BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 52, No. 2, 2004

C^1-Stably Positively Expansive Maps

by

Kazuhiro SAKAI

Presented by Andrzej LASOTA

Summary. The notion of C^1-stably positively expansive differentiable maps on closed C^∞ manifolds is introduced, and it is proved that a differentiable map f is C^1-stably positively expansive if and only if f is expanding. Furthermore, for such maps, the ε-time dependent stability is shown. As a result, every expanding map is ε-time dependent stable.

Let X be a compact metric space with metric d, and let f be a continuous map of X onto itself. We say that f is *positively expansive* if there exists a constant $c > 0$ such that $d(f^i(x), f^i(y)) \le c$ $(x, y \in X)$ for all $i \ge 0$ implies $x = y$. Such a number c is called an *expansive constant*. This property (although not c) is independent of the metric. It is well known that every expanding map on a C^∞ closed manifold is positively expansive (see [1, 4, 7]).

As usual, a sequence $\{x_i\}_{i=0}^n$ $(1 \le n \le \infty)$ of points in X is called a δ-*pseudo-orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for $0 \le i \le n-1$. We say that f has the *shadowing property* if for every $\varepsilon > 0$, there is $\delta > 0$ such that for every δ-pseudo-orbit $\{x_i\}_{i=0}^\infty$, there exists $y \in X$ satisfying $d(f^i(y), x_i) < \varepsilon$ for all $i \ge 0$. This property is also independent of the metric. The shadowing property usually plays an important role in the modern stability theory of dynamical systems. It is also well known that every positively expansive open map has the shadowing property, and the set of all periodic points $P(f)$ of f is dense in the non-wandering set $\Omega(f)$ of f (see [4, 5, 6]).

Let M be a closed C^∞ manifold, and let d be the distance on M induced from a Riemannian metric $\|\cdot\|$ on TM. Let $C^1(M)$ denote the set of all C^1-differentiable maps on M endowed with the C^1-topology. We say that

2000 *Mathematics Subject Classification*: Primary 37C50, 37C75, 37D20.

Key words and phrases: C^1-stably positively expansive, positively expansive, expanding, pseudo-orbit, shadowing property, time dependent stable.

$f \in C^1(M)$ is *expanding* if there are constants $C > 0$ and $\mu > 1$ such that

$$\|D_x f^n(v)\| \geq C\mu^n \|v\|$$

for all $v \in TM$ and $n \geq 0$. Every expanding map on M is structurally stable (see [7]) and, as was stated, is positively expansive. But, there exists an example of a positively expansive map on the unit circle that is not expanding (see [1]).

In this paper, we introduce the notion of C^1-stably positively expansive differentiable maps on M, and prove that a differentiable map f is C^1-stably positively expansive if and only if f is expanding. Furthermore, we show the ε-time dependent stability for such maps, so that every expanding map is ε-time dependent stable.

For any sequence $\{g_i\}_{i=1}^n \subset C^1(M)$ $(1 \leq n \leq \infty)$ and for any $x \in M$, we set

$$x_0 = x, \qquad x_i = g_i \circ \cdots \circ g_1(x)$$

for all $1 \leq i \leq n$. Such a sequence of points $\{x_i\}_{i=0}^n$ $(x_0 = x)$ is called the $\{g_i\}_{i=1}^n$-*orbit of* x; we denote by $\mathcal{O}(\{g_i\}_{i=1}^n)$ the set of all $\{g_i\}_{i=1}^n$-orbits.

We say that $f \in C^1(M)$ is C^1-*stably positively expansive* (with constants ν and c) if there are constants $\nu > 0$ and $c > 0$ such that for any sequence $\{g_i\}_{i=1}^\infty \subset C^1(M)$ with $\varrho_{C^1}(f, g_i) < \nu$ $(i \in \mathbb{N})$ and for any $\{x_i\}_{i=0}^\infty, \{y_i\}_{i=0}^\infty \in \mathcal{O}(\{g_i\}_{i=1}^\infty)$, $d(x_i, y_i) \leq c$ for all $i \geq 0$ implies $x_0 = y_0$. Here ϱ_{C^1} is the usual C^1-metric on $C^1(M)$. Hereafter, we denote by $\mathcal{SPE}^1(M)$ the set of all C^1-stably positively expansive differentiable maps on M.

