BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 52, No. 2, 2004

FUNCTIONAL ANALYSIS

A Note on the Measure of Solvability

by

D. CAPONETTI and G. TROMBETTA

Presented by Aleksander PEŁCZYŃSKI

Summary. Let X be an infinite-dimensional Banach space. The measure of solvability $\nu(I)$ of the identity operator I is equal to 1.

Let X be an infinite-dimensional normed space, and let ψ denote a measure of noncompactness on X. In this note we show that for any given $\varepsilon > 0$ there exists a $(\psi)(1+\varepsilon)$-set contractive mapping of a nonempty, convex and non-totally-bounded subset of X having positive minimal displacement.

Then the fact that in any infinite-dimensional Banach space for any given $\varepsilon > 0$ there exists a fixed point free $(\psi)(1+\varepsilon)$-set contraction of the unit ball implies that the measure of solvability $\nu(I)$ of the identity operator I is equal to 1. This result gives a positive answer to a question posed by M. Väth in [11].

1. Preliminaries. Let X be an infinite-dimensional normed space, and let $B = \{x \in X : \|x\| \le 1\}$ and $S = \{x \in X : \|x\| = 1\}$ be, respectively, the unit ball and unit sphere of X. Let C denote a set in X, and $T : C \to C$ be a given mapping. The *minimal displacement* $\eta(T)$ of T is the number defined by

$$\eta(T) = \inf\{\|Tx - x\| : x \in C\}.$$

A mapping T for which $\eta(T) > 0$ is *without approximate fixed points*. The first study of Lipschitz mappings without approximate fixed points was done by K. Goebel [6]. We refer the reader to [7] for a collection of results on this and related problems.

2000 *Mathematics Subject Classification*: 47H09, 47H10.

Key words and phrases: minimal displacement, measure of noncompactness, $(\psi)k$-set contraction, measure of solvability.

In [8] P. K. Lin and Y. Sternfeld, following the work of B. Nowak [9] and Y. Benyamini and Y. Sternfeld [2], proved

THEOREM 1.1. *Let X be an infinite-dimensional normed space. For any nonempty, convex and non-totally-bounded subset C of X there exists a Lipschitz mapping $T : C \to C$ for which $\eta(T) > 0$.*

2. $(\psi)k$-set contractions and the measure of solvability. A mapping ψ defined on the family of all bounded subsets of an infinite-dimensional normed space X is called a *measure of noncompactness* on X (see [1]) if it has the following properties:

(1) $\psi(A) = 0$ if and only if A is precompact.
(2) $\psi(\overline{\text{co}}\, A) = \psi(A)$, where $\overline{\text{co}}\, A$ denotes the closed convex hull of A.
(3) $\psi(A_1 \cup A_2) = \max\{\psi(A_1), \psi(A_2)\}$.
(4) $\psi(A_1 + A_2) \le \psi(A_1) + \psi(A_2)$.
(5) $\psi(\lambda A) = |\lambda|\psi(A)$ for every real number λ.

Let D be a nonempty subset of X. A continuous mapping $T : D \to X$ is called a *$(\psi)k$-set contraction* if for any bounded subset A of D,

$$\psi(T(A)) \le k\psi(A).$$

For a bounded subset A of X, the *Kuratowski measure of noncompactness* $\alpha(A)$ is the infimum of all $\varepsilon > 0$ such that A admits a finite covering by sets of diameter less than ε.

By combining Theorem 1.1 with a previous result of Furi and Martelli [5] we obtain the existence of an $(\alpha)(1 + \varepsilon)$-set contraction having a positive minimal displacement.

THEOREM 2.1. *Let X be an infinite-dimensional normed space. For any nonempty, convex and non-totally-bounded subset C of X and any given $\varepsilon > 0$ there exists an $(\alpha)(1 + \varepsilon)$-set contraction $F_\varepsilon : C \to C$ for which $\eta(F_\varepsilon) > 0$.*

