}{\hookrightarrow} K(\ell_p,\ell_q)\), \(\ell_\infty \hookrightarrow L(\ell_p,\ell_q)\), and \(K(\ell_p,\ell_q) \overset{c}{\hookrightarrow} L(\ell_p,\ell_q)\) whenever \(1 < p \leq q < \infty\).

Since \(X \hookrightarrow K_w^*(X^*,Y)\), obviously \(c_0 \hookrightarrow K_w^*(\ell_1,Y)\) for every Banach space \(Y\). By Theorem 18 in Emmanuele [13], \(c_0 \overset{c}{\hookrightarrow} K_w^*(\ell_1,Y)\) whenever \(Y\) has the Gelfand–Phillips property. Thus \(c_0\) is complemented in \(K_w^*(\ell_1,\ell_1)\). Further, Theorem 13, as well as the Emmanuele result just cited, show \(c_0 \overset{c}{\hookrightarrow} K_w^*(\ell_1,\ell_1)\), \(1 < p < \infty\). In fact, we can conclude more. Suppose that \(Z\) contains an infinite-dimensional subspace \(Y\) which has a shrinking and semi-normalized basis \((y_n)\). Let \((y_n^*)\) be the associated sequence of coefficient functionals. Define \(L : \ell_1 \rightarrow Y\) by \(L(\lambda) = \sum_{i=1}^{\infty} \lambda_i y_i\). Then \(L^*(y_k^*) = e_k \in c_0\) for each \(k\). Since \((y_n)\) is shrinking, \(L\) is a \(w^*-w\) continuous operator and satisfies the hypotheses of Theorem 13. (Theorems 14 and 20 also apply to this setting.) In fact, if one defines \(\widehat{L} : \ell_1 \rightarrow Z\) by \(\widehat{L}(\lambda) = L(\lambda)\), then \(\widehat{L} \in L_w^*(\ell_1,Z)\). Thus \(c_0 \overset{c}{\hookrightarrow} K_w^*(\ell_1,Z)\), \(\ell_\infty \hookrightarrow L_w^*(\ell_1,Z)\), and \(K_w^*(\ell_1,Z) \overset{c}{\hookrightarrow} L_w^*(\ell_1,Z)\).
We note that $K_{w^*}(\ell_1, Z)$ may also contain copies of $c_0$ which fail to be complemented in this space of operators as well as copies of $c_0$ which are complemented. For example, $\ell_\infty$ contains all spaces with shrinking bases, and thus $c_0 \hookrightarrow K_{w^*}(\ell_1, \ell_\infty)$. However, $\ell_\infty$ naturally (and isometrically) embeds in $K_{w^*}(\ell_1, \ell_\infty)$, and thus the canonical copy of $c_0$ contained in $\ell_\infty$ cannot be complemented in this space of operators.

Similar arguments show that if $1 < p \leq q < \infty$ and $\ell_q \hookrightarrow Z$, then $c_0 \hookrightarrow K_{w^*}(\ell_p, Z) = K(\ell_p, Z)$, $\ell_\infty \hookrightarrow L_{w^*}(\ell_p, Z) = L(\ell_p, Z)$ and $K(\ell_p, Z) \hookrightarrow L(\ell_p, Z)$. Note also that $K(\ell_p, \ell_\infty)$ contains both complemented and uncomplemented copies of $c_0$.

**Examples.** The first example shows that there are Banach spaces $X$ and $Y$ such that $c_0 \hookrightarrow X, Y$, $c_0 \hookrightarrow K_{w^*}(X^*, Y)$, but $K_{w^*}(X^*, Y) \neq L_{w^*}(X^*, Y)$. Clearly $c_0$ does not embed in $\ell_2$. A direct application of Theorem 20 shows that $c_0 \hookrightarrow K_{w^*}(\ell_2, \ell_2)$ and the identity operator from $\ell_2$ to $\ell_2$ shows that $K_{w^*}(\ell_2, \ell_2) \neq L_{w^*}(\ell_2, \ell_2)$.

The next example [15] shows that we can find Banach spaces $X$ and $Y$ such that $c_0 \hookrightarrow K_{w^*}(X^*, Y)$, but $K_{w^*}(X^*, Y) = L_{w^*}(X^*, Y)$. Let $E = F$ be the Bourgain–Delbaen space which is an $L_\infty$ space with RNP and such that $E^*$ is a Schur space even though $c_0 \hookrightarrow E$. Assume that $c_0 \hookrightarrow K_{w^*}(E^{**}, E)$ and let $(T_n)$ be a copy of $c_0$ in $K_{w^*}(E^{**}, E)$. Define $\phi : \ell_\infty \to L(E^{**}, E)$ by $\phi(b)(x^{**}) = \sum b_n T_n(x^{**})$. Since $c_0 \hookrightarrow E^*$, the series $\sum b_n T_n(y^*)$ converges unconditionally for each $y^* \in E^*$, hence $\phi(b) \in L_{w^*}(E^{**}, E)$. Note that $\|\phi(e_n)\| = \|T_n\| \to 0$. A result of Rosenthal [28] implies that $\ell_\infty \hookrightarrow L_{w^*}(E^{**}, E)$. On the other hand, $K_{w^*}(E^{**}, E) = L_{w^*}(E^{**}, E)$ since $E^*$ is a Schur space. By Drewnowski’s result, $\ell_\infty \hookrightarrow E$ or $\ell_\infty \hookrightarrow E^*$, a contradiction. Hence $c_0 \hookrightarrow K_{w^*}(E^{**}, E)$. Thus the spaces $X = E^*$ and $Y = E$ are as desired.

Alternatively, for $1 \leq q < p$, $L(\ell_p, \ell_q) = K(\ell_p, \ell_q)$ (Pitt). Kalton showed that for $1 \leq q < p$, $L(\ell_p, \ell_q)$ is reflexive [23]. Thus $c_0 \hookrightarrow K(\ell_p, \ell_q) \simeq K_{w^*}(\ell_p^*, \ell_q)$, and the spaces $X = \ell_p^*$ and $Y = \ell_q$ are as desired.

We conclude the paper by asking the following question.

**Question.** Are there Banach spaces $X, Y$ such that $K_{w^*}(X^*, Y) \neq L_{w^*}(X^*, Y)$ and $c_0 \hookrightarrow K_{w^*}(X^*, Y)$?

**References**


Embeddability of $c_0$ in Spaces of Operators 255


Ioana Ghenciu
Department of Mathematics
University of Wisconsin–River Falls
River Falls, WI 54022-5001, U.S.A.
E-mail: ioana.ghenciu@uwrf.edu

Paul Lewis
Department of Mathematics
University of North Texas
Box 311430
Denton, TX 76203-1430, U.S.A.
E-mail: lewis@unt.edu

Received November 14, 2008