The first result of the present paper will be proved by combining an idea involved in [3] with the shadowing property.

THEOREM A. *The set $\mathcal{SPE}^1(M)$ is characterized as the set of all expanding maps.*

The notion of time dependent stability for diffeomorphisms is introduced in [3, p. 163] and it is proved that if a C^2-diffeomorphism satisfies both Axiom A and the strong transversality condition, then the map is time dependent stable. Conversely, it is also proved therein that if a C^1-diffeomorphism g is time dependent stable, then g satisfies both Axiom A and the strong transversality condition. In this paper, we introduce a slightly stronger version of the time dependent stability for differentiable maps as follows.

We say that $f \in C^1(M)$ is ε-*time dependent stable* if for any $\varepsilon > 0$, there exists a C^1-neighborhood $\mathcal{U}(f) \subset C^1(M)$ of f such that for any finite sequence $\{g_i\}_{i=1}^n \subset \mathcal{U}(f)$, there exists a homeomorphism $h : M \to M$ satisfying

$$f^n \circ h(x) = h \circ g_n \circ \cdots \circ g_1(x), \qquad d(h(x), x) < \varepsilon \qquad (x \in M).$$

The neighborhood $\mathcal{U}(f)$ is independent of n.

The following is also proved.

Theorem B. *Every $f \in \mathcal{SPE}^1(M)$ is ε-time dependent stable.*

Recall that every expanding map is structurally stable. As a corollary, we have the following stronger result for expanding maps. We remark that to prove the above result, we do not assume C^2-differentiability for the maps.

Corollary. *Every expanding map is ε-time dependent stable.*

1. Proof of Theorem A. Let us recall that M is a closed C^∞ manifold, and d is the distance on M induced from $\|\cdot\|$ on TM. First of all we show that every expanding map $f \in C^1(M)$ is in $\mathcal{SPE}^1(M)$.

Suppose that f is an expanding map; that is, there are constants $C > 0$ and $\mu > 1$ such that for any $v \in T_xM$ and $n \geq 0$, $\|D_xf(v)\| \geq C\mu^n\|v\|$ for all $x \in M$. Fix $m > 0$ such that $C\mu^m > 1$, and put $\lambda = C^{1/m}\mu > 1$.

Mather's trick. *Under the above notations, set*

$$\|v\|' = \sum_{i=0}^{m-1} \frac{1}{\lambda^i} \|D_xf^i(v)\|$$

for $v \in T_xM$ and $x \in M$. Then $\|D_xf(v)\|' \geq \lambda\|v\|'$ for any $v \in T_xM$ $(x \in M)$.

Lemma 1. *Let f, $\lambda > 1$ and $\|\cdot\|'$ be as above. Then there exist $\nu > 0$ and $\lambda > \lambda' > 1$ such that if $\varrho_{C^1}(f, g) < \nu$, then $\|D_xg(v)\|' \geq \lambda'\|v\|'$ for any $v \in T_xM$ $(x \in M)$.*

Proof. Pick $K > 1$ such that $\sup_{x\in M} \|D_xf\| < K$. Obviously

$$\|v\| \leq \|v\|' \leq \sum_{i=0}^{m-1} \left(\frac{K}{\lambda}\right)^i \|v\|$$

for all $v \in TM$ by construction. Fix any $\varepsilon > 0$ small enough so that $\lambda - \varepsilon \sum_{i=0}^{m-1} (K/\lambda)^i > 1$, and take $\nu > 0$ such that for $g \in C^1(M)$, $\varrho_{C^1}(f, g) < \nu$ implies

$$\big|\|Df^i(Df(v))\| - \|Df^i(Dg(v))\|\big| \leq \|Df^i(Df(v)) - Df^i(Dg(v))\| < \varepsilon K^i\|v\|$$