Proof. Let $\varepsilon > 0$. We show that the set

$$S_\varepsilon = \{F : C \to C : F \text{ is an } (\alpha)(1+\varepsilon)\text{-set contraction and } \eta(F) > 0\}$$

is nonempty. By Theorem 1.1 there exists a Lipschitz mapping $F : C \to C$, with Lipschitz constant $L > 1$, such that $\eta(F) > 0$. Then F is an $(\alpha)L$-set contraction. If $\varepsilon \ge L - 1$, then $F \in S_\varepsilon$. If $\varepsilon < L - 1$, we define $F_\varepsilon : C \to C$ by setting

$$F_\varepsilon(x) = \left(1 - \frac{\varepsilon}{L-1}\right)x + \frac{\varepsilon}{L-1}\, F(x).$$

It is easy to check that F_ε is an $(\alpha)(1+\varepsilon)$-set contraction. Moreover $\eta(F_\varepsilon) = \frac{\varepsilon}{L-1}\eta(F)$, so that $\eta(F_\varepsilon) > 0$ and the proof is complete. ∎

We say that two measures of noncompactness φ and ψ are *equivalent* if there exist two positive constants c_1 and c_2 such that, for any bounded subset A of X,

$$c_1\psi(A) \leq \varphi(A) \leq c_2\psi(A).$$

For a bounded subset A of X, let $\chi(A)$ denote the *Hausdorff measure of noncompactness*, i.e. the infimum of all $\varepsilon > 0$ such that A has a finite ε-net in X, and $\beta(A)$ the *lattice measure of noncompactness*, i.e. the supremum of all $\varepsilon > 0$ such that A contains a sequence $\{x_n\}$ such that $\|x_n - x_k\| \geq \varepsilon$ for $n \neq k$. Then the inequalities (see [10])

$$\chi(A) \leq \beta(A) \leq \alpha(A) \leq 2\chi(A)$$

imply that χ and β are equivalent to the Kuratowski measure of noncompactness α.

In the classical Lebesgue spaces $L_p[0,1]$ ($1 \leq p < \infty$), with the usual norm denoted by $\|\cdot\|_p$, let ω_p be the measure of noncompactness defined, for a bounded subset A of $L_p[0,1]$, by the formula (see [1])

$$\omega_p(A) = \lim_{\delta\to 0} \sup_{f\in A} \max_{0<h\leq\delta} \|f - f_h\|_p,$$

where f_h denotes the Steklov function of f. Then ω_p is a measure of noncompactness on $L_p[0,1]$ equivalent to the Kuratowski measure of noncompactness α.

Remark 2.2. With slight changes in the proof, Theorem 2.1 holds when α is replaced by any measure of noncompactness ψ equivalent to α. Indeed, if T is an $(\alpha)(L)$-set contractive mapping, then T is $(\psi)(\frac{c_2}{c_1}L)$-set contractive for some $0 < c_1 \leq c_2$.

We now focus our attention on $(\psi)k$-set contractions of the unit ball without fixed points, for a measure of noncompactness ψ equivalent to α. For a given mapping $G : B \to X$ we denote by $G|_S$ the restriction of G to S. We recall the following proposition proved in [11].

Proposition 2.3 ([11, Proposition 3]). *Let $k \geq 0$, and $F : B \to B$ be a $(\psi)k$-set contraction without fixed points. Then there exists a $(\psi)k$-set contraction $G : B \to B$ without fixed points which satisfies $G|_S = 0$.*

The next corollary improves a result obtained by M. Väth in [11, Corollary 2], stating the existence of a fixed point free mapping F of the unit ball whose measure of noncompactness, i.e. $\inf\{k \geq 0 : \gamma(F(A)) \leq k\gamma(A)\}$, is bounded by 2, where $\gamma = \alpha$, χ or β.

Corollary 2.4. *Let X be an infinite-dimensional normed space and ψ a measure of noncompactness on X equivalent to α. Then for any given $\varepsilon > 0$, there exists a fixed point free $(\psi)(1+\varepsilon)$-set contraction $F : B \to B$ with the additional property of vanishing on S.*

We observe that, as a consequence of Darbo's fixed point theorem, whenever X is an infinite-dimensional Banach space, if $F : B \to B$ is a (ψ)1-set contraction then $\eta(F) = 0$. Nevertheless, fixed point free (ψ)1-set contractions of the unit ball may exist in infinite-dimensional Banach spaces, and in [11] it is proved that for a large class of Banach spaces the best possible bound 1 is attained. It remains an open problem, posed by M. Väth, if the best possible bound 1 for fixed point free mappings is achieved in every infinite-dimensional Banach space X.

