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MATHEMATICAL ANALYSIS

Non-MSF Wavelets for the Hardy Space $H^2(\mathbb{R})$

by

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Summary. All wavelets constructed so far for the Hardy space $H^2(\mathbb{R})$ are MSF wavelets. We construct a family of H^2-wavelets which are not MSF. An equivalence relation on H^2-wavelets is introduced and it is shown that the corresponding equivalence classes are non-empty. Finally, we construct a family of H^2-wavelets with Fourier transform not vanishing in any neighbourhood of the origin.

1. Introduction. The classical Hardy space $H^2(\mathbb{R})$ is the collection of all square integrable functions whose Fourier transform is supported in $\mathbb{R}^+ = (0, \infty)$:

$$H^2(\mathbb{R}) := \{f \in L^2(\mathbb{R}) : \widehat{f}(\xi) = 0 \text{ for a.e. } \xi \le 0\},$$

where $\widehat{f}$ is the Fourier transform of f defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-i\xi x}\, dx.$$

Clearly, $H^2(\mathbb{R})$ is a closed subspace of $L^2(\mathbb{R})$. A function $\psi \in H^2(\mathbb{R})$ is said to be a *wavelet* for $H^2(\mathbb{R})$ if the system of functions $\{\psi_{j,k} = 2^{j/2}\psi(2^j \cdot - k) : j,k \in \mathbb{Z}\}$ forms an orthonormal basis for $H^2(\mathbb{R})$. We shall call such a ψ an *H^2-wavelet.*

Two basic equations characterize all H^2-wavelets. The proof of the following theorem can be obtained from the corresponding result for the usual case of $L^2(\mathbb{R})$ (see Theorem 6.4 in Chapter 7 of [6]).

THEOREM 1.1. *A function $\psi \in H^2(\mathbb{R})$ with $\|\psi\|_2 = 1$ is an H^2-wavelet if and only if*

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$$\sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2 = \chi_{\mathbb{R}^+}(\xi) \quad \textit{for a.e. } \xi \in \mathbb{R}$$

and

$$\sum_{j\geq 0} \widehat{\psi}(2^j\xi)\,\overline{\widehat{\psi}(2^j(\xi+2q\pi))} = 0 \quad \textit{for a.e. } \xi \in \mathbb{R} \textit{ and for all } q \in 2\mathbb{Z}+1.$$

From the Paley–Wiener theorem it follows that there is no compactly supported function in $H^2(\mathbb{R})$ apart from the zero function, hence, there is no compactly supported H^2-wavelet. On the other hand, there exist H^2-wavelets with compactly supported Fourier transform. One such example is given by $\widehat{\psi} = \chi_{[2\pi,4\pi]}$, which is the analogue of the Shannon wavelet for $L^2(\mathbb{R})$. P. Auscher [2] proved that there is no H^2-wavelet ψ satisfying the following regularity condition: $|\widehat{\psi}|$ is continuous on $\mathbb{R}$ and $|\widehat{\psi}(\xi)| = O((1+|\xi|)^{-\alpha-1/2})$ at ∞, for some $\alpha > 0$. In particular, $H^2(\mathbb{R})$ does not have a wavelet ψ with $|\widehat{\psi}|$ continuous and $\widehat{\psi}$ compactly supported.

Analogously to the L^2 case, an H^2-wavelet ψ will be called a *minimally supported frequency* (MSF) wavelet if $|\widehat{\psi}| = \chi_K$ for some $K \subset \mathbb{R}^+$. Such wavelets were called *unimodular wavelets* in [5]. The associated set K will be called an H^2*-wavelet set*. In this situation the set K has Lebesgue measure 2π.

There is a simple characterization of H^2-wavelet sets analogous to the L^2 case.

THEOREM 1.2. *A set $K \subset \mathbb{R}^+$ is an H^2-wavelet set if and only if the following two conditions hold*:

(i) $\{K + 2k\pi : k \in \mathbb{Z}\}$ *is a partition of* $\mathbb{R}$.
(ii) $\{2^jK : j \in \mathbb{Z}\}$ *is a partition of* $\mathbb{R}^+$.