for all $v \in TM$ and $0 \leq i \leq m-1$. Thus for any $v \in TM$,

$$\begin{aligned}
\|Dg(v)\|' &= \|Dg(v)\| + \frac{1}{\lambda}\|Df(Dg(v))\| + \frac{1}{\lambda^2}\|Df^2(Dg(v))\| + \cdots \\
&\quad + \frac{1}{\lambda^{m-1}}\|Df^{m-1}(Dg(v))\| \\
&\geq (\|Df(v)\| - \varepsilon\|v\|) + \frac{1}{\lambda}(\|Df(Df(v))\| - \varepsilon K\|v\|) \\
&\quad + \frac{1}{\lambda^2}(\|Df^2(Df(v))\| - \varepsilon K^2\|v\|) + \cdots \\
&\quad + \frac{1}{\lambda^{m-1}}(\|Df^{m-1}(Df(v))\| - \varepsilon K^{m-1}\|v\|)
\end{aligned}$$

$$= \sum_{i=0}^{m-1} \frac{1}{\lambda^i} \|Df^{i+1}(v)\| - \varepsilon \sum_{i=0}^{m-1} \left(\frac{K}{\lambda}\right)^i \|v\|$$

$$= \|Df(v)\|' - \varepsilon \sum_{i=0}^{m-1} \left(\frac{K}{\lambda}\right)^i \|v\|$$

$$\geq \lambda\|v\|' - \varepsilon \sum_{i=0}^{m-1} \left(\frac{K}{\lambda}\right)^i \|v\|' = \lambda'\|v\|',$$

where $\lambda' = \lambda - \varepsilon \sum_{i=0}^{m-1} \left(K/\lambda\right)^i$. The lemma is proved. ∎

For the sake of simplicity, let $\|\cdot\| = \|\cdot\|'$ and $\lambda = \lambda'$ (notice that any Riemannian metric can be approximated by C^∞-metrics). We denote by $\exp : TM \to M$ the exponential map, and again by d the metric on M arising from the above new metric on TM. Then we have $\|D_x g(v)\| \geq \lambda\|v\|$ for any $v \in T_xM$ $(x \in M)$ provided that $\varrho_{C^1}(f,g) < \nu$ $(g \in C^1(M))$ by Lemma 1. Thus, there are constants $0 < \nu_1 < \nu$, $c > 0$ and $\lambda > \eta > 1$ such that

$$d(g(x), g(y)) \geq \eta d(x,y)$$

whenever $d(x,y) \leq c$ $(x, y \in M)$ if $\varrho_{C^1}(f,g) < \nu_1$ $(g \in C^1(M))$.

Indeed, for sufficiently small $\varepsilon > 0$ such that $\lambda - \varepsilon > 1$, there are $0 < \nu_1 < \nu$ and $c > 0$ such that for any $x \in M$,

$$\|\exp_{g(x)}^{-1} \circ g \circ \exp_x(v) - D_x g(v)\| \leq \varepsilon\|v\|$$

if $\|v\| < c$ whenever $\varrho_{C^1}(f,g) < \nu_1$ $(g \in C^1(M))$. Thus, if $d(x,y) = \|\exp_x^{-1} y\| \leq c$ $(x, y \in M)$, then

$$\|\exp_{g(x)}^{-1} \circ g \circ \exp_x(\exp_x^{-1} y)\| \geq \|D_x g(\exp_x^{-1} y)\| - \varepsilon\|\exp_x^{-1} y\|.$$

Thus $\|\exp_{g(x)}^{-1} g(y)\| \geq (\lambda - \varepsilon)\|\exp_x^{-1} y\|$. Since $d(g(x), g(y)) = \|\exp_{g(x)}^{-1} g(y)\|$, we see that $d(g(x), g(y)) \geq \eta d(x,y)$, where $\eta = \lambda - \varepsilon$.

Therefore, for any sequence $\{g_i\}_{i=1}^\infty \subset C^1(M)$ with $\varrho_{C^1}(f, g_i) < \nu_1$ $(i \in \mathbb{N})$ and for any $\{x_i\}_{i=0}^\infty, \{y_i\}_{i=0}^\infty \in \mathcal{O}(\{g_i\}_{i=1}^\infty)$, if $d(x_i, y_i) \leq c$ for all $i \geq 0$, then it can be easily seen that

$$d(x_0, y_0) \leq d(x_i, y_i)/\eta^i < c/\eta^i$$

for all $i \geq 0$. Thus we have $x_0 = y_0$, and so the expanding map f is C^1-stably positively expansive with constants ν_1 and c, that is, $f \in \mathcal{SPE}^1(M)$.

