Non-MSF Wavelets for the Hardy Space $H^2(\mathbb{R})$

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Summary. All wavelets constructed so far for the Hardy space $H^2(\mathbb{R})$ are MSF wavelets. We construct a family of $H^2$-wavelets which are not MSF. An equivalence relation on $H^2$-wavelets is introduced and it is shown that the corresponding equivalence classes are non-empty. Finally, we construct a family of $H^2$-wavelets with Fourier transform not vanishing in any neighbourhood of the origin.

1. Introduction. The classical Hardy space $H^2(\mathbb{R})$ is the collection of all square integrable functions whose Fourier transform is supported in $\mathbb{R}^+ = (0, \infty)$:

$$H^2(\mathbb{R}) := \{ f \in L^2(\mathbb{R}) : \hat{f}(\xi) = 0 \text{ for a.e. } \xi \leq 0 \},$$

where $\hat{f}$ is the Fourier transform of $f$ defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-ix\xi} \, dx.$$

Clearly, $H^2(\mathbb{R})$ is a closed subspace of $L^2(\mathbb{R})$. A function $\psi \in H^2(\mathbb{R})$ is said to be a wavelet for $H^2(\mathbb{R})$ if the system of functions $\{ \psi_{j,k} = 2^{j/2}\psi(2^j \cdot -k) : j,k \in \mathbb{Z} \}$ forms an orthonormal basis for $H^2(\mathbb{R})$. We shall call such a $\psi$ an $H^2$-wavelet.

Two basic equations characterize all $H^2$-wavelets. The proof of the following theorem can be obtained from the corresponding result for the usual case of $L^2(\mathbb{R})$ (see Theorem 6.4 in Chapter 7 of [6]).

**Theorem 1.1.** A function $\psi \in H^2(\mathbb{R})$ with $\|\psi\|_2 = 1$ is an $H^2$-wavelet if and only if

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\[ \sum_{j \in \mathbb{Z}} |\hat{\psi}(2^j \xi)|^2 = \chi_{\mathbb{R}^+}(\xi) \quad \text{for a.e. } \xi \in \mathbb{R} \]

and
\[ \sum_{j \geq 0} \hat{\psi}(2^j \xi) \hat{\psi}(2^j (\xi + 2q)) = 0 \quad \text{for a.e. } \xi \in \mathbb{R} \text{ and for all } q \in 2\mathbb{Z} + 1. \]

From the Paley–Wiener theorem it follows that there is no compactly supported function in \( H^2(\mathbb{R}) \) apart from the zero function, hence, there is no compactly supported \( H^2 \)-wavelet. On the other hand, there exist \( H^2 \)-wavelets with compactly supported Fourier transform. One such example is given by \( \hat{\psi} = \chi_{[2\pi, 4\pi]} \), which is the analogue of the Shannon wavelet for \( L^2(\mathbb{R}) \). P. Auscher [2] proved that there is no \( H^2 \)-wavelet \( \psi \) satisfying the following regularity condition: \( |\hat{\psi}| \) is continuous on \( \mathbb{R} \) and \( |\hat{\psi}(\xi)| = O((1 + |\xi|)^{-\alpha - 1/2}) \) at \( \infty \), for some \( \alpha > 0 \). In particular, \( H^2(\mathbb{R}) \) does not have a wavelet \( \psi \) with \( |\hat{\psi}| \) continuous and \( \hat{\psi} \) compactly supported.

Analogously to the \( L^2 \) case, an \( H^2 \)-wavelet \( \psi \) will be called a minimally supported frequency (MSF) wavelet if \( |\hat{\psi}| = \chi_K \) for some \( K \subset \mathbb{R}^+ \). Such wavelets were called unimodular wavelets in [5]. The associated set \( K \) will be called an \( H^2 \)-wavelet set. In this situation the set \( K \) has Lebesgue measure \( \pi \).

There is a simple characterization of \( H^2 \)-wavelet sets analogous to the \( L^2 \) case.

**Theorem 1.2.** A set \( K \subset \mathbb{R}^+ \) is an \( H^2 \)-wavelet set if and only if the following two conditions hold:

(i) \( \{ K + 2k\pi : k \in \mathbb{Z} \} \) is a partition of \( \mathbb{R} \).