In [5], the authors proved that the only H^2-wavelet set which is an interval is $[2\pi, 4\pi]$. They also characterized all H^2-wavelet sets consisting of two disjoint intervals. In [3] (see also [1]) we proved a result on the structure of H^2-wavelet sets consisting of a finite number of intervals and, as an application, characterized 3-interval H^2-wavelet sets. All these wavelet sets depend on a finite number of integral parameters, which proves that there are countably many H^2-wavelet sets which are unions of at most three disjoint intervals. We also constructed a family of 4-interval H^2-wavelet sets with some of the endpoints depending on a continuous real parameter, thereby proving the uncountability of such sets (see [1]). In the proof of Theorem 3.2 below, we exhibit a family of H^2-wavelet sets with some of the endpoints depending on two independent continuous parameters. Some more H^2-wavelet sets were constructed in [7] where the author also proves the existence of an H^2-MSF wavelet ψ such that $\psi \notin L^p(\mathbb{R})$ for $p < 2$.

All wavelets for $H^2(\mathbb{R})$ known to date have been MSF, i.e., their Fourier transform is the characteristic function of a subset of $\mathbb{R}^+$. In the next section, we construct a family of non-MSF H^2-wavelets. In Section 3, we introduce an equivalence relation on the set of H^2-wavelets and explicitly construct examples of H^2-wavelets in each of the corresponding equivalence classes. In the last section, we construct a family of H^2-wavelets with Fourier transform discontinuous at the origin.

2. The construction of non-MSF wavelets. Our strategy of constructing the family of non-MSF wavelets of $H^2(\mathbb{R})$ is the following. We start with an H^2-MSF wavelet so that $|\widehat{\psi}|$ assumes the value 1 on its support. Then we add some more sets to the support of $\widehat{\psi}$ and reassign values to $\widehat{\psi}$ in such a manner that the equalities $\sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2 = \chi_{\mathbb{R}^+}(\xi)$ a.e. and $\sum_{k\in\mathbb{Z}} |\widehat{\psi}(\xi + 2k\pi)|^2 = 1$ a.e. are preserved, which are necessary conditions for ψ to be an H^2-wavelet.

Fix $r \in \mathbb{N}$ and let k be an integer satisfying $1 \le k < 2(2^r - 1)$. Define

$$K_{r,k} = \left[\frac{2(k+1)}{2^{r+1}-1}\,\pi, \frac{2k}{2^r-1}\,\pi\right] \cup \left[\frac{2^{r+1}k}{2^r-1}\,\pi, \frac{2^{r+2}(k+1)}{2^{r+1}-1}\pi\right] = A \cup B, \qquad \text{say.}$$

Observe that the sets $A + 2k\pi$ and B are disjoint and their union is an interval of length 2π so that (i) in Theorem 1.2 is satisfied. Similarly, $2^r A$ and B are disjoint and their union is the interval $[a, 2a]$, where $a = 2^{r+1}(k+1)\pi/(2^{r+1}-1)$, hence, (ii) in Theorem 1.2 is also satisfied. Therefore, $K_{r,k}$ is an H^2-wavelet set. In fact, $\{K_{r,k} : r \in \mathbb{N},\ 1 \le k < 2(2^r - 1)\}$ is precisely the collection of all H^2-wavelet sets consisting of two disjoint intervals, as observed in [5].

In particular, for $k = 2^r - 1$, we get the following family of H^2-wavelet sets:

$$K_r = \left[\frac{2^{r+1}}{2^{r+1}-1}\,\pi, 2\pi\right] \cup \left[2^{r+1}\pi, \frac{2^{2r+2}}{2^{r+1}-1}\,\pi\right], \quad r \in \mathbb{N}. \tag{2.1}$$

Denote the intervals on the right hand side of (2.1) by I_r and J_r respectively. Note that $2\pi/3 \le |I_r| < \pi$ and $\pi < |J_r| \le 4\pi/3$. We denote the Lebesgue measure of a set S by $|S|$. First of all, we observe that $2^{-1}I_r + 2^{r+1}\pi \subset J_r$.