To prove the converse, we need some preparations. The following so-called Franks Lemma (see [2, Lemma 1.1]) will play an essential role several times in the proof. This is a fundamental C^1-perturbation lemma working well only for the C^1-topology.

LEMMA 2. *Let $f \in C^1(M)$. For any $\varepsilon_0 > 0$, there exists $\delta_0 > 0$ such that for a given finite set $\{x_1, \dots, x_N\} \subset M$, a neighborhood U of $\{x_1, \dots, x_N\}$ and linear maps $L_i : T_{x_i}M \to T_{x_{i+1}}M$, if $\|L_i - D_{x_i}f\| < \delta_0$ for all $1 \le i \le N$, then there exists $g \in C^1(M)$ with $\varrho_{C^1}(f, g) < \varepsilon_0$ such that $g(x) = f(x)$ if $x \in \{x_1, \dots, x_N\} \cup (M \setminus U)$ and $D_{x_i}g = L_i$ for all $1 \le i \le N$.*

Let $\{g_i\}_{i=1}^n \subset C^1(M)$ $(1 \le n < \infty)$ be any finite sequence, and let $\{x_i\}_{i=0}^n \in \mathcal{O}(\{g_i\}_{i=1}^n)$ be as defined before. We say that $\{x_i\}_{i=0}^n$ is a *periodic* $\{g_i\}_{i=1}^n$-*orbit* if $x_n = x_0$. Let $\{x_i\}_{i=0}^n$ be a periodic $\{g_i\}_{i=1}^n$-orbit. We say that $\{x_i\}_{i=0}^n$ is *expanding* if all the eigenvalues λ of

$$D_{x_0}(g_n \circ \cdots \circ g_1) = D_{x_{n-1}}g_n \circ \cdots \circ D_{x_0}g_1$$

satisfy $|\lambda| > 1$. Then the following is true.

PROPOSITION 1. *Suppose $f \in \mathcal{SPE}^1(M)$ with constants ν and c, and let $\{p_i\}_{i=0}^n$ be any periodic $\{g_i\}_{i=1}^n$-orbit. If $\varrho_{C^1}(g_i, f) < \nu/2$ for $1 \le i \le n$, then $\{p_i\}_{i=0}^n$ is expanding.*

Proof. Assuming that there is an eigenvalue λ of $D_{p_0}(g_n \circ \cdots \circ g_1)$ with $|\lambda| \le 1$, we shall derive a contradiction.

Let $\delta_0 > 0$ be given by Lemma 2 for $\varepsilon_0 = \nu/2$. Then, from the proof of the Franks Lemma 2 (see [2, p. 303]) there exists $\alpha > 0$ such that for any $i = 1, \dots, n$, there is $g_i' \in C^1(M)$ satisfying $\varrho_{C^1}(g_i', g_i) < \nu/2$ and

$$g_i'(x) = \begin{cases} \exp_{p_i} \circ D_{p_{i-1}}g_i \circ \exp_{p_{i-1}}^{-1}(x) & \text{if } x \in B_{\alpha/4}(p_{i-1}), \\ g_i(x) & \text{if } x \notin B_\alpha(p_{i-1}) \end{cases}$$

(we apply the lemma to the pair $\{p_{i-1}, D_{p_{i-1}}g_i\}$ for each $1 \le i \le n$ individually). Here $B_\alpha(x)$ $(x \in M)$ is the closed ball centered at x with radius $\alpha > 0$.

Note that

$$g'_{i|B_{\alpha/4}(p_{i-1})} = \exp_{p_i} \circ D_{p_{i-1}}g_i \circ \exp_{p_{i-1}}^{-1}$$

and $g_i'(p_{i-1}) = p_i$ for all i. Since there is an eigenvalue λ of $D_{p_0}(g_n \circ \cdots \circ g_1)$ with $|\lambda| \le 1$, for the number c, we can take $x \in B_{\alpha/4}(p_0) \setminus \{p_0\}$ and a sequence of points $\{x_j\}_{j=0}^\infty$ $(x_0 = x)$ such that

$$x_{nk+i} = g'_{nk+i} \circ \cdots \circ g'_1(x_0)$$

and $d(x_{nk+i-1}, p_{i-1}) \le c$ for all $1 \le i \le n$ and $k \ge 0$. Here $j = nk + i - 1$. This is a contradiction because $d(g'_{nk+i}, f) < \nu$ for all $1 \le i \le n$ and $k \ge 0$. ∎

REMARK. (1) By mimicking the argument used above, it is easily checked that if $f \in \mathcal{SPE}^1(M)$, then there are no singular points; that is, f is a local diffeomorphism.

