Transitive Properties of Ideals on Generalized Cantor Spaces

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Summary. We compute transitive cardinal coefficients of ideals on generalized Cantor spaces. In particular, we show that there exists a null set \( A \subseteq 2^{\omega_1} \) such that for every null set \( B \subseteq 2^{\omega_1} \) we can find \( x \in 2^{\omega_1} \) such that \( A \cup (A + x) \) cannot be covered by any translation of \( B \).

1. Introduction, definitions and basic properties. In 2001 Kraszewski [5] defined a class of productive \( \sigma \)-ideals of subsets of the Cantor space \( 2^\omega \) and observed that both \( \sigma \)-ideals of meagre sets and of null sets are in this class. Next, from every productive \( \sigma \)-ideal \( J \) one can produce a \( \sigma \)-ideal \( J_\kappa \) of subsets of the generalized Cantor space \( 2^\kappa \). In particular, starting from the meagre sets and null sets in \( 2^\omega \) we obtain the meagre sets and null sets in \( 2^\kappa \), respectively. This description gives us a powerful tool for investigating combinatorial properties of ideals on \( 2^\kappa \), which was done in [5]. In this paper we continue this research, focusing on transitive cardinal coefficients of ideals of subsets of \( 2^\kappa \).

We use standard set-theoretical notation and terminology from [2]. Let \((G, +)\) be an infinite abelian group. We consider a \( \sigma \)-ideal \( J \) of subsets of \( G \) which is proper, contains all singletons and is invariant (under group operations).

For an ideal \( J \) we consider the following transitive cardinal numbers:

\[
\text{add}_t(J) = \min \{|A| : A \subseteq J \land \neg(\exists B \in J)(\forall A \in A)(\exists g \in G) A \subseteq B + g\},
\]

\[
\text{add}_t^*(J) = \min \{|T| : T \subseteq G \land (\exists A \in J) A + T \notin J\},
\]

2000 Mathematics Subject Classification: 03E05, 03E17.

Key words and phrases: generalized Cantor spaces, transitive cardinal coefficients.

[115]
\[ \text{cov}_t(\mathcal{J}) = \min \{|T| : T \subseteq G \land (\exists A \in \mathcal{J}) A + T = G \}, \]
\[ \text{cof}_t(\mathcal{J}) = \min \{|\mathcal{B}| : \mathcal{B} \subseteq \mathcal{J} \land (\forall A \in \mathcal{J})(\exists B \in \mathcal{B})(\exists g \in G) A \subseteq B + g \}. \]

The first two are both called the \textit{transitive additivity}. The remaining two are called the \textit{transitive covering number} and \textit{transitive cofinality}, respectively.

We say that an ideal \( \mathcal{J} \) is \( \kappa \)-\textit{translatable} if
\[ (\forall A \in \mathcal{J})(\exists B_A \in \mathcal{J})(\forall S \in [G]^\kappa)(\exists t_S \in G) A + S \subseteq B_A + t_S. \]

We define the \textit{translatability number} of \( \mathcal{J} \) as follows:
\[ \tau(\mathcal{J}) = \min \{\kappa : \mathcal{J} \text{ is not } \kappa\text{-translatable} \}. \]

For more information about relations between classical and transitive cardinal coefficients of ideals, see [2]. For more about translatability, see [1], [3] and [6].

From now on we deal with the generalized Cantor space \( 2^\kappa \) interpreted as the set of all functions from an infinite cardinal number \( \kappa \) into \{0, 1\}. This space is endowed with the standard product topology. Moreover, we consider the standard product measure and product group structure on \( 2^\kappa \).

We introduce some extra notation in order to simplify further considerations. Let \( \kappa \) be an infinite cardinal number. We put \( \text{Inj}(\omega, \kappa) = \{\varphi \in \kappa^\omega : \varphi \text{ is an injection}\} \). For \( A \subseteq 2^\kappa, B \subseteq 2^n \) and \( \varphi \in \text{Inj}(\omega, \kappa) \) we put
\[ \varphi \ast A = \{ x \circ \varphi : x \in A \}, \quad B_\varphi = \{ x \in 2^\kappa : x \circ \varphi \in B \}. \]

Obviously, \( \varphi \ast A \subseteq 2^\omega \) and \( B_\varphi \subseteq 2^\kappa \). Another simple observation is that for \( A \subseteq 2^\kappa, B \subseteq 2^\omega \) and \( \varphi \in \text{Inj}(\omega, \kappa) \) we have \( A \subseteq (\varphi \ast A)_\varphi \) and \( \varphi \ast B_\varphi = B \).