For $r \in \mathbb{N}$, define the function ψ_r by

$$\widehat{\psi_r}(\xi) = \begin{cases} 1/\sqrt{2} & \text{if } \xi \in I_r \cup (2^{-1}I_r) \cup (2^{-1}I_r + 2^{r+1}\pi), \\ -1/\sqrt{2} & \text{if } \xi \in I_r + 2^{r+2}\pi, \\ 1 & \text{if } \xi \in J_r \setminus (2^{-1}I_r + 2^{r+1}\pi), \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

THEOREM 2.1. *For each $r \in \mathbb{N}$, ψ_r is a wavelet for the Hardy space $H^2(\mathbb{R})$.*

Some preparation is needed before we prove Theorem 2.1. Define the maps τ and d as follows:

$$\tau : \mathbb{R} \to [2\pi, 4\pi], \qquad \tau(x) = x + 2k(x)\pi,$$
$$d : \mathbb{R}^+ \to [2\pi, 4\pi], \qquad d(x) = 2^{j(x)}x,$$

where $k(x)$ and $j(x)$ are unique integers such that $x + 2k(x)\pi$ and $2^{j(x)}x$ belong to $[2\pi, 4\pi]$.

We first prove the following lemma which gives useful information on the support of $\widehat{\psi}_r$. This will be crucial for proving Theorem 2.1.

LEMMA 2.2. *Let* $E_r = \operatorname{supp} \widehat{\psi}_r = (2^{-1}I_r) \cup I_r \cup J_r \cup (I_r + 2^{r+2}\pi)$.

(i) *If* $\xi \in 2^{-1}I_r$, *then* $\xi + 2k\pi \in E_r$ *if and only if* $k = 0, 2^r$, *and* $2^j\xi \in E_r$ *if and only if* $j = 0, 1$.
(ii) *If* $\xi \in I_r$, *then* $\xi + 2k\pi \in E_r$ *if and only if* $k = 0, 2^{r+1}$, *and* $2^j\xi \in E_r$ *if and only if* $j = 0, -1$.
(iii) *If* $\xi \in 2^{-1}I_r + 2^{r+1}\pi$, *then* $\xi + 2k\pi \in E_r$ *if and only if* $k = 0, -2^r$, *and* $2^j\xi \in E_r$ *if and only if* $j = 0, 1$.
(iv) *If* $\xi \in J_r \setminus (2^{-1}I_r + 2^{r+1}\pi)$, *then* $\xi + 2k\pi \in E_r$ *if and only if* $k = 0$, *and* $2^j\xi \in E_r$ *if and only if* $j = 0$.
(v) *If* $\xi \in I_r + 2^{r+2}\pi$, *then* $\xi + 2k\pi \in E_r$ *if and only if* $k = 0, -2^{r+1}$, *and* $2^j\xi \in E_r$ *if and only if* $j = 0, -1$.

Proof. Observe that $\tau(E) = \tau(E + 2k\pi)$ and $d(F) = d(2^jF)$ for every $j, k \in \mathbb{Z}$ and every $E \subset \mathbb{R}$, $F \subset \mathbb{R}^+$. Hence,

$$\tau(2^{-1}I_r + 2^{r+1}\pi) = \tau(2^{-1}I_r), \qquad \tau(I_r + 2^{r+2}\pi) = \tau(I_r), \tag{2.3}$$
$$d(2^{-1}I_r) = d(I_r), \qquad d(2^{-1}I_r + 2^{r+1}\pi) = d(I_r + 2^{r+2}\pi). \tag{2.4}$$

It also follows from the definition of the maps τ and d that if W is an H^2-wavelet set and $E, F \subset W$, then $\tau(E) \cap \tau(F) = \emptyset$ and $d(E) \cap d(F) = \emptyset$. Since $I_r \cup J_r$ is an H^2-wavelet set and $2^{-1}I_r + 2^{r+1}\pi \subset J_r$, we have

$$\tau(I_r) \cap \tau(2^{-1}I_r + 2^{r+1}\pi) = \emptyset, \tag{2.5}$$
$$\tau(J_r \setminus (2^{-1}I_r + 2^{r+1}\pi)) \cap \tau(2^{-1}I_r + 2^{r+1}\pi) = \emptyset. \tag{2.6}$$