(2) Since $f \in \mathcal{SPE}^1(M)$ is an open map, f has the shadowing property. Thus, $P(f)$ is dense in $\Omega(f)$ (cf. [4, 5, 6]). Furthermore, f is *topologically*

transitive, that is, f has a dense orbit (since M is connected, see [4, 6]). Hence, $M = \Omega(f)$.

The next technically important lemma was proved in [3] for a homeomorphism g on a compact metric space. Roughly speaking, by compactness, for every $\varepsilon > 0$ we can find a natural number $n_0 = n_0(\varepsilon)$ such that for any given g-orbit $\{x, g(x), g^2(x), \ldots, g^k(x)\}$ with arbitrarily large k, there exists a shortcut from x to $g^k(x)$ by a pseudo-orbit of g with length less than n_0. The proof is similar for continuous maps.

LEMMA 3 (cf. [3, Lemma 3]). *Let $g : (X, d) \to (X, d)$ be a continuous map on a compact metric space. For $\varepsilon > 0$, there exists an integer $n_0 = n_0(\varepsilon) > 0$ such that if $g^k(x) = y$ for some $k > 0$, then there is a sequence of points $\{x_i\}_{i=0}^{\overline{n}} \subset X$ with $x_0 = x$, $x_{\overline{n}} = y$, $d(g(x_i), x_{i+1}) < \varepsilon$ $(0 \le i \le \overline{n} - 1)$ and $\overline{n} \le n_0$.*

Let $f \in C^1(M)$ be a local diffeomorphism. A *backward orbit* $\mathbf{x} = \{x_{-i}\}_{i=0}^{\infty}$ of f is a sequence of points in M such that $f(x_{-i}) = x_{-i+1}$ for $i \ge 0$. Denote by $\mathcal{O}_f^-$ the set of backward orbits of f (notice that every local diffeomorphism maps M onto itself, since M is connected). The inverse map

$$(D_{x_{-i-1}} f)^{-1} : T_{x_{-i}} M \to T_{x_{-i-1}} M$$

can be defined for $i \ge 0$. Hence, for any $\mathbf{x} = \{x_{-i}\}_{i=0}^{\infty} \in \mathcal{O}_f^-$ and $v \in T_{x_0} M$, we put

$$D_{\mathbf{x}} f^{-n}(v) = (D_{x_{-n}} f)^{-1} \circ \cdots \circ (D_{x_{-1}} f)^{-1}(v) \quad \text{for } n \ge 1.$$

Let SM be the unit circle in TM, that is, $SM = \{v \in TM : \|v\| = 1\}$, and define a metric D on SM by

$$D(v_x, w_y) = \max\{d(x, y), \|v_x - A_{x,y}(w_y)\|\}$$

for $v_x \in T_x M \cap SM$ and $w_y \in T_y M \cap SM$ $(x, y \in M)$. Here $A_{x,y} : T_y M \to T_x M$ is the parallel transformation defined as usual. If we denote by $\widehat{f}$ the local diffeomorphism of SM defined by

$$\widehat{f}(v) = \frac{D_x f(v)}{\|D_x f(v)\|} \quad \text{for } v \in T_x M \cap SM \ (x \in M),$$

then $\widehat{f^n}(v) = D_x f^n(v)/\|D_x f^n(v)\|$ for all $n \ge 0$. The inverse along a given backward orbit can be defined as follows. For any $\mathbf{x} = \{x_{-i}\}_{i=0}^{\infty} \in \mathcal{O}_f^-$, we let

$$\widehat{f}_{\mathbf{x}}^{-1}(v) = \frac{(D_{x_{-1}} f)^{-1}(v)}{\|(D_{x_{-1}} f)^{-1}(v)\|} \quad \text{for } v \in T_{x_0} M \cap SM \quad \text{etc.}$$

