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Transitive Properties of Ideals on Generalized Cantor Spaces

by

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Summary. We compute transitive cardinal coefficients of ideals on generalized Cantor spaces. In particular, we show that there exists a null set $A \subseteq 2^{\omega_1}$ such that for every null set $B \subseteq 2^{\omega_1}$ we can find $x \in 2^{\omega_1}$ such that $A \cup (A + x)$ cannot be covered by any translation of B.

1. Introduction, definitions and basic properties. In 2001 Kraszewski [5] defined a class of *productive* σ-ideals of subsets of the Cantor space 2^ω and observed that both σ-ideals of meagre sets and of null sets are in this class. Next, from every productive σ-ideal $\mathcal{J}$ one can produce a σ-ideal $\mathcal{J}_\kappa$ of subsets of the generalized Cantor space 2^κ. In particular, starting from the meagre sets and null sets in 2^ω we obtain the meagre sets and null sets in 2^κ, respectively. This description gives us a powerful tool for investigating combinatorial properties of ideals on 2^κ, which was done in [5]. In this paper we continue this research, focusing on transitive cardinal coefficients of ideals of subsets of 2^κ.

We use standard set-theoretical notation and terminology from [2]. Let $(G, +)$ be an infinite abelian group. We consider a σ-ideal $\mathcal{J}$ of subsets of G which is proper, contains all singletons and is invariant (under group operations).

For an ideal $\mathcal{J}$ we consider the following transitive cardinal numbers:

$$\mathrm{add}_t(\mathcal{J}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J} \wedge \neg(\exists B \in \mathcal{J})(\forall A \in \mathcal{A})(\exists g \in G)\, A \subseteq B + g\},$$

$$\mathrm{add}_t^*(\mathcal{J}) = \min\{|T| : T \subseteq G \wedge (\exists A \in \mathcal{J})\, A + T \notin \mathcal{J}\},$$

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$$\mathrm{cov}_t(\mathcal{J}) = \min\{|T| : T \subseteq G \wedge (\exists A \in \mathcal{J})\, A + T = G\},$$
$$\mathrm{cof}_t(\mathcal{J}) = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathcal{J} \wedge (\forall A \in \mathcal{J})(\exists B \in \mathcal{B})(\exists g \in G)\, A \subseteq B + g\}.$$

The first two are both called the *transitive additivity*. The remaining two are called the *transitive covering number* and *transitive cofinality*, respectively.

We say that an ideal $\mathcal{J}$ is *κ-translatable* if

$$(\forall A \in \mathcal{J})(\exists B_A \in \mathcal{J})(\forall S \in [G]^\kappa)(\exists t_S \in G)\, A + S \subseteq B_A + t_S.$$

We define the *translatability number* of $\mathcal{J}$ as follows:

$$\tau(\mathcal{J}) = \min\{\kappa : \mathcal{J} \text{ is not } \kappa\text{-translatable}\}.$$

For more information about relations between classical and transitive cardinal coefficients of ideals, see [2]. For more about translatability, see [1], [3] and [6].

From now on we deal with the generalized Cantor space 2^κ interpreted as the set of all functions from an infinite cardinal number κ into $\{0,1\}$. This space is endowed with the standard product topology. Moreover, we consider the standard product measure and product group structure on 2^κ.

We introduce some extra notation in order to simplify further considerations. Let κ be an infinite cardinal number. We put $\mathrm{Inj}(\omega,\kappa) = \{\varphi \in \kappa^\omega : \varphi \text{ is an injection}\}$. For $A \subseteq 2^\kappa$, $B \subseteq 2^\omega$ and $\varphi \in \mathrm{Inj}(\omega,\kappa)$ we put

$$\varphi * A = \{x \circ \varphi : x \in A\}, \qquad B_\varphi = \{x \in 2^\kappa : x \circ \varphi \in B\}.$$

Obviously, $\varphi * A \subseteq 2^\omega$ and $B_\varphi \subseteq 2^\kappa$. Another simple observation is that for $A \subseteq 2^\kappa, B \subseteq 2^\omega$ and $\varphi \in \mathrm{Inj}(\omega,\kappa)$ we have $A \subseteq (\varphi * A)_\varphi$ and $\varphi * B_\varphi = B$.

Let $\mathcal{J}$ be a σ-ideal of subsets of 2^ω. We say that $\mathcal{J}$ is *productive* if

$$(\forall A \subseteq 2^\omega)(\forall \varphi \in \mathrm{Inj}(\omega,\omega))(\varphi * A \in \mathcal{J} \Rightarrow A \in \mathcal{J}).$$

It is easy to show that $\mathcal{J}$ is productive if and only if for every $A \subseteq 2^\omega$ and $\varphi \in \mathrm{Inj}(\omega,\omega)$ if $A \in \mathcal{J}$ then $A_\varphi \in \mathcal{J}$.