We now apply Corollary 2.4 to show that the measure of solvability $\nu(I)$ of the identity operator, in any infinite-dimensional Banach space, is equal to 1. The measure of solvability has been introduced in [4] (see also [11]), and has applications in problems of spectral theory for nonlinear operators. Let $B_r = \{x \in X : \|x\| \le r\}$ and $S_r = \{x \in X : \|x\| = r\}$; then $B = B_1$ and $S = S_1$. Given $F : X \to X$ with $F(x) \neq 0$ for $x \neq 0$ define

$$\begin{aligned}\nu_r(F) = \inf\{k \ge 0 : &\text{ there exists an } (\alpha)k\text{-set contraction } G : B_r \to X \\ &\text{with } G|_{S_r} = 0, \text{ and } F(x) \neq G(x) \text{ for all } x \in B_r\}.\end{aligned}$$

The *measure of solvability* $\nu(F)$ of F is defined by setting

$$\nu(F) = \inf\{\nu_r(F) : r > 0\}.$$

In [11, Corollary 3] it is shown that in any infinite-dimensional Banach space $1 \le \nu(I) \le 2$. The author of [11] conjectures that $\nu(I) = 1$. We prove this conjecture:

THEOREM 2.5. *In any infinite-dimensional Banach space*, $\nu(I) = 1$.

Proof. As pointed out in [11] the inequality $\nu(I) \ge 1$ follows from Rothe's variant of Darbo's fixed point theorem (see [3]).

On the other hand, let $r = 1$, and let $\varepsilon > 0$ be given. By Corollary 2.4 there exists a fixed point free $(\alpha)(1 + \varepsilon)$-set contraction $F_\varepsilon : B \to B$ such that $F_\varepsilon|_S = 0$. Then we have

$$1 \le \nu(I) \le \nu_1(I) \le 1 + \varepsilon.$$

The theorem follows by the arbitrariness of ε. ■

Clearly the above theorem holds true when the measure of solvability $\nu(I)$ is defined with respect to any measure of noncompactness ψ equivalent to α, instead of α itself.

References

[1] R. R. Akhmerov, M. I. Kamenskiĭ, A. S. Potapov, A. E. Rodkina and B. N. Sadovskii, *Measures of Noncompactness and Condensing Operators*, Birkhäuser, Basel, 1992.

[2] Y. Benyamini and Y. Sternfeld, *Spheres in infinite-dimensional normed spaces are Lipschitz contractible*, Proc. Amer. Math. Soc. 88 (1983), 439–445.

[3] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, New York, 1985.
[4] W. Feng, *A new spectral theory for nonlinear operators and its applications*, Abstr. Appl. Anal. 2 (1997), 163–183.
[5] M. Furi and M. Martelli, *On α-Lipschitz retractions of the unit closed ball onto its boundary*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 57 (1974), 61–65.
[6] K. Goebel, *On the minimal displacement of points under Lipschitzian mappings*, Pacific J. Math. 45 (1973), 135–140.
[7] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Univ. Press, Cambridge, 1990.
[8] P. K. Lin and Y. Sternfeld, *Convex sets with the Lipschitz fixed point property are compact*, Proc. Amer. Math. Soc. 93 (1985), 633–639.
[9] B. Nowak, *On the Lipschitzian retraction of the unit ball in infinite-dimensional Banach spaces onto its boundary*, Bull. Acad. Polon. Sci. 27 (1979), 861–864.
[10] M. Väth, *Volterra and Integral Equations of Vector Functions*, Monogr. Textbooks Pure Applied Math. 224, Dekker, New York.
[11] —, *Fixed point free maps of a closed ball with small measures of noncompactness*, Collect. Math. 52 (2001), 101–116.

Diana Caponetti
Dipartimento di Matematica e Applicazioni
Università di Palermo
Via Archirafi 34
I-90123 Palermo, Italy
E-mail: d.caponetti@math.unipa.it

Giulio Trombetta
Dipartimento di Matematica
Università della Calabria
I-87036 Arcavacata di Rende (CS), Italy
E-mail: trombetta@unical.it

Received August 22, 2003;
received in final form February 12, 2004 (7350)