From (2.3), (2.5) and (2.6), we get

$$\tau(2^{-1}I_r) \cap \tau(I_r) = \emptyset, \qquad \tau(2^{-1}I_r) \cap \tau(I_r + 2^{r+2}\pi) = \emptyset,$$
$$\tau(2^{-1}I_r) \cap \tau(J_r \setminus (2^{-1}I_r + 2^{r+1}\pi)) = \emptyset.$$

Therefore, if $\xi \in 2^{-1}I_r$, then $\xi + 2k\pi \in E_r$ if and only if $k = 0, 2^r$.

Similarly, we have

$$d(I_r) \cap d(J_r) = \emptyset, \qquad d(I_r) \cap d(2^{-1}I_r + 2^{r+1}\pi) = \emptyset. \tag{2.7}$$

From (2.4) and (2.7) we get

$$d(2^{-1}I_r) \cap d(J_r) = \emptyset, \qquad d(2^{-1}I_r) \cap d(I_r + 2^{r+2}\pi) = \emptyset.$$

From this we deduce that if $\xi \in 2^{-1}I_r$, then $2^j\xi \in E_r$ if and only if $j = 0, 1$.

We have proved (i) of the lemma. The proof of (ii)–(v) is similar. ∎

Proof of Theorem 2.1. In view of the characterization of H^2-wavelets (see Theorem 1.1), we need to show the following:

(a) $\|\psi_r\|_2 = 1$.

(b) $\varrho(\xi) := \sum_{j\in\mathbb{Z}} |\widehat{\psi}_r(2^j\xi)|^2 = \chi_{\mathbb{R}^+}(\xi)$ for a.e. $\xi \in \mathbb{R}$.

(c) $t_q(\xi) := \sum_{j\geq 0} \widehat{\psi}_r(2^j\xi)\,\overline{\widehat{\psi}_r(2^j(\xi + 2q\pi))} = 0$

for a.e. $\xi \in \mathbb{R}$ and all $q \in 2\mathbb{Z} + 1$.

Proof of (a). We have

$$\begin{aligned}\|\widehat{\psi}_r\|_2^2 &= \int_{\mathbb{R}} |\widehat{\psi}_r(\xi)|^2\,d\xi = \tfrac{1}{2}(|I_r| + \tfrac{1}{2}|I_r| + \tfrac{1}{2}|I_r| + |I_r|) + |J_r| - \tfrac{1}{2}|I_r| \\ &= |I_r| + |J_r| = 2\pi.\end{aligned}$$

Hence, $\|\psi_r\|_2^2 = \frac{1}{2\pi}\|\widehat{\psi}_r\|_2^2 = 1$.

Proof of (b). Observe that $\varrho(\xi) = 0$ if $\xi \leq 0$. Since $\varrho(2\xi) = \varrho(\xi)$ for a.e. ξ, it is enough to show that $\varrho(\xi) = 1$ on any set E such that $d(E) = [2\pi, 4\pi]$; $I_r \cup J_r$ is such a set since it is an H^2-wavelet set.

If $\xi \in I_r$, then by Lemma 2.2(ii), $2^j\xi \in \operatorname{supp} \widehat{\psi}_r$ if and only if $j = -1, 0$. Hence, $\varrho(\xi) = |\widehat{\psi}_r(\xi/2)|^2 + |\widehat{\psi}_r(\xi)|^2 = (1/\sqrt{2})^2 + (1/\sqrt{2})^2 = 1$.

We write $J_r = (2^{-1}I_r + 2^{r+1}\pi) \cup \{J_r \setminus (2^{-1}I_r + 2^{r+1}\pi)\} = M \cup L$, say. If $\xi \in M$, then $2^j\xi \in \operatorname{supp}\widehat{\psi}_r$ if and only if $j = 0, 1$ (see Lemma 2.2(iii)) so that $\varrho(\xi) = |\widehat{\psi}_r(\xi)|^2 + |\widehat{\psi}_r(2\xi)|^2 = (1/\sqrt{2})^2 + (-1/\sqrt{2})^2 = 1$. For $\xi \in L$, no other dilate of ξ is in the support of $\widehat{\psi}_r$, hence, $\varrho(\xi) = 1$ a.e. on L.