Directly from the definitions we deduce that the σ-ideals of meagre subsets and of null subsets of 2^ω are productive. Also the σ-ideal generated by closed null subsets of 2^ω is productive. Moreover, the ideal $\mathbb{S}_2$ investigated in [4] is the least non-trivial productive σ-ideal of subsets of the Cantor space.

For any productive σ-ideal $\mathcal{J}$ we define

$$\mathcal{J}_\kappa = \{A \subseteq 2^\kappa : (\exists \varphi \in \mathrm{Inj}(\omega,\kappa))\, \varphi * A \in \mathcal{J}\}.$$

A standard consideration shows that $\mathcal{J}_\kappa$ is a σ-ideal of subsets of 2^κ. If $\mathcal{J}$ is invariant then so is $\mathcal{J}_\kappa$. If $A \in \mathcal{J}_\kappa$ then any $\varphi \in \mathrm{Inj}(\omega,\kappa)$ such that $\varphi * A \in \mathcal{J}$ is called a *witness* for A.

Let us also recall one useful definition from [5]. We say that an ideal $\mathcal{J}$ of subsets of 2^ω has *WFP* (*Weak Fubini Property*) if for every $\varphi \in \mathrm{Inj}(\omega,\omega)$ and every $A \subseteq 2^\omega$ if A_φ is in $\mathcal{J}$ then so is A.

The σ-ideals of subsets of 2^ω mentioned above obviously have WFP. We will need the following technical lemma proved in [5].

LEMMA 1.1. *If $\mathcal{J}$ is a productive ideal of subsets of 2^ω having WFP then for every $\varphi \in \mathrm{Inj}(\omega,\kappa)$ and every $A \subseteq 2^\omega$ if $A_\varphi \in \mathcal{J}_\kappa$ then $A \in \mathcal{J}$.* ∎

2. Transitive cardinal coefficients of ideals on 2^κ. From now on we assume that $\mathcal{J}$ is a proper, invariant and productive σ-ideal of subsets of 2^ω containing all singletons and that $\kappa \geq \omega_1$. We investigate relations between transitive cardinal coefficients of $\mathcal{J}$ and those of $\mathcal{J}_\kappa$. Some of them are similar to relations between standard cardinal coefficients of $\mathcal{J}$ and $\mathcal{J}_\kappa$ proved in [5]. We omit the proofs, as they are also analogous.

THEOREM 2.1. $\mathrm{add}_t(\mathcal{J}_\kappa) = \omega_1$. ∎

THEOREM 2.2. $\mathrm{cof}_t(\mathcal{J}_\kappa) \leq \max\{\mathrm{cof}([\kappa]^{\leq\omega}), \mathrm{cof}_t(\mathcal{J})\}$. *Moreover, if $\mathcal{J}$ has WFP then* $\mathrm{cof}_t(\mathcal{J}_\kappa) \geq \mathrm{cof}_t(\mathcal{J})$. ∎

However, other transitive cardinal coefficients behave in a radically different way.

THEOREM 2.3. *If $\mathcal{J}$ has WFP then* $\mathrm{add}_t^*(\mathcal{J}_\kappa) = \mathrm{add}_t^*(\mathcal{J})$.

Proof. Let $T \subseteq 2^\kappa$ be such that $A + T \notin \mathcal{J}_\kappa$ for some $A \in \mathcal{J}_\kappa$ and let φ be a witness for A. Then $\varphi * A \in \mathcal{J}$ and $\varphi * A + \varphi * T = \varphi * (A + T) \notin \mathcal{J}$. Hence $\mathrm{add}_t^*(\mathcal{J}_\kappa) \geq \mathrm{add}_t^*(\mathcal{J})$.

To show the other inequality, let us fix $T \subseteq 2^\omega$ such that $A + T \notin \mathcal{J}$ for some $A \in \mathcal{J}$. We have $A_{\mathrm{id}_\omega} \in \mathcal{J}_\kappa$ (because $\mathrm{id}_\omega \in \mathrm{Inj}(\omega,\kappa)$ and $\mathcal{J}$ is productive). We define $T' = \{t \in 2^\kappa : t \restriction \omega \in T \wedge t \restriction (\kappa \setminus \omega) \equiv 0\}$. Then $A_{\mathrm{id}_\omega} + T' = (A+T)_{\mathrm{id}_\omega}$ and from Lemma 1.1 we get $(A+T)_{\mathrm{id}_\omega} \notin \mathcal{J}_\kappa$, which ends the proof. ∎

THEOREM 2.4. $\mathrm{cov}_t(\mathcal{J}_\kappa) = \mathrm{cov}_t(\mathcal{J})$.