(ii) \( \{ 2^j K : j \in \mathbb{Z} \} \) is a partition of \( \mathbb{R}^+ \).

In [5], the authors proved that the only \( H^2 \)-wavelet set which is an interval is \( [2\pi, 4\pi] \). They also characterized all \( H^2 \)-wavelet sets consisting of two disjoint intervals. In [3] (see also [1]) we proved a result on the structure of \( H^2 \)-wavelet sets consisting of a finite number of intervals and, as an application, characterized 3-interval \( H^2 \)-wavelet sets. All these wavelet sets depend on a finite number of integral parameters, which proves that there are countably many \( H^2 \)-wavelet sets which are unions of at most three disjoint intervals. We also constructed a family of 4-interval \( H^2 \)-wavelet sets with some of the endpoints depending on a continuous real parameter, thereby proving the uncountability of such sets (see [1]). In the proof of Theorem 3.2 below, we exhibit a family of \( H^2 \)-wavelet sets with some of the endpoints depending on two independent continuous parameters. Some more \( H^2 \)-wavelet sets were constructed in [7] where the author also proves the existence of an \( H^2 \)-MSF wavelet \( \psi \) such that \( \psi \not\in L^p(\mathbb{R}) \) for \( p < 2 \).
All wavelets for $H^2(\mathbb{R})$ known to date have been MSF, i.e., their Fourier transform is the characteristic function of a subset of $\mathbb{R}^+$. In the next section, we construct a family of non-MSF $H^2$-wavelets. In Section 3, we introduce an equivalence relation on the set of $H^2$-wavelets and explicitly construct examples of $H^2$-wavelets in each of the corresponding equivalence classes. In the last section, we construct a family of $H^2$-wavelets with Fourier transform discontinuous at the origin.

2. The construction of non-MSF wavelets. Our strategy of constructing the family of non-MSF wavelets of $H^2(\mathbb{R})$ is the following. We start with an $H^2$-MSF wavelet so that $|\hat{\psi}|$ assumes the value 1 on its support. Then we add some more sets to the support of $\hat{\psi}$ and reassign values to $\hat{\psi}$ in such a manner that the equalities $\sum_{j \in \mathbb{Z}} |\hat{\psi}(2^j \xi)|^2 = \chi_{\mathbb{R}^+}(\xi)$ a.e. and $\sum_{k \in \mathbb{Z}} |\hat{\psi}(\xi + 2k\pi)|^2 = 1$ a.e. are preserved, which are necessary conditions for $\hat{\psi}$ to be an $H^2$-wavelet.

Fix $r \in \mathbb{N}$ and let $k$ be an integer satisfying $1 \leq k < 2(2^r - 1)$. Define

$$K_{r,k} = \left[ \frac{2(k + 1)}{2^{r+1} - 1}, \frac{2k}{2^r - 1} \right] \cup \left[ \frac{2^{r+1}k}{2^r - 1}, \frac{2^{r+2}(k + 1)}{2^{r+1} - 1} \right] = A \cup B,$$

say. Observe that the sets $A + 2k\pi$ and $B$ are disjoint and their union is an interval of length $2\pi$ so that (i) in Theorem 1.2 is satisfied. Similarly, $2^r A$ and $B$ are disjoint and their union is the interval $[a, 2a]$, where $a = 2^{r+1}(k + 1)/(2^{r+1} - 1)$, hence, (ii) in Theorem 1.2 is also satisfied. Therefore, $K_{r,k}$ is an $H^2$-wavelet set. In fact, $\{K_{r,k} : r \in \mathbb{N}, 1 \leq k < 2(2^r - 1)\}$ is precisely the collection of all $H^2$-wavelet sets consisting of two disjoint intervals, as observed in [5].