A pair of points $(v, \{x_{-i}\}_{i=0}^{\infty}) \in SM \times \mathcal{O}_f^-$ $(v \in T_{x_0} M)$ will be called an *ε-non-wandering pair* of f (cf. [3]) if for any integer $N > 0$ and $0 < \delta < \varepsilon$, there are an integer $n > N$ and a point $w \in T_{x_0} M \cap SM$ such that

$$D(v, w) < \delta \quad \text{and} \quad D(v, \widehat{f}_{\mathbf{x}}^{-n}(w)) < \varepsilon.$$

Here $\mathbf{x} = \{x_{-i}\}_{i=0}^{\infty}$. The notion of ε-non-wandering pair is slightly stronger than the original one defined by [3, p. 168]. The set of all ε-non-wandering pairs will be denoted by $\Sigma_\varepsilon(f) \subset SM \times \mathcal{O}_f^-$.

Recall that every $f \in \mathcal{SPE}^1(M)$ is a local diffeomorphism. The following proposition will be proved by combining an idea involved in [3, Lemma 4] and the shadowing property.

PROPOSITION 2. *Let $f \in \mathcal{SPE}^1(M)$. Then there exist an integer $m_0 > 0$ and numbers $\varepsilon > 0$, $0 < \varrho < 1$ such that for any pair $(v, \{x_{-i}\}_{i=0}^{\infty}) \in \Sigma_\varepsilon(f)$, we have $\|D_{\mathbf{x}} f^{-n}(v)\| < \varrho^n$ provided that $n \geq m_0$. Here $\mathbf{x} = \{x_{-i}\}_{i=0}^{\infty}$.*

Proof. Let $f \in \mathcal{SPE}^1(M)$ with constants ν and c. Fix $\varepsilon_0 = \nu/2$ and choose a constant $K > 1$ such that $\sup_{x\in M} \|D_x f\| < K$.

Let $0 < \delta_0 = \delta_0(\varepsilon_0/2) < \varepsilon_0/2$ be as in Lemma 2, and take $0 < \varepsilon < \delta_0$ such that

$$(1) \qquad \frac{\varepsilon K}{1+\varepsilon} < \delta_0/2 \quad \text{and} \quad \|D_x f - A_{f(x),f(y)} \circ D_y f \circ (A_{x,y})^{-1}\| < \frac{\delta_0}{2}$$

if $d(x,y) < \varepsilon$ $(x, y \in M)$. Since f has the shadowing property (see Remark (2)), for ε, there exists $0 < \delta_1 = \delta_1(\varepsilon) < \varepsilon/2$ such that every δ_1-pseudo-orbit of f is ε-shadowed by some point. Let $n_0 = n_0(\delta_1) > 0$ be as in Lemma 3, and fix an integer $\ell > 0$ large enough so that

$$\frac{1}{\ell} \log K^{n_0} < \frac{1}{2} \log(1+\varepsilon).$$

Now, assuming that the proposition is false, we shall derive a contradiction. If the assertion is not true, then for all $n > 0$, there exist $(v_n, \{x_{-i}^n\}_{i=0}^{\infty}) \in \Sigma_{\delta_1}(f)$ and $m_n > n$ such that

$$\left(1 - \frac{1}{n}\right)^{m_n} \leq \|D_{\mathbf{x}_n} f^{-m_n}(v_n)\|.$$

Here $\mathbf{x}_n = \{x_{-i}^n\}_{i=0}^{\infty}$. Fix $n > 0$ such that $m_n > \max\{\ell, n_0\}$ and

$$-\frac{1}{2} \log(1+\varepsilon) < \frac{1}{m_n} \log \|D_{\mathbf{x}_n} f^{-m_n}(v_n)\|.$$

For simplicity, we denote $\mathbf{x}_n$ by $\mathbf{x} = \{x_{-i}\}_{i=0}^{\infty}$ (that is, $x_{-i} = x_{-i}^n$ for $i \geq 0$), $v' = v_n$ and $m = m_n$. By the definition of v', for the above m and any $0 < \delta < \delta_1$, there are $v_{x_0} \in T_{x_0}M \cap SM$ and $k > m$ such that $D(v_{x_0}, v') < \delta$ and $D(\widehat{f}_{\mathbf{x}}^{-k}(v_{x_0}), v') < \delta_1$. We may assume

$$(2) \qquad -\frac{1}{2} \log(1+\varepsilon) < \frac{1}{m} \log \|D_{\mathbf{x}} f^{-m}(v_{x_0})\|$$

choosing v_{x_0} sufficiently near to v'.