Let \( \mathcal{J} \) be a \( \sigma \)-ideal of subsets of \( 2^\omega \). We say that \( \mathcal{J} \) is \textit{productive} if
\[ (\forall A \subseteq 2^\omega)(\forall \varphi \in \text{Inj}(\omega, \omega))(\varphi \ast A \in \mathcal{J} \Rightarrow A \in \mathcal{J}). \]

It is easy to show that \( \mathcal{J} \) is productive if and only if for every \( A \subseteq 2^\omega \) and \( \varphi \in \text{Inj}(\omega, \omega) \) if \( A \in \mathcal{J} \) then \( A_\varphi \in \mathcal{J} \).

Directly from the definitions we deduce that the \( \sigma \)-ideals of meagre subsets and of null subsets of \( 2^\omega \) are productive. Also the \( \sigma \)-ideal generated by closed null subsets of \( 2^\omega \) is productive. Moreover, the ideal \( S_2 \) investigated in [4] is the least non-trivial productive \( \sigma \)-ideal of subsets of the Cantor space.

For any productive \( \sigma \)-ideal \( \mathcal{J} \) we define
\[ \mathcal{J}_\kappa = \{ A \subseteq 2^\kappa : (\exists \varphi \in \text{Inj}(\omega, \kappa)) \varphi \ast A \in \mathcal{J} \}. \]

A standard consideration shows that \( \mathcal{J}_\kappa \) is a \( \sigma \)-ideal of subsets of \( 2^\kappa \). If \( \mathcal{J} \) is invariant then so is \( \mathcal{J}_\kappa \). If \( A \in \mathcal{J}_\kappa \) then any \( \varphi \in \text{Inj}(\omega, \kappa) \) such that \( \varphi \ast A \in \mathcal{J} \) is called a \textit{witness} for \( A \).

Let us also recall one useful definition from [5]. We say that an ideal \( \mathcal{J} \) of subsets of \( 2^\omega \) has \textit{WFP (Weak Fubini Property)} if for every \( \varphi \in \text{Inj}(\omega, \omega) \) and every \( A \subseteq 2^\omega \) if \( A_\varphi \in \mathcal{J} \) then so is \( A \).
The $\sigma$-ideals of subsets of $2^\omega$ mentioned above obviously have WFP. We will need the following technical lemma proved in [5].

**Lemma 1.1.** If $\mathcal{J}$ is a productive ideal of subsets of $2^\omega$ having WFP then for every $\varphi \in \text{Inj}(\omega, \kappa)$ and every $A \subseteq 2^\omega$ if $A_\varphi \in \mathcal{J}_\kappa$ then $A \in \mathcal{J}$. ■

2. **Transitive cardinal coefficients of ideals on** $2^\kappa$. From now on we assume that $\mathcal{J}$ is a proper, invariant and productive $\sigma$-ideal of subsets of $2^\omega$ containing all singletons and that $\kappa \geq \omega_1$. We investigate relations between transitive cardinal coefficients of $\mathcal{J}$ and those of $\mathcal{J}_\kappa$. Some of them are similar to relations between standard cardinal coefficients of $\mathcal{J}$ and $\mathcal{J}_\kappa$ proved in [5]. We omit the proofs, as they are also analogous.

**Theorem 2.1.** $\text{add}_t(\mathcal{J}_\kappa) = \omega_1$. ■

**Theorem 2.2.** $\text{cof}_t(\mathcal{J}_\kappa) \leq \max\{\text{cof}([\kappa]^{<\omega}), \text{cof}(\mathcal{J})\}$. Moreover, if $\mathcal{J}$ has WFP then $\text{cof}_t(\mathcal{J}_\kappa) \geq \text{cof}_t(\mathcal{J})$. ■

However, other transitive cardinal coefficients behave in a radically different way.

**Theorem 2.3.** If $\mathcal{J}$ has WFP then $\text{add}_t^*(\mathcal{J}_\kappa) = \text{add}_t^*(\mathcal{J})$.

**Proof.** Let $T \subseteq 2^\kappa$ be such that $A + T \notin \mathcal{J}_\kappa$ for some $A \in \mathcal{J}_\kappa$ and let $\varphi$ be a witness for $A$. Then $\varphi \ast A \in \mathcal{J}$ and $\varphi \ast A + \varphi \ast T = \varphi \ast (A + T) \notin \mathcal{J}$. Hence $\text{add}_t(\mathcal{J}_\kappa) \geq \text{add}_t^*(\mathcal{J})$.