In particular, for $k = 2^r - 1$, we get the following family of $H^2$-wavelet sets:

$$K_r = \left[ \frac{2^{r+1}}{2^{r+1} - 1}, \frac{2}{2^r - 1} \right] \cup \left[ \frac{2^{r+1}}{2^r - 1}, \frac{2^{r+2} + 1}{2^{r+1} - 1} \right], \quad r \in \mathbb{N}.\quad (2.1)$$

Denote the intervals on the right hand side of (2.1) by $I_r$ and $J_r$ respectively. Note that $2\pi/3 \leq |I_r| < \pi$ and $\pi < |J_r| \leq 4\pi/3$. We denote the Lebesgue measure of a set $S$ by $|S|$. First of all, we observe that $2^{-1}I_r + 2^{r+1}\pi \subset J_r$. For $r \in \mathbb{N}$, define the function $\psi_r$ by

$$\hat{\psi}_r(\xi) = \begin{cases} 
1/\sqrt{2} & \text{if } \xi \in I_r \cup (2^{-1}I_r) \cup (2^{-1}I_r + 2^{r+1}\pi), \\
-1/\sqrt{2} & \text{if } \xi \in I_r + 2^{r+2}\pi, \\
1 & \text{if } \xi \in J_r \setminus (2^{-1}I_r + 2^{r+1}\pi), \\
0 & \text{otherwise}.
\end{cases} \quad (2.2)$$

**Theorem 2.1.** For each $r \in \mathbb{N}$, $\psi_r$ is a wavelet for the Hardy space $H^2(\mathbb{R})$. 
Some preparation is needed before we prove Theorem 2.1. Define the maps $\tau$ and $d$ as follows:

$$
\tau : \mathbb{R} \rightarrow [2\pi, 4\pi], \quad \tau(x) = x + 2k(x)\pi,
$$

$$
d : \mathbb{R}^+ \rightarrow [2\pi, 4\pi], \quad d(x) = 2^j(x)x,
$$

where $k(x)$ and $j(x)$ are unique integers such that $x + 2k(x)\pi$ and $2^j(x)x$ belong to $[2\pi, 4\pi]$.

We first prove the following lemma which gives useful information on the support of $\hat{\psi}_r$. This will be crucial for proving Theorem 2.1.

**Lemma 2.2.** Let $E_r = \text{supp } \hat{\psi}_r = (2^{-1}I_r) \cup I_r \cup J_r \cup (I_r + 2^r+2\pi)$.

(i) If $\xi \in 2^{-1}I_r$, then $\xi + 2k\pi \in E_r$ if and only if $k = 0, 2^r$, and $2^j\xi \in E_r$ if and only if $j = 0, 1$.

(ii) If $\xi \in I_r$, then $\xi + 2k\pi \in E_r$ if and only if $k = 0, 2^r+1$, and $2^j\xi \in E_r$ if and only if $j = 0, 1$.

(iii) If $\xi \in 2^{-1}I_r + 2^{r+1}\pi$, then $\xi + 2k\pi \in E_r$ if and only if $k = 0, -2^r$, and $2^j\xi \in E_r$ if and only if $j = 0, 1$.

(iv) If $\xi \in J_r \setminus (2^{-1}I_r + 2^{r+1}\pi)$, then $\xi + 2k\pi \in E_r$ if and only if $k = 0$, and $2^j\xi \in E_r$ if and only if $j = 0$.

(v) If $\xi \in I_r + 2^{r+2}\pi$, then $\xi + 2k\pi \in E_r$ if and only if $k = 0, -2^{r+1}$, and $2^j\xi \in E_r$ if and only if $j = 0, 1$.

**Proof.** Observe that $\tau(E) = \tau(E + 2\pi)$ and $d(F) = d(2^jF)$ for every $j, k \in \mathbb{Z}$ and every $E \subset \mathbb{R}$, $F \subset \mathbb{R}^+$. Hence,

$$
(2.3) \quad \tau(2^{-1}I_r + 2^r+1\pi) = \tau(2^{-1}I_r), \quad \tau(I_r + 2^{r+2}\pi) = \tau(I_r),
$$

$$
(2.4) \quad d(2^{-1}I_r) = d(I_r), \quad d(2^{-1}I_r + 2^{r+1}\pi) = d(I_r + 2^{r+2}\pi).
$$

It also follows from the definition of the maps $\tau$ and $d$ that if $W$ is an $H^2$-wavelet set and $E, F \subset W$, then $\tau(E) \cap \tau(F) = \emptyset$ and $d(E) \cap d(F) = \emptyset$.