Let $v_{y_0} = \widehat{f}_{\mathbf{x}}^{-k}(v_{x_0})$ (and so $y_0 = x_{-k}$). By an application of Lemma 3 for $\widehat{f} : SM \to SM$ and $\widehat{f}^{k-m}(v_{y_0}) = \widehat{f}_{\mathbf{x}}^{-m}(v_{x_0})$, there exists a sequence $\{v_{y_j}\}_{j=0}^{\overline{n}} \subset SM$ $(\overline{n} \leq n_0)$ such that $v_{y_{\overline{n}}} = \widehat{f}_{\mathbf{x}}^{-m}(v_{x_0})$ and $D(\widehat{f}(v_{y_j}), v_{y_{j+1}}) < \delta_1$;

hence,

(3) $$d(f(y_j), y_{j+1}) < \delta_1$$

and

(4) $$\|\widehat{f}(v_{y_j}) - A_{f(y_j),y_{j+1}}(v_{y_{j+1}})\| < \delta_1$$

for all $0 \leq j \leq \overline{n} - 1$. Clearly, $D_{x_{-m}} f^m(v_{y_{\overline{n}}}) = \|D_{x_{-m}} f^m(v_{y_{\overline{n}}})\| v_{x_0}$ since $\widehat{f}^m(v_{y_{\overline{n}}}) = v_{x_0}$, and thus

$$\frac{1}{m} \log \|D_{x_{-m}} f^m(v_{y_{\overline{n}}})\| < \frac{1}{2} \log(1+\varepsilon)$$

by (2) (notice that $y_{\overline{n}} = x_{-m}$). By (3), it is easy to see that the sequence of points

(5) $$\{y_0, y_1, y_2, \dots, y_{\overline{n}-1}, x_{-m}, x_{-m+1}, \dots, x_{-1}\}$$

is a cyclic δ_1-pseudo-orbit of f since $y_{\overline{n}} = x_{-m}$ and $d(x_0, y_0) < \delta_1$. Since f has the shadowing property and is positively expansive ($2\delta_1 < c$), it is easy to see that there exists a point $p = f^q(p) \in P(f)$ ($q = m + \overline{n}$) whose f-orbit is ε-shadowing (5).

Next, let us perturb f along the periodic orbit of p. We define linear maps $L_{f^i(p)} : T_{f^i(p)}M \to T_{f^{i+1}(p)}M$ such that

$$\|L_{f^i(p)} - D_{f^i(p)} f\| < \delta_0 \quad \text{for } 0 \leq i \leq q-1$$

as follows. By (4), there exist linear isomorphisms $B_{f(y_j)} : T_{f(y_j)}M \to T_{f(y_j)}M$ such that

$$B_{f(y_j)}(\widehat{f}(v_{y_j})) = A_{f(y_j),y_{j+1}}(v_{y_{j+1}}) \quad \text{and} \quad \|B_{f(y_j)} - I\| < \varepsilon$$

for $0 \leq j \leq \overline{n} - 1$, where $I : T_{f(y_j)}M \to T_{f(y_j)}M$ is the identity map. Put

(6) $$L_{f^j(p)} = A_{f^{j+1}(p),f(y_j)} \circ B_{f(y_j)} \circ D_{y_j} f \circ (A_{f^j(p),y_j})^{-1}$$

for $0 \leq j \leq \overline{n} - 1$. For $\overline{n} \leq i \leq q-2$, put

(7) $$L_{f^i(p)} = A_{f^{i+1}(p),x_{-m-\overline{n}+i+1}} \circ D_{x_{-m-\overline{n}+i}} f \circ (A_{f^i(p),x_{-m-\overline{n}+i}})^{-1}.$$