Proof. Similar to the proof of Theorem 2.3. ∎

THEOREM 2.5. *If $\mathcal{J}$ has WFP then $\tau(\mathcal{J}_\kappa) = \tau(\mathcal{J})$.*

Proof. Suppose that $\mathcal{J}$ is ξ-translatable. We consider any $A \in \mathcal{J}_\kappa$ and $\varphi \in \mathrm{Inj}(\omega,\kappa)$ being its witness. Then $\varphi * A \in \mathcal{J}$; let us fix $B_{\varphi*A} \in \mathcal{J}$. If $S \in [2^\kappa]^\xi$ then without loss of generality we can assume that $\varphi * S \in [2^\omega]^\xi$ and thus there exists $t_{\varphi*S} \in 2^\omega$ such that $\varphi * A + \varphi * S \subseteq B_{\varphi*A} + t_{\varphi*S}$. Then

$$A + S \subseteq (\varphi * (A+S))_\varphi \subseteq (B_{\varphi*A} + t_{\varphi*S})_\varphi = (B_{\varphi*A})_\varphi + t$$

for some $t \in 2^\kappa$. Hence $\mathcal{J}_\kappa$ is ξ-translatable.

On the other hand, let us assume that $\mathcal{J}_\kappa$ is ξ-translatable and consider any $A \in \mathcal{J}$. Then $A' = A_{\mathrm{id}_\omega} \in \mathcal{J}_\kappa$; let us fix $B_{A'} \in \mathcal{J}_\kappa$. If $T \in [2^\omega]^\xi$ then we define $T' \in [2^\kappa]^\xi$ as in the proof of Theorem 2.3. There exists an appropriate $t_{T'} \in 2^\kappa$ such that $A' + T' \subseteq B_{A'} + t_{T'}$. But $A' + T' = (A+T)_{\mathrm{id}_\omega}$ and

$$(A+T+t_{T'}\restriction\omega)_{\mathrm{id}_\omega} = (A+T)_{\mathrm{id}_\omega} + t_{T'} \subseteq B.$$

Let us define

$$C = \bigcup_{T\in[2^\omega]^\xi} (A+T+t_{T'}\restriction\omega).$$

Then $C \subseteq 2^\omega$ and

$$C_{\mathrm{id}_\omega} = \bigcup_{T\in[2^\omega]^\xi} (A+T+t_{T'}\restriction\omega)_{\mathrm{id}_\omega} \subseteq B \in \mathcal{J}_\kappa.$$

Thus $C_{\mathrm{id}_\omega} \in \mathcal{J}_\kappa$ and from Lemma 1.1 we know that $C \in \mathcal{J}$.

Let us consider any $S \in [2^\omega]^\xi$ and put $t_S = t_{T'} \restriction \omega$. Then $A+S = A+S+t_S+t_S \subseteq C+t_S$ and we are done. ∎

As an immediate corollary we obtain the following interesting result.

Corollary 2.6. *There exists a null set $A \subseteq 2^{\omega_1}$ such that for every null set $B \subseteq 2^{\omega_1}$ we can find $x \in 2^{\omega_1}$ such that $A \cup (A+x)$ cannot be covered by any translation of B.*

Proof. From [1] we know that $\tau(\mathcal{N}) = 2$, where $\mathcal{N}$ stands for the ideal of null subsets of 2^ω. In [5] it is shown that $\mathcal{N}_{\omega_1}$ is exactly the ideal of null subsets of 2^{ω_1}. But from Theorem 2.5 we know that $\tau(\mathcal{N}_{\omega_1}) = 2$, and this is what we have been supposed to show. ∎

References

[1] T. Bartoszyński, *A note on duality between measure and category*, Proc. Amer. Math. Soc. 128 (2000), 2745–2748.

[2] T. Bartoszyński and H. Judah, *Set Theory*: *On the Structure of the Real Line*, A. K. Peters, Wellesley, MA, 1995.

[3] T. J. Carlson, *Strong measure zero and strongly meager sets*, Proc. Amer. Math. Soc. 118 (1993), 577–586.

[4] J. Cichoń and J. Kraszewski, *On some new ideals on the Cantor and Baire spaces*, ibid. 126 (1998), 1549–1555.

[5] J. Kraszewski, *Properties of ideals on generalized Cantor spaces*, J. Symbolic Logic 66 (2001), 1303–1320.

[6] M. Kysiak, *On Erdős–Sierpiński duality for Lebesgue measure and Baire category*, Master's thesis, Warszawa, 2000 (in Polish).

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