Since $I_r \cup J_r$ is an $H^2$-wavelet set and $2^{-1}I_r + 2^{r+1}\pi \subset J_r$, we have

$$
(2.5) \quad \tau(I_r) \cap \tau(2^{-1}I_r + 2^{r+1}\pi) = \emptyset,
$$

$$
(2.6) \quad \tau(J_r \setminus (2^{-1}I_r + 2^{r+1}\pi)) \cap \tau(2^{-1}I_r + 2^{r+1}\pi) = \emptyset.
$$

From (2.3), (2.5) and (2.6), we get

$$
\tau(2^{-1}I_r) \cap \tau(I_r) = \emptyset, \quad \tau(2^{-1}I_r) \cap \tau(I_r + 2^{r+2}\pi) = \emptyset,
$$

$$
\tau(2^{-1}I_r) \cap \tau(J_r \setminus (2^{-1}I_r + 2^{r+1}\pi)) = \emptyset.
$$

Therefore, if $\xi \in 2^{-1}I_r$, then $\xi + 2k\pi \in E_r$ if and only if $k = 0, 2^r$.

Similarly, we have

$$
(2.7) \quad d(I_r) \cap d(J_r) = \emptyset, \quad d(I_r) \cap d(2^{-1}I_r + 2^{r+1}\pi) = \emptyset.
$$
From (2.4) and (2.7) we get
\[ d(2^{-1}I_r) \cap d(J_r) = \emptyset, \quad d(2^{-1}I_r) \cap d(I_r + 2r^2\pi) = \emptyset. \]
From this we deduce that if \( \xi \in 2^{-1}I_r \), then \( 2^j \xi \in E_r \) if and only if \( j = 0, 1 \).

We have proved (i) of the lemma. The proof of (ii)-(v) is similar. \( \blacksquare \)

**Proof of Theorem 2.1.** In view of the characterization of \( H^2 \)-wavelets (see Theorem 1.1), we need to show the following:

(a) \( \|\psi_r\|_2 = 1. \)

(b) \( g(\xi) := \sum_{j \in \mathbb{Z}} |\hat{\psi}_r(2^j \xi)|^2 = \chi_{\mathbb{R}^+}(\xi) \quad \text{for a.e.} \ \xi \in \mathbb{R}. \)

(c) \( t_q(\xi) := \sum_{j \geq 0} \hat{\psi}_r(2^j \xi) \overline{\psi}_r(2^j(\xi + 2q\pi)) = 0 \)
for a.e. \( \xi \in \mathbb{R} \) and all \( q \in 2\mathbb{Z} + 1. \)

**Proof of (a).** We have
\[
||\hat{\psi}_r||_2^2 = \int_\mathbb{R} |\hat{\psi}_r(\xi)|^2 d\xi = \frac{1}{2}(|I_r| + \frac{1}{2}|I_r| + \frac{1}{2}|I_r|) + |J_r| - \frac{1}{2}|I_r|
\]
Hence, \( ||\psi_r||_2^2 = \frac{1}{2\pi}||\hat{\psi}_r||_2^2 = 1. \)

**Proof of (b).** Observe that \( g(\xi) = 0 \) if \( \xi \leq 0. \) Since \( g(2\xi) = g(\xi) \) for a.e. \( \xi, \) it is enough to show that \( g(\xi) = 1 \) on any set \( E \) such that \( d(E) = [2\pi, 4\pi]; \)
\( I_r \cup J_r \) is such a set since it is an \( H^2 \)-wavelet set.

If \( \xi \in I_r, \) then by Lemma 2.2(ii), \( 2^j \xi \in \text{supp} \ \hat{\psi}_r \) if and only if \( j = -1, 0. \)
Hence, \( g(\xi) = |\hat{\psi}_r(\xi/2)|^2 + |\hat{\psi}_r(\xi)|^2 = (1/\sqrt{2})^2 + (1/\sqrt{2})^2 = 1. \)

We write \( J_r = (2^{-1}I_r + 2r^2\pi) \cup \{J_r \setminus (2^{-1}I_r + 2r^2\pi)\} = M \cup L, \) say.
If \( \xi \in M, \) then \( 2^j \xi \in \text{supp} \ \hat{\psi}_r \) if and only if \( j = 0, 1 \) (see Lemma 2.2(iii)) so that \( g(\xi) = |\hat{\psi}_r(\xi)|^2 + |\hat{\psi}_r(2\xi)|^2 = (1/\sqrt{2})^2 + (-1/\sqrt{2})^2 = 1. \) For \( \xi \in L, \) no other dilate of \( \xi \) is in the support of \( \hat{\psi}_r, \) hence, \( g(\xi) = 1 \) a.e. on \( L. \)