Proof of (c). Since $t_{-q}(\xi) = \overline{t_q(\xi - 2q\pi)}$, it is enough to prove that $t_q = 0$ a.e. for all positive and odd integer q. The term

$$\widehat{\psi}_r(2^j\xi)\,\overline{\widehat{\psi}_r(2^j(\xi + 2q\pi))}$$

is non-zero when both $2^j\xi$ and $2^j\xi + 2 \cdot 2^jq\pi$ are in the support of $\widehat{\psi}_r$. Referring again to Lemma 2.2, we observe that this is possible if either $2^jq = 2^r$ or $2^jq = 2^{r+1}$. Since the integer q is odd, either $j = r, q = 1$ or $j = r+1, q = 1$. In the first case, $2^j\xi \in 2^{-1}I_r$ so that $2^j(\xi + 2q\pi) \in 2^{-1}I_r + 2^{r+1}\pi$, $2^{j+1}\xi \in I_r$, and $2^{j+1}(\xi + 2q\pi) \in I_r + 2^{r+2}\pi$. Hence, $t_q(\xi) = (1/\sqrt{2})(1/\sqrt{2}) + (1/\sqrt{2})(-1/\sqrt{2}) = 0$. The second case is treated similarly. This completes the proof of the theorem. ∎

3. An equivalence relation. In this section we shall introduce an equivalence relation on the collection of all wavelets of $H^2(\mathbb{R})$ and, by explicit construction, show that each of the corresponding equivalence classes is non-empty.

Let ψ be an H^2-wavelet. For $j \in \mathbb{Z}$, define the following closed subspaces of $H^2(\mathbb{R})$: $V_j = \overline{\operatorname{span}}\{\psi_{l,k} : l < j,\ k \in \mathbb{Z}\}$. It is easy to verify that these subspaces have the following properties:

(i) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$,
(ii) $f \in V_j$ if and only if $f(2\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$,
(iii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $H^2(\mathbb{R})$, $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$, and
(iv) V_0 is invariant under the group of translations by integers.

In view of property (iv), it is natural to ask the following question: Does there exist other groups of translations under which V_0 remains invariant? We shall answer this question by considering the groups of translations by dyadic rationals. For $y \in \mathbb{R}$, let T_y be the (unitary) translation operator defined by $T_y f(x) = f(x-y)$. Consider the following groups of translation operators:

$$\mathcal{G}_r = \{T_{m/2^r} : m \in \mathbb{Z}\},\ r \geq 0,\ r \in \mathbb{Z}, \qquad \mathcal{G}_\infty = \{T_y : y \in \mathbb{R}\}.$$

Let $\mathcal{G}$ be a set of bounded linear operators on $H^2(\mathbb{R})$ and V a closed subspace of $H^2(\mathbb{R})$. We say that V is $\mathcal{G}$-*invariant* if $Tf \in V$ for every $f \in V$ and $T \in \mathcal{G}$.

Denote by $\mathcal{L}_r$ the collection of all H^2-wavelets such that the corresponding space V_0 is $\mathcal{G}_r$-invariant. Clearly, $\mathcal{L}_0$ is the set of all H^2-wavelets, and we have the following inclusions:

$$\mathcal{L}_0 \supset \mathcal{L}_1 \supset \mathcal{L}_2 \supset \cdots \supset \mathcal{L}_r \supset \mathcal{L}_{r+1} \supset \cdots \supset \mathcal{L}_\infty.$$

We now define an equivalence relation on H^2-wavelets, where the equivalence classes are given by $\mathcal{M}_r = \mathcal{L}_r \setminus \mathcal{L}_{r+1}$, with $\mathcal{M}_\infty = \mathcal{L}_\infty$. Thus, $\mathcal{M}_r$, $r \geq 0$, consists of those H^2-wavelets for which V_0 is $\mathcal{G}_r$-invariant but not $\mathcal{G}_{r+1}$-invariant.

