A Note on the Measure of Solvability

by

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Summary. Let $X$ be an infinite-dimensional Banach space. The measure of solvability $\nu(I)$ of the identity operator $I$ is equal to 1.

Let $X$ be an infinite-dimensional normed space, and let $\psi$ denote a measure of noncompactness on $X$. In this note we show that for any given $\varepsilon > 0$ there exists a $(\psi)(1 + \varepsilon)$-set contractive mapping of a nonempty, convex and non-totally-bounded subset of $X$ having positive minimal displacement.

Then the fact that in any infinite-dimensional Banach space for any given $\varepsilon > 0$ there exists a fixed point free $(\psi)(1 + \varepsilon)$-set contraction of the unit ball implies that the measure of solvability $\nu(I)$ of the identity operator $I$ is equal to 1. This result gives a positive answer to a question posed by M. Väth in [11].

1. Preliminaries. Let $X$ be an infinite-dimensional normed space, and let $B = \{x \in X : \|x\| \leq 1\}$ and $S = \{x \in X : \|x\| = 1\}$ be, respectively, the unit ball and unit sphere of $X$. Let $C$ denote a set in $X$, and $T : C \to C$ be a given mapping. The minimal displacement $\eta(T)$ of $T$ is the number defined by

$$\eta(T) = \inf \{\|Tx - x\| : x \in C\}.$$

A mapping $T$ for which $\eta(T) > 0$ is without approximate fixed points. The first study of Lipschitz mappings without approximate fixed points was done by K. Goebel [6]. We refer the reader to [7] for a collection of results on this and related problems.

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**Theorem 1.1.** Let $X$ be an infinite-dimensional normed space. For any nonempty, convex and non-totally-bounded subset $C$ of $X$ there exists a Lipschitz mapping $T : C \to C$ for which $\eta(T) > 0$.

2. $(\psi)k$-set contractions and the measure of solvability. A mapping $\psi$ defined on the family of all bounded subsets of an infinite-dimensional normed space $X$ is called a *measure of noncompactness* on $X$ (see [1]) if it has the following properties:

1. $\psi(A) = 0$ if and only if $A$ is precompact.
2. $\psi(\overline{A}) = \psi(A)$, where $\overline{A}$ denotes the closed convex hull of $A$.
3. $\psi(A_1 \cup A_2) = \max\{\psi(A_1), \psi(A_2)\}$.
4. $\psi(A_1 + A_2) \leq \psi(A_1) + \psi(A_2)$.
5. $\psi(\lambda A) = |\lambda|\psi(A)$ for every real number $\lambda$.

Let $D$ be a nonempty subset of $X$. A continuous mapping $T : D \to X$ is called a $(\psi)k$-set contraction if for any bounded subset $A$ of $D$,

$$\psi(T(A)) \leq k\psi(A).$$

For a bounded subset $A$ of $X$, the *Kuratowski measure of noncompactness* $\alpha(A)$ is the infimum of all $\varepsilon > 0$ such that $A$ admits a finite covering by sets of diameter less than $\varepsilon$.

By combining Theorem 1.1 with a previous result of Furi and Martelli [5] we obtain the existence of an $(\alpha)(1 + \varepsilon)$-set contraction having a positive minimal displacement.

**Theorem 2.1.** Let $X$ be an infinite-dimensional normed space. For any nonempty, convex and non-totally-bounded subset $C$ of $X$ and any given $\varepsilon > 0$ there exists an $(\alpha)(1 + \varepsilon)$-set contraction $F_\varepsilon : C \to C$ for which $\eta(F_\varepsilon) > 0$.

**Proof.** Let $\varepsilon > 0$. We show that the set

$$S_\varepsilon = \{ F : C \to C : F \text{ is an } (\alpha)(1 + \varepsilon)\text{-set contraction and } \eta(F) > 0 \}$$

is nonempty. By Theorem 1.1 there exists a Lipschitz mapping $F : C \to C$, with Lipschitz constant $L > 1$, such that $\eta(F) > 0$. Then $F$ is an $(\alpha)L$-set contraction. If $\varepsilon \geq L - 1$, then $F \in S_\varepsilon$. If $\varepsilon < L - 1$, we define $F_\varepsilon : C \to C$ by setting

$$F_\varepsilon(x) = \left(1 - \frac{\varepsilon}{L - 1}\right)x + \frac{\varepsilon}{L - 1}F(x).$$

It is easy to check that $F_\varepsilon$ is an $(\alpha)(1 + \varepsilon)$-set contraction. Moreover $\eta(F_\varepsilon) = \frac{\varepsilon}{L - \varepsilon} \eta(F)$, so that $\eta(F_\varepsilon) > 0$ and the proof is complete. ■
We say that two measures of noncompactness \( \varphi \) and \( \psi \) are equivalent if there exist two positive constants \( c_1 \) and \( c_2 \) such that, for any bounded subset \( A \) of \( X \),
\[
c_1 \psi(A) \leq \varphi(A) \leq c_2 \psi(A).
\]
For a bounded subset \( A \) of \( X \), let \( \chi(A) \) denote the Hausdorff measure of noncompactness, i.e. the infimum of all \( \varepsilon \) such that \( A \) has a finite \( \varepsilon \)-net in \( X \), and \( \beta(A) \) the lattice measure of noncompactness, i.e. the supremum of all \( \varepsilon > 0 \) such that \( A \) contains a sequence \( \{x_n\} \) such that \( \|x_n - x_k\| \geq \varepsilon \) for \( n \neq k \). Then the inequalities (see [10])
\[
\chi(A) \leq \beta(A) \leq \alpha(A) \leq 2 \chi(A)
\]
implies that \( \chi \) and \( \beta \) are equivalent to the Kuratowski measure of noncompactness \( \alpha \).

