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Locally Nilpotent Monomial Derivations

by

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Summary. We prove that every locally nilpotent monomial k-derivation of $\mathrm{k}[X_1, \ldots, X_n]$ is triangular, whenever k is a ring of characteristic zero. A method of testing monomial k-derivations for local nilpotency is also presented.

1. Introduction. Let A be a ring. A *derivation* on A is an additive map $D : A \to A$ satisfying the Leibniz rule: $D(ab) = aD(b) + bD(a)$ for all $a, b \in A$. A derivation D on a ring A is called *locally nilpotent* if for every $a \in A$ there is an $n \in \mathbb{N}$ such that $D^n(a) = 0$. If $A = \mathrm{k}[X_1, \ldots, X_n]$, where k is a ring, then a derivation D is called a k-*derivation* if D is a k-linear map (in other words $D(c) = 0$ for all $c \in \mathrm{k}$). A k-derivation D on $\mathrm{k}[X_1, \ldots, X_n]$ is called *triangular* (resp. *monomial*) if $D(X_i) \in \mathrm{k}[X_1, \ldots, X_{i-1}]$ (resp. $D(X_i)$ is a monomial).

Locally nilpotent k-derivations on the ring $\mathrm{k}[X_1, \ldots, X_n]$ play an important role in the study of several famous problems. The Cancellation Problem, Hilbert's 14th Problem, the Linearization Problems, the Abyankar–Sathaye Conjecture and the Jacobian Conjecture are a few examples. For more comments and references see e.g. [4].

In this note we prove the following:

THEOREM 1. *Let* k *be an integral ring of characteristic zero. Then every locally nilpotent monomial* k*-derivation on* $\mathrm{k}[X_1, \ldots, X_n]$ *is triangular* (*possibly after a permutation of variables*).

As an application, we can rephrase the following theorem of Maubach, in which k is assumed to be a field of characteristic zero.

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Theorem 2 ([2, Corollary 3.4]). *Triangular monomial derivations on* $A = \mathrm{k}[X_1, X_2, X_3, X_4]$ *have kernel generated by at most four elements.*

Theorem 2 can be written in the following form:

Theorem 3. *Locally nilpotent monomial derivations on* $A = \mathrm{k}[X_1, X_2, X_3, X_4]$ *have kernel generated by at most four elements.*

By Theorem 1 we also have the following:

Corollary 4. *Let* D *be a monomial* k*-derivation on* $\mathrm{k}[X_1, \dots, X_n]$, *where* k *is an integral ring of characteristic zero. Then* D *is locally nilpotent if and only if* D *is triangular* (*possibly after a permutation of variables*).

This gives a very easy way to check if a given monomial k-derivation on $\mathrm{k}[X_1, \dots, X_n]$ is locally nilpotent. This also provides an answer to a question of Arno van den Essen (so-called Nilpotency Problem).

2. Degree function associated with a locally nilpotent derivation. Let D be a fixed locally nilpotent derivation on an integral ring A of characteristic zero (not necessarily equal to $\mathrm{k}[X_1, \dots, X_n]$). Put

$$\begin{aligned} A_d &= \{\, a \in A \mid D(a) \in A_{d-1} \,\}, \qquad d \geq 1, \\ A_0 &= \{\, a \in A \mid D(a) = 0 \,\}, \\ A_{-\infty} &= \{0\}. \end{aligned}$$

Notice that, for $d \in \mathbb{N}$, $f \in A_d$ if and only if $D^{d+1}(f) = 0$. Thus $A_d \subset A_{d+1}$ for all d, and

$$\bigcup_{d \in \mathbb{N}} A_d = A,$$

since D is locally nilpotent. Thus we can define the following function:

$$\deg_D : A \ni f \mapsto \min\{d \in \mathbb{N} \cup \{-\infty\} \mid f \in A_d\} \in \mathbb{N} \cup \{-\infty\}.$$

This function has the following usual properties of degree functions (cf. [1] or [3, Proposition 6.1.1]):

$$\deg_D(f \cdot g) = \deg_D f + \deg_D g, \tag{1}$$

$$\deg_D(f + g) \leq \max\{\deg_D f, \deg_D g\}, \tag{2}$$

$$\text{if} \quad \deg_D f \neq \deg_D g, \quad \text{then} \quad \deg D(f + g) = \max\{\deg_D f, \deg_D g\}. \tag{3}$$

Besides these usual properties the definition of $\deg_D$ yields

$$\deg_D(D(f)) = \deg_D f - 1 \tag{4}$$

for all $f \in A \setminus A_0$.

3. Proof of Theorem. Let D be any locally nilpotent monomial k-derivation on $\mathrm{k}[X_1, \ldots, X_n]$, say

$$D = c_1 X^{\alpha_1} \frac{\partial}{\partial X_1} + \cdots + c_n X^{\alpha_n} \frac{\partial}{\partial X_n}, \tag{5}$$

where $c_1, \ldots, c_n \in \mathrm{k}$, $\alpha_i = (\alpha_{i,1}, \ldots, \alpha_{i,n})$ and $X^{\alpha_i} = X_1^{\alpha_{i,1}} \cdots X_n^{\alpha_{i,n}}$. Let $\deg_D$ be the associated degree function. Rearranging variables if necessary, we can assume that

$$\deg_D X_1 \leq \cdots \leq \deg_D X_n. \tag{6}$$

Let $i \in \{1, \ldots, n\}$. We can assume that $X_i \notin A_0$ (otherwise $D(X_i) \in \mathrm{k}[X_1, \ldots, X_{i-1}]$). By (5) and (1) we have

$$\deg_D(D(X_i)) = \alpha_{i,1} \cdot \deg_D X_1 + \cdots + \alpha_{i,n} \cdot \deg_D X_n. \tag{7}$$

But, by (4), we also have

$$\deg_D(D(X_i)) = \deg_D X_i - 1. \tag{8}$$

From (7) and (8) we obtain

$$\alpha_{i,1} \cdot \deg_D X_1 + \cdots + \alpha_{i,n} \cdot \deg_D X_n = \deg_D X_i - 1.$$

Thus $\alpha_{i,j} = 0$ for i and j such that $\deg_D X_i \leq \deg_D X_j$. In particular, by (6), $\alpha_{i,j} = 0$ for $j \geq i$. This means, by (5), that $D(X_i) \in \mathrm{k}[X_1, \ldots, X_{i-1}]$.

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