This equivalence relation was first defined in [9] for the classical case of wavelets of $L^2(\mathbb{R})$. In the same paper the equivalence classes were characterized in terms of the support of the Fourier transform of the wavelets. It was also proved that $\mathcal{M}_r$, $r = 0, 1, 2, 3$, are non-empty. Later, in [4], [8], examples of wavelets of $L^2(\mathbb{R})$ were constructed for each of these equivalence classes, by different methods.

The characterization of $\mathcal{M}_r$ can be easily carried over to the case of $H^2(\mathbb{R})$. First of all we introduce some notation.

Let ψ be an H^2-wavelet and let $E = \operatorname{supp} \widehat{\psi}$. For $k \in \mathbb{Z}$, define $E(\psi, k) = \{\xi \in E : \xi + 2k\pi \in E\} = E \cap (E + 2k\pi)$ and $\mathcal{E}_\psi = \{k \in \mathbb{Z} : E(\psi, k) \neq \emptyset\}$. Then the characterization of the equivalence classes is the following.

THEOREM 3.1. (i) *$\mathcal{M}_\infty$ is precisely the collection of all H^2-MSF wavelets.*

(ii) *An H^2-wavelet ψ is in $\mathcal{M}_r$, $r \geq 1$, if and only if every element of $\mathcal{E}_\psi$ is divisible by 2^r but there is an element of $\mathcal{E}_\psi$ not divisible by 2^{r+1}.*

(iii) *An H^2-wavelet ψ is in $\mathcal{M}_0$ if and only if $\mathcal{E}_\psi$ contains an odd integer.*

The proof of the above theorem is an easy adaptation of the corresponding result proved in [9] for $L^2(\mathbb{R})$. The purpose of this section is to show that all the equivalence classes are non-empty. Indeed, we show that, the non-MSF H^2-wavelets constructed in the previous section serve as examples in $\mathcal{M}_r$, $r \geq 1$. For the class $\mathcal{M}_0$, it is natural to consider the case $r = 0$ in (2.2). Unfortunately this does not work since we get $\widehat{\psi_0} = \chi_{[2\pi,4\pi]}$. Hence, ψ_0 is in $\mathcal{M}_\infty$, being an MSF wavelet. To show that $\mathcal{M}_0$ is non-empty, we produce an interesting family of H^2-wavelet sets consisting of five disjoint intervals.

THEOREM 3.2. *The equivalence classes $\mathcal{M}_r$, $r \in \mathbb{N} \cup \{0, \infty\}$, defined above, are non-empty.*

Proof. We mentioned in the introduction that all previously known H^2-wavelets are MSF. Hence, $\mathcal{M}_\infty$ is non-empty.

Now, fix $r \in \mathbb{N}$ and consider the H^2-wavelet ψ_r defined in (2.2). From Lemma 2.2, we notice that $\mathcal{E}_{\psi_r} = \{0, \pm 2^r, \pm 2^{r+1}\}$. By Theorem 3.1(ii), $\psi_r \in \mathcal{M}_r$.

We now construct a family of wavelets in $\mathcal{M}_0$. Let $\pi < x < y < 2\pi$ and $x + 2\pi > 2y$. That is, (x, y) is in the interior of the triangle with vertices $(\pi, \frac{3}{2}\pi)$, $(\pi, 2\pi)$ and $(2\pi, 2\pi)$. Consider the following set:

$$K_{x,y} = [x, y] \cup [2\pi, 2x] \cup [2y, x + 2\pi] \cup [y + 2\pi, 4\pi] \cup [2x + 4\pi, 2y + 4\pi].$$

Denote the intervals on the right by $I_1, \ldots, I_5$. The conditions on x and y ensure that these intervals are non-empty. Observe that I_1, $I_4 - 2\pi$, I_2, $I_5 - 4\pi$, I_3 are pairwise disjoint, and their union is $[x, x + 2\pi]$. Similarly, I_1, $2^{-1}I_3$, $2^{-2}I_5$, $2^{-1}I_4$, I_2 are pairwise disjoint, and their union is $[x, 2x]$. Hence, by Theorem 1.2, $K_{x,y}$ is an H^2-wavelet set.