**Proof of (c).** Since \( t_{-q}(\xi) = t_q(\xi - 2q\pi), \) it is enough to prove that \( t_q = 0 \)
a.e. for all positive and odd integer \( q. \) The term
\[ \hat{\psi}_r(2^j \xi) \overline{\psi}_r(2^j(\xi + 2q\pi)) \]
is non-zero when both \( 2^j \xi \) and \( 2^j \xi + 2 \cdot 2^j q\pi \) are in the support of \( \hat{\psi}_r. \)
Referring again to Lemma 2.2, we observe that this is possible if either \( 2^j q = 2^r \) or \( 2^j q = 2^{r+1}. \) Since the integer \( q \) is odd, either \( j = r, q = 1 \)
or \( j = r + 1, q = 1. \) In the first case, \( 2^j \xi \in 2^{-1}I_r \) so that \( 2^j(\xi + 2q\pi) \in \)
\( 2^{-1}I_r \cup 2^{r+2}\pi, 2^j 2q \xi \in I_r, \) and \( 2^j + 1(\xi + 2q\pi) \in I_r + 2^{r+2}\pi. \) Hence, \( t_q(\xi) = (1/\sqrt{2})(1/\sqrt{2}) + (1/\sqrt{2})(-1/\sqrt{2}) = 0. \) The second case is treated similarly.
This completes the proof of the theorem. \( \blacksquare \)
3. An equivalence relation. In this section we shall introduce an equivalence relation on the collection of all wavelets of $H^2(\mathbb{R})$ and, by explicit construction, show that each of the corresponding equivalence classes is non-empty.

Let $\psi$ be an $H^2$-wavelet. For $j \in \mathbb{Z}$, define the following closed subspaces of $H^2(\mathbb{R})$: $V_j = \overline{\text{span}}\{\psi_{l,k} : l < j, \; k \in \mathbb{Z}\}$. It is easy to verify that these subspaces have the following properties:

(i) $V_j \subseteq V_{j+1}$ for all $j \in \mathbb{Z}$,
(ii) $f \in V_j$ if and only if $f(2\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$,
(iii) $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $H^2(\mathbb{R})$, $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$, and
(iv) $V_0$ is invariant under the group of translations by integers.

In view of property (iv), it is natural to ask the following question: Does there exist other groups of translations under which $V_0$ remains invariant? We shall answer this question by considering the groups of translations by dyadic rationals. For $y \in \mathbb{R}$, let $T_y$ be the (unitary) translation operator defined by $T_y f(x) = f(x - y)$. Consider the following groups of translation operators:

$G_r = \{T_{m/2^r} : m \in \mathbb{Z}\}, \quad r \geq 0, \quad r \in \mathbb{Z}, \quad G_\infty = \{T_y : y \in \mathbb{R}\}.$

Let $\mathcal{G}$ be a set of bounded linear operators on $H^2(\mathbb{R})$ and $V$ a closed subspace of $H^2(\mathbb{R})$. We say that $V$ is $\mathcal{G}$-invariant if $Tf \in V$ for every $f \in V$ and $T \in \mathcal{G}$.

Denote by $\mathcal{L}_r$ the collection of all $H^2$-wavelets such that the corresponding space $V_0$ is $G_r$-invariant. Clearly, $\mathcal{L}_0$ is the set of all $H^2$-wavelets, and we have the following inclusions:

$\mathcal{L}_0 \supset \mathcal{L}_1 \supset \mathcal{L}_2 \supset \cdots \supset \mathcal{L}_r \supset \mathcal{L}_{r+1} \supset \cdots \supset \mathcal{L}_\infty.$

We now define an equivalence relation on $H^2$-wavelets, where the equivalence classes are given by $\mathcal{M}_r = \mathcal{L}_r \setminus \mathcal{L}_{r+1}$, with $\mathcal{M}_\infty = \mathcal{L}_\infty$. Thus, $\mathcal{M}_r$, $r \geq 0$, consists of those $H^2$-wavelets for which $V_0$ is $G_r$-invariant but not $G_{r+1}$-invariant.