To show the other inequality, let us fix $T \subseteq 2^\omega$ such that $A + T \notin \mathcal{J}$ for some $A \in \mathcal{J}$. We have $A_{id_\omega} \in \mathcal{J}_\kappa$ (because $id_\omega \in \text{Inj}(\omega, \kappa)$ and $\mathcal{J}$ is productive). We define $T' = \{t \in 2^\kappa : t \upharpoonright \omega \in T \wedge t \upharpoonright (\kappa \setminus \omega) \equiv 0\}$. Then $A_{id_\omega} + T' = (A + T)_{id_\omega}$ and from Lemma 1.1 we get $(A + T)_{id_\omega} \notin \mathcal{J}_\kappa$, which ends the proof. ■

**Theorem 2.4.** $\text{cov}_t(\mathcal{J}_\kappa) = \text{cov}_t(\mathcal{J})$.

**Proof.** Similar to the proof of Theorem 2.3. ■

**Theorem 2.5.** If $\mathcal{J}$ has WFP then $\tau(\mathcal{J}_\kappa) = \tau(\mathcal{J})$.

**Proof.** Suppose that $\mathcal{J}$ is $\xi$-translatable. We consider any $A \in \mathcal{J}_\kappa$ and $\varphi \in \text{Inj}(\omega, \kappa)$ being its witness. Then $\varphi \ast A \in \mathcal{J}$; let us fix $B_{\varphi \ast A} \in \mathcal{J}$. If $S \in [2^\kappa]^\xi$ then without loss of generality we can assume that $\varphi \ast S \in [2^\omega]^\xi$ and thus there exists $t_{\varphi \ast S} \in 2^\omega$ such that $\varphi \ast A + \varphi \ast S \subseteq B_{\varphi \ast A} + t_{\varphi \ast S}$. Then

$$A + S \subseteq (\varphi \ast (A + S)) \subseteq (B_{\varphi \ast A} + t_{\varphi \ast S}) \varphi = (B_{\varphi \ast A}) \varphi + t$$

for some $t \in 2^\kappa$. Hence $\mathcal{J}_\kappa$ is $\xi$-translatable.

On the other hand, let us assume that $\mathcal{J}_\kappa$ is $\xi$-translatable and consider any $A \in \mathcal{J}$. Then $A' = A_{id_\omega} \in \mathcal{J}_\kappa$; let us fix $B_{A'} \in \mathcal{J}_\kappa$. If $T \in [2^\omega]^\xi$ then we define $T' \in [2^\kappa]^\xi$ as in the proof of Theorem 2.3. There exists an appropriate $t_{T'} \in 2^\kappa$ such that $A' + T' \subseteq B_{A'} + t_{T'}$. But $A' + T' = (A + T)_{id_\omega}$ and

...
\[(A + T + t_T | \omega)_{\text{id}_\omega} = (A + T)_{\text{id}_\omega} + t_T \subseteq B.\]

Let us define
\[C = \bigcup_{T \in [2^\omega]^\xi} (A + T + t_T | \omega).\]

Then \(C \subseteq 2^\omega\) and
\[C_{\text{id}_\omega} = \bigcup_{T \in [2^\omega]^\xi} (A + T + t_T | \omega)_{\text{id}_\omega} \subseteq B \in \mathcal{J}_K.\]

Thus \(C_{\text{id}_\omega} \in \mathcal{J}_K\) and from Lemma 1.1 we know that \(C \in \mathcal{J}\).

Let us consider any \(S \in [2^\omega]^\xi\) and put \(t_S = t_T | \omega\). Then \(A + S = A + S + t_S + t_S \subseteq C + t_S\) and we are done. \(\blacksquare\)

As an immediate corollary we obtain the following interesting result.

**Corollary 2.6.** There exists a null set \(A \subseteq 2^{\omega_1}\) such that for every null set \(B \subseteq 2^{\omega_1}\) we can find \(x \in 2^{\omega_1}\) such that \(A \cup (A + x)\) cannot be covered by any translation of \(B\).

**Proof.** From [1] we know that \(\tau(\mathcal{N}) = 2\), where \(\mathcal{N}\) stands for the ideal of null subsets of \(2^{\omega_1}\). In [5] it is shown that \(\mathcal{N}_{\omega_1}\) is exactly the ideal of null subsets of \(2^{\omega_1}\). But from Theorem 2.5 we know that \(\tau(\mathcal{N}_{\omega_1}) = 2\), and this is what we have been supposed to show. \(\blacksquare\)

**References**


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Received May 20, 2003;
received in final form January 13, 2004 (7339)