In particular, we obtain a family of 5-interval H^2-wavelet sets where the endpoints of the intervals depend on two independent real parameters.

Note that $2^{-1}I_3 \cap K_{x,y} = \emptyset$, $I_3 + 4\pi \cap K_{x,y} = \emptyset$, and $2^{-1}I_3 + 2\pi$ is properly contained in I_4. Now, define the function ψ_0 by

$$\widehat{\psi_0}(\xi) = \begin{cases} 1/\sqrt{2} & \text{if } \xi \in I_3 \cup (2^{-1}I_3) \cup (2^{-1}I_3 + 2\pi), \\ -1/\sqrt{2} & \text{if } \xi \in I_3 + 4\pi, \\ 1 & \text{if } \xi \in K_{x,y} \setminus (2^{-1}I_3 + 2\pi), \\ 0 & \text{otherwise.} \end{cases}$$

It can be proved that ψ_0 is an H^2-wavelet. The proof is similar to that of Theorem 2.1 and we skip it to avoid repetition. It is also clear that $\mathcal{E}_{\psi_0} = \{0, \pm 1, \pm 2\}$. Hence by Theorem 3.1(iii), $\psi_0 \in \mathcal{M}_0$. This completes the proof. ■

4. H^2-wavelets with Fourier transform discontinuous at the origin. In this section we construct a family of wavelets for $H^2(\mathbb{R})$ whose Fourier transforms are discontinuous at the origin. First we recall a result proved in [6] for wavelets of $L^2(\mathbb{R})$ (see Theorem 2.7 in Chapter 3 of [6]).

THEOREM 4.1. *Let ψ be a wavelet for $L^2(\mathbb{R})$ such that $\widehat{\psi}$ has compact support and $|\widehat{\psi}|$ is continuous at 0. Then $\widehat{\psi} = 0$ a.e. in an open neighbourhood of the origin.*

This result also holds for $H^2(\mathbb{R})$ with essentially the same proof. We are interested in the following question: Does there exist an H^2-wavelet such that $\widehat{\psi}$ has compact support and does not vanish in any neighbourhood of the origin? In this section we shall give a positive answer to this question. We need the following concepts.

DEFINITION 4.2. A set A is said to be *translation equivalent* to a set B if there exists a partition $\{A_n : n \in \mathbb{Z}\}$ of A and $k_n \in \mathbb{Z}$ such that $\{A_n + 2k_n\pi : n \in \mathbb{Z}\}$ is a partition of B. Similarly, A is *dilation equivalent* to B if there exists another partition $\{A'_n : n \in \mathbb{Z}\}$ of A and $j_n \in \mathbb{Z}$ such that $\{2^{j_n} A'_n : n \in \mathbb{Z}\}$ is a partition of B.

Theorem 1.2 has the following simple but useful consequence.

COROLLARY 4.3. *Let $K_1, K_2 \subset \mathbb{R}^+$ and suppose K_1 is both translation and dilation equivalent to K_2. Then K_1 is an H^2-wavelet set if and only if K_2 is.*

Let $r \in \mathbb{N}$ and $t_r = 2^{r+1}\pi/(2^{r+1} - 1)$. Then we know that

$$K_r = [t_r, 2\pi] \cup [2^{r+1}\pi, 2^{r+1}t_r] = I_r \cup J_r$$

is an H^2-wavelet set (see (2.1)). For $\varepsilon > 0$ such that $\varepsilon < (2^r - 1)\pi/(2^{r+1} - 1)$, let

$$\begin{aligned} S_1 &= [t_r/2 + \varepsilon/2^{r+1}, t_r/2 + \varepsilon], \\ S_2 &= [t_r + 2\varepsilon, 2\pi], \\ S_3 &= [2^{r+1}t_r, 2^{r+1}t_r + 2\varepsilon]. \end{aligned}$$