This equivalence relation was first defined in [9] for the classical case of wavelets of $L^2(\mathbb{R})$. In the same paper the equivalence classes were characterized in terms of the support of the Fourier transform of the wavelets. It was also proved that $\mathcal{M}_r$, $r = 0, 1, 2, 3$, are non-empty. Later, in [4], [8], examples of wavelets of $L^2(\mathbb{R})$ were constructed for each of these equivalence classes, by different methods.

The characterization of $\mathcal{M}_r$ can be easily carried over to the case of $H^2(\mathbb{R})$. First of all we introduce some notation.

Let $\psi$ be an $H^2$-wavelet and let $E = \text{supp } \hat{\psi}$. For $k \in \mathbb{Z}$, define $E(\psi, k) = \{\xi \in E : \xi + 2k\pi \in E\} = E \cap (E + 2k\pi)$ and $\mathcal{E}_\psi = \{k \in \mathbb{Z} : E(\psi, k) \neq \emptyset\}$. Then the characterization of the equivalence classes is the following.
Theorem 3.1. (i) $\mathcal{M}_\infty$ is precisely the collection of all $H^2$-MSF wavelets.

(ii) An $H^2$-wavelet $\psi$ is in $\mathcal{M}_r$, $r \geq 1$, if and only if every element of $\mathcal{E}_\psi$ is divisible by $2^r$ but there is an element of $\mathcal{E}_\psi$ not divisible by $2^{r+1}$.

(iii) An $H^2$-wavelet $\psi$ is in $\mathcal{M}_0$ if and only if $\mathcal{E}_\psi$ contains an odd integer.

The proof of the above theorem is an easy adaptation of the corresponding result proved in [9] for $L^2(\mathbb{R})$. The purpose of this section is to show that all the equivalence classes are non-empty. Indeed, we show that, the non-MSF $H^2$-wavelets constructed in the previous section serve as examples in $\mathcal{M}_r$, $r \geq 1$. For the class $\mathcal{M}_0$, it is natural to consider the case $r = 0$ in (2.2). Unfortunately this does not work since we get $\hat{\psi}_0 = \chi_{[2\pi, 4\pi]}$. Hence, $\psi_0$ is in $\mathcal{M}_\infty$, being an MSF wavelet. To show that $\mathcal{M}_0$ is non-empty, we produce an interesting family of $H^2$-wavelet sets consisting of five disjoint intervals.

Theorem 3.2. The equivalence classes $\mathcal{M}_r$, $r \in \mathbb{N} \cup \{0, \infty\}$, defined above, are non-empty.

Proof. We mentioned in the introduction that all previously known $H^2$-wavelets are MSF. Hence, $\mathcal{M}_\infty$ is non-empty.

Now, fix $r \in \mathbb{N}$ and consider the $H^2$-wavelet $\psi_r$ defined in (2.2). From Lemma 2.2, we notice that $\mathcal{E}_{\psi_r} = \{0, \pm 2^r, \pm 2^{r+1}\}$. By Theorem 3.1(ii), $\psi_r \in \mathcal{M}_r$.

We now construct a family of wavelets in $\mathcal{M}_0$. Let $\pi < x < y < 2\pi$ and $x + 2\pi > 2y$. That is, $(x, y)$ is in the interior of the triangle with vertices $(\pi, \frac{3}{2}\pi)$, $(\pi, 2\pi)$ and $(2\pi, 2\pi)$. Consider the following set:

$$K_{x,y} = [x, y] \cup [2\pi, 2x] \cup [2y, x + 2\pi] \cup [y + 2\pi, 4\pi] \cup [2x + 4\pi, 2y + 4\pi].$$

Denote the intervals on the right by $I_1, \ldots, I_5$. The conditions on $x$ and $y$ ensure that these intervals are non-empty. Observe that $I_1$, $I_4 - 2\pi$, $I_2$, $I_5 - 4\pi$, $I_3$ are pairwise disjoint, and their union is $[x, x + 2\pi]$. Similarly, $I_1$, $2^{-1}I_3$, $2^{-2}I_5$, $2^{-1}I_4$, $I_2$ are pairwise disjoint, and their union is $[x, 2x]$. Hence, by Theorem 1.2, $K_{x,y}$ is an $H^2$-wavelet set.