Finally, we put

(8) $$L_{f^{q-1}(p)} = B_p \circ A_{f^{q-1}(p),x_{-1}} \circ D_{x_{-1}} f \circ (A_{f^{q-1}(p),x_{-1}})^{-1},$$

where $B_p : T_pM \to T_pM$ is a linear isomorphism satisfying $B_p(A_{p,x_0}(v_{x_0})) = A_{p,y_0}(v_{y_0})$ and $\|B_p - I\| < \varepsilon$. Then, by the choice of ε (see (1)), we can see that $\|L_{f^i(p)} - D_{f^i(p)} f\| < \delta_0$ for $0 \leq i \leq q-1$. Thus, by Lemma 2, for all $0 \leq i \leq q-1$ there exists $g_{i+1} \in C^1(M)$ with $\varrho_{C^1}(f, g_{i+1}) < \varepsilon_0/2$ such that

$$g_{i+1}(f^i(p)) = f^{i+1}(p) \quad \text{and} \quad D_{f^i(p)} g_{i+1} = L_{f^i(p)}.$$

Notice that $g_{i+1} = f$ outside a small neighborhood of $f^i(p)$.

If we put $\overline{v} = A_{p,y_0}(v_{y_0})$, then

$$D_p(g_q \circ \dots \circ g_1)(\overline{v}) = \gamma \overline{v}$$

for some $\gamma \in \mathbb{R}$ by construction. Hence $|\gamma| > 1$ (see Proposition 1). Furthermore, from (6)–(8),

$$\begin{aligned}
&\frac{1}{q}\log\|D_p(g_q\circ\cdots\circ g_1)(\overline{v})\| \\
&\quad= \frac{1}{q}\log\|D_{y_0}f(v_{y_0})\|\cdots\|D_{y_{\overline{n}-1}}f(v_{y_{\overline{n}-1}})\|\cdot\|D_{x_{-m}}f^m(v_{y_{\overline{n}}})\| \\
&\quad\le \frac{1}{q}\log K^{\overline{n}}\cdot\|D_{x_{-m}}f^m(v_{y_{\overline{n}}})\| = \frac{1}{q}\log K^{\overline{n}} + \frac{1}{q}\log\|D_{x_{-m}}f^m(v_{y_{\overline{n}}})\| \\
&\quad< \frac{1}{2}\log(1+\varepsilon) + \frac{1}{2}\log(1+\varepsilon) = \log(1+\varepsilon)
\end{aligned}$$

since

$$\frac{1}{m}\log\|D_{x_{-m}}f^m(v_{y_{\overline{n}}})\| < \frac{1}{2}\log(1+\varepsilon).$$

Pick γ' such that

$$\log(1+\gamma') = \frac{1}{q}\log\|D_p(g_q\circ\cdots\circ g_1)(\overline{v})\|.$$

Then $0 < \gamma' < \varepsilon$. If we define a new sequence of linear isomorphisms $L'_{f^i(p)} : T_{f^i(p)}M \to T_{f^{i+1}(p)}M$ by

$$L'_{f^i(p)} = (1+\gamma')^{-1}D_{f^i(p)}g_{i+1} \qquad \text{for } 0 \le i \le q-1,$$

then, by (1) and Lemma 2, for any $0 \le i \le q-1$ there exists $g'_{i+1} \in C^1(M)$ such that $\varrho_{C^1}(g'_{i+1}, g_{i+1}) < \varepsilon_0/2$, $g'_{i+1}(f^i(p)) = f^{i+1}(p)$ and

$$D_{f^i(p)}g'_{i+1} = L'_{f^i(p)}.$$

Hence

$$\begin{aligned}
\|D_p(g'_q\circ\cdots\circ g'_1)(\overline{v})\| &= (1+\gamma')^{-q}\|D_p(g_q\circ\cdots\circ g_1)(\overline{v})\| \\
&= (1+\gamma')^{-q}(1+\gamma')^q = 1
\end{aligned}$$

so that

$$D_p(g'_q\circ\cdots\circ g'_1)(\overline{v}) = \overline{v}.$$

Thus, there exists a non-expanding $\{g'_i\}_{i=1}^q$-periodic orbit starting from p. However, since the C^1-distance from g'_i to f is less than ε_0 for all i, it follows from Proposition 1 that $\{g'_i\}_{i=1}^q$ should have only expanding periodic orbits. This is a contradiction. ∎

We are in a position to prove the opposite direction of Theorem A; that is, if $f \in \mathcal{SPE}^1(M)$, then f is expanding. The proof follows