In the classical Lebesgue spaces \( L^p[0,1] \) (\( 1 \leq p < \infty \)), with the usual norm denoted by \( \| \cdot \|_p \), let \( \omega_p \) be the measure of noncompactness defined, for a bounded subset \( A \) of \( L^p[0,1] \), by the formula (see [1])
\[
\omega_p(A) = \limsup_{\delta \to 0} \sup_{f \in A} \max_{0 < h \leq \delta} \|f - f_h\|_p,
\]
where \( f_h \) denotes the Steklov function of \( f \). Then \( \omega_p \) is a measure of noncompactness on \( L^p[0,1] \) equivalent to the Kuratowski measure of noncompactness \( \alpha \).

**Remark 2.2.** With slight changes in the proof, Theorem 2.1 holds when \( \alpha \) is replaced by any measure of noncompactness \( \psi \) equivalent to \( \alpha \). Indeed, if \( T \) is an \((\alpha)(L)\)-set contractive mapping, then \( T \) is \((\psi)(\frac{c_2}{c_1}L)\)-set contractive for some \( 0 < c_1 \leq c_2 \).

We now focus our attention on \((\psi)k\)-set contractions of the unit ball without fixed points, for a measure of noncompactness \( \psi \) equivalent to \( \alpha \). For a given mapping \( G : B \to X \) we denote by \( G|_S \) the restriction of \( G \) to \( S \). We recall the following proposition proved in [11].

**Proposition 2.3 ([11, Proposition 3]).** Let \( k \geq 0 \), and \( F : B \to B \) be a \((\psi)k\)-set contraction without fixed points. Then there exists a \((\psi)k\)-set contraction \( G : B \to B \) without fixed points which satisfies \( G|_S = 0 \).

The next corollary improves a result obtained by M. Väth in [11, Corollary 2], stating the existence of a fixed point free mapping \( F \) of the unit ball whose measure of noncompactness, i.e. \( \inf \{ k \geq 0 : \gamma(F(A)) \leq k \gamma(A) \} \), is bounded by 2, where \( \gamma = \alpha, \chi \) or \( \beta \).

**Corollary 2.4.** Let \( X \) be an infinite-dimensional normed space and \( \psi \) a measure of noncompactness on \( X \) equivalent to \( \alpha \). Then for any given \( \varepsilon > 0 \), there exists a fixed point free \((\psi)(1 + \varepsilon)\)-set contraction \( F : B \to B \) with the additional property of vanishing on \( S \).
We observe that, as a consequence of Darbo’s fixed point theorem, whenever $X$ is an infinite-dimensional Banach space, if $F : B \to B$ is a $\psi$1-set contraction then $\eta(F) = 0$. Nevertheless, fixed point free ($\psi$)1-set contractions of the unit ball may exist in infinite-dimensional Banach spaces, and in [11] it is proved that for a large class of Banach spaces the best possible bound 1 is attained. It remains an open problem, posed by M. Väth, if the best possible bound 1 for fixed point free mappings is achieved in every infinite-dimensional Banach space $X$.

We now apply Corollary 2.4 to show that the measure of solvability $\nu(I)$ of the identity operator, in any infinite-dimensional Banach space, is equal to 1. The measure of solvability has been introduced in [4] (see also [11]), and has applications in problems of spectral theory for nonlinear operators. Let $B_r = \{ x \in X : \|x\| \leq r \}$ and $S_r = \{ x \in X : \|x\| = r \}$; then $B = B_1$ and $S = S_1$. Given $F : X \to X$ with $F(x) \neq 0$ for $x \neq 0$ define

$$
\nu_r(F) = \inf \{ k \geq 0 : \text{there exists an } \alpha k \text{-set contraction } G : B_r \to X \\
\text{with } G|_{S_r} = 0, \text{ and } F(x) \neq G(x) \text{ for all } x \in B_r \}.
$$

The measure of solvability $\nu(F)$ of $F$ is defined by setting

$$
\nu(F) = \inf \{ \nu_r(F) : r > 0 \}.
$$

In [11, Corollary 3] it is shown that in any infinite-dimensional Banach space $1 \leq \nu(I) \leq 2$. The author of [11] conjectures that $\nu(I) = 1$. We prove this conjecture:

**Theorem 2.5.** In any infinite-dimensional Banach space, $\nu(I) = 1$.

**Proof.** As pointed out in [11] the inequality $\nu(I) \geq 1$ follows from Rothe’s variant of Darbo’s fixed point theorem (see [3]).

On the other hand, let $r = 1$, and let $\varepsilon > 0$ be given. By Corollary 2.4 there exists a fixed point free $\alpha(1 + \varepsilon)$-set contraction $F_{\varepsilon} : B \to B$ such that $F_{\varepsilon}|_S = 0$. Then we have

$$
1 \leq \nu(I) \leq \nu_1(I) \leq 1 + \varepsilon.
$$

The theorem follows by the arbitrariness of $\varepsilon$. ■

Clearly the above theorem holds true when the measure of solvability $\nu(I)$ is defined with respect to any measure of noncompactness $\psi$ equivalent to $\alpha$, instead of $\alpha$ itself.

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