In particular, we obtain a family of 5-interval $H^2$-wavelet sets where the endpoints of the intervals depend on two independent real parameters.

Note that $2^{-1}I_3 \cap K_{x,y} = \emptyset$, $I_3 + 4\pi \cap K_{x,y} = \emptyset$, and $2^{-1}I_3 + 2\pi$ is properly contained in $I_4$. Now, define the function $\hat{\psi}_0$ by

$$\hat{\psi}_0(\xi) = \begin{cases} 1/\sqrt{2} & \text{if } \xi \in I_3 \cup (2^{-1}I_3) \cup (2^{-1}I_3 + 2\pi), \\ -1/\sqrt{2} & \text{if } \xi \in I_3 + 4\pi, \\ 1 & \text{if } \xi \in K_{x,y} \setminus (2^{-1}I_3 + 2\pi), \\ 0 & \text{otherwise}. \end{cases}$$
It can be proved that $\psi_0$ is an $H^2$-wavelet. The proof is similar to that of Theorem 2.1 and we skip it to avoid repetition. It is also clear that $E_{\psi_0} = \{0, \pm 1, \pm 2\}$. Hence by Theorem 3.1(iii), $\psi_0 \in \mathcal{M}_0$. This completes the proof. ■

4. $H^2$-wavelets with Fourier transform discontinuous at the origin. In this section we construct a family of wavelets for $H^2(\mathbb{R})$ whose Fourier transforms are discontinuous at the origin. First we recall a result proved in [6] for wavelets of $L^2(\mathbb{R})$ (see Theorem 2.7 in Chapter 3 of [6]).

**Theorem 4.1.** Let $\psi$ be a wavelet for $L^2(\mathbb{R})$ such that $\hat{\psi}$ has compact support and $|\hat{\psi}|$ is continuous at 0. Then $\hat{\psi} = 0$ a.e. in an open neighbourhood of the origin.

This result also holds for $H^2(\mathbb{R})$ with essentially the same proof. We are interested in the following question: Does there exist an $H^2$-wavelet such that $\hat{\psi}$ has compact support and does not vanish in any neighbourhood of the origin? In this section we shall give a positive answer to this question. We need the following concepts.

**Definition 4.2.** A set $A$ is said to be translation equivalent to a set $B$ if there exists a partition $\{A_n : n \in \mathbb{Z}\}$ of $A$ and $k_n \in \mathbb{Z}$ such that $\{A_n + k_n \pi : n \in \mathbb{Z}\}$ is a partition of $B$. Similarly, $A$ is dilation equivalent to $B$ if there exists another partition $\{A'_n : n \in \mathbb{Z}\}$ of $A$ and $j_n \in \mathbb{Z}$ such that $\{2^{j_n} A'_n : n \in \mathbb{Z}\}$ is a partition of $B$.

Theorem 1.2 has the following simple but useful consequence.

**Corollary 4.3.** Let $K_1, K_2 \subset \mathbb{R}^+$ and suppose $K_1$ is both translation and dilation equivalent to $K_2$. Then $K_1$ is an $H^2$-wavelet set if and only if $K_2$ is.

Let $r \in \mathbb{N}$ and $t_r = 2^{r+1} \pi / (2^{r+1} - 1)$. Then we know that

$$K_r = [t_r, 2\pi] \cup [2^{r+1} \pi, 2^{r+1} t_r] = I_r \cup J_r$$

is an $H^2$-wavelet set (see (2.1)). For $\varepsilon > 0$ such that $\varepsilon < (2^r - 1)\pi / (2^{r+1} - 1)$, let

$$S_1 = [t_r/2 + \varepsilon / 2^{r+1}, t_r/2 + \varepsilon],$$

$$S_2 = [t_r + 2\varepsilon, 2\pi],$$

$$S_3 = [2^{r+1} t_r, 2^{r+1} t_r + 2\varepsilon].$$

The condition on $\varepsilon$ ensures that $S_2$ is non-empty. Let

$$E_0 = S_1 + 2^{r+1} \pi, \quad F_0 = 2^{-(r+2)} E_0,$$

$$E_n = F_{n-1} + 2^{r+1} \pi, \quad F_n = 2^{-(n+r+2)} E_n, \quad n \geq 1.$$
Define

\[(4.1) \quad K_{r,\varepsilon} = \left( J_r \setminus \bigcup_{n=0}^{\infty} E_n \right) \cup \left( \bigcup_{n=0}^{\infty} F_n \right) \cup (S_1 \cup S_2 \cup S_3). \]