The condition on ε ensures that S_2 is non-empty. Let

$$\begin{aligned} E_0 &= S_1 + 2^{r+1}\pi, & F_0 &= 2^{-(r+2)}E_0, \\ E_n &= F_{n-1} + 2^{r+1}\pi, & F_n &= 2^{-(n+r+2)}E_n, \quad n \geq 1. \end{aligned}$$

Define

$$K_{r,\varepsilon} = \Big(J_r \setminus \bigcup_{n=0}^{\infty} E_n\Big) \cup \Big(\bigcup_{n=0}^{\infty} F_n\Big) \cup (S_1 \cup S_2 \cup S_3). \tag{4.1}$$

THEOREM 4.4. *For each $r \in \mathbb{N}$, the set $K_{r,\varepsilon}$ defined in* (4.1) *is an H^2-wavelet set.*

Proof. The result will follow from Corollary 4.3 once we show that $K_{r,\varepsilon}$ is both translation and dilation equivalent to the wavelet set K_r. First of all, we show by induction that $E_n \subset J_r$ for all $n \geq 0$.

Observe that $t_r + 2^{r+1}\pi = 2^{r+1}t_r$, hence $[0, t_r] + 2^{r+1}\pi = J_r$. Therefore $E_0 = S_1 + 2^{r+1}\pi \subset [0, t_r] + 2^{r+1}\pi = J_r$. Now assume that $E_m \subset J_r$. Then $F_m = 2^{-(m+r+2)}E_m \subset 2^{-(m+1)}[0, t_r] \subset [0, t_r]$, hence $E_{m+1} = F_m + 2^{r+1}\pi \subset [0, t_r] + 2^{r+1}\pi = J_r$.

The intervals E_n, $n \geq 0$, lie inside J_r and E_{n+1} lies to the left of E_n for all $n \geq 0$. Similarly, the intervals F_n, $n \geq 0$, lie in $2^{-(n+1)}[\pi, t_r]$ so that F_{n+1} lies to the left of F_n for $n \geq 0$.

We now show that $K_{r,\varepsilon}$ is dilation equivalent to K_r. We have

$$\begin{aligned} 2S_1 \cup S_2 \cup \frac{1}{2^{r+1}} S_3 &= [t_r + \varepsilon/2^r, t_r + 2\varepsilon] \cup [t_r + 2\varepsilon, 2\pi] \cup [t_r, t_r + \varepsilon/2^r] \\ &= [t_r, 2\pi] = I_r, \end{aligned}$$

and

$$\Big(J_r \setminus \bigcup_{n=0}^{\infty} E_n\Big) \cup \Big(\bigcup_{n=0}^{\infty} 2^{n+r+2}F_n\Big) = \Big(J_r \setminus \bigcup_{n=0}^{\infty} E_n\Big) \cup \Big(\bigcup_{n=0}^{\infty} E_n\Big) = J_r,$$

since $E_n \subset J_r$ for all $n \geq 0$, which proves the dilation equivalence.

Finally, we show that $K_{r,\varepsilon}$ is translation equivalent to K_r. Observe that

$$S_2 \cup (S_3 - 2^{r+1}\pi) = [t_r + 2\varepsilon, 2\pi] \cup [t_r, t_r + 2\varepsilon] = I_r,$$

and

$$\begin{aligned} &\Big(J_r \setminus \bigcup_{n=0}^{\infty} E_n\Big) \cup \Big(\bigcup_{n=0}^{\infty} (F_n + 2^{r+1}\pi)\Big) \cup (S_1 + 2^{r+1}\pi) \\ &\qquad = \Big(J_r \setminus \bigcup_{n=0}^{\infty} E_n\Big) \cup \Big(\bigcup_{n=1}^{\infty} E_n\Big) \cup E_0 = J_r. \ \blacksquare \end{aligned}$$

Let $\widehat{\psi}_{r,\varepsilon}$ be the characteristic function of the set $K_{r,\varepsilon}$ so that $\psi_{r,\varepsilon}$ is an H^2-wavelet. Since $F_n \subset 2^{-(n+1)}[\pi, t_r]$ for all $n \geq 0$, $\widehat{\psi}_{r,\varepsilon}$ does not vanish in any neighbourhood of 0. In particular, it is discontinuous at the origin.

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