**Theorem 4.4.** For each \( r \in \mathbb{N} \), the set \( K_{r,\varepsilon} \) defined in (4.1) is an \( H^2 \)-wavelet set.

**Proof.** The result will follow from Corollary 4.3 once we show that \( K_{r,\varepsilon} \) is both translation and dilation equivalent to the wavelet set \( K_r \). First of all, we show by induction that \( E_n \subset J_r \) for all \( n \geq 0 \).

Observe that \( t_r + 2^{r+1}\pi = 2^{r+1}t_r \), hence \([0, t_r] + 2^{r+1}\pi = J_r \). Therefore \( E_0 = S_1 + 2^{r+1}\pi \subset [0, t_r] + 2^{r+1}\pi = J_r \). Now assume that \( E_m \subset J_r \). Then \( F_m = 2^{-(m+2)}E_m \subset 2^{-(m+1)}[0, t_r] \subset [0, t_r] \), hence \( E_{m+1} = F_m + 2^{r+1}\pi \subset [0, t_r] + 2^{r+1}\pi = J_r \).

The intervals \( E_n, n \geq 0 \), lie inside \( J_r \) and \( E_{n+1} \) lies to the left of \( E_n \) for all \( n \geq 0 \). Similarly, the intervals \( F_n, n \geq 0 \), lie in \( 2^{-(n+1)}[\pi, t_r] \) so that \( F_{n+1} \) lies to the left of \( F_n \) for \( n \geq 0 \).

We now show that \( K_{r,\varepsilon} \) is dilation equivalent to \( K_r \). We have

\[ 2S_1 \cup S_2 \cup \frac{1}{2^{r+1}} S_3 = [t_r + \varepsilon/2^r, t_r + 2\varepsilon] \cup [t_r + 2\varepsilon, 2\pi] \cup [t_r, t_r + \varepsilon/2^r] = [t_r, 2\pi] = I_r, \]

and

\[ \left( J_r \setminus \bigcup_{n=0}^{\infty} E_n \right) \cup \left( \bigcup_{n=0}^{\infty} 2^{n+r+2} F_n \right) = \left( J_r \setminus \bigcup_{n=0}^{\infty} E_n \right) \cup \left( \bigcup_{n=0}^{\infty} E_n \right) = J_r, \]

since \( E_n \subset J_r \) for all \( n \geq 0 \), which proves the dilation equivalence.

Finally, we show that \( K_{r,\varepsilon} \) is translation equivalent to \( K_r \). Observe that

\[ S_2 \cup (S_3 - 2^{r+1}\pi) = [t_r + 2\varepsilon, 2\pi] \cup [t_r, t_r + 2\varepsilon] = I_r, \]

and

\[ \left( J_r \setminus \bigcup_{n=0}^{\infty} E_n \right) \cup \left( \bigcup_{n=0}^{\infty} (F_n + 2^{r+1}\pi) \right) \cup (S_1 + 2^{r+1}\pi) \]

\[ = \left( J_r \setminus \bigcup_{n=0}^{\infty} E_n \right) \cup \left( \bigcup_{n=1}^{\infty} E_n \right) \cup E_0 = J_r. \]

Let \( \hat{\psi}_{r,\varepsilon} \) be the characteristic function of the set \( K_{r,\varepsilon} \) so that \( \psi_{r,\varepsilon} \) is an \( H^2 \)-wavelet. Since \( F_n \subset 2^{-(n+1)}[\pi, t_r] \) for all \( n \geq 0 \), \( \hat{\psi}_{r,\varepsilon} \) does not vanish in any neighbourhood of 0. In particular, it is discontinuous at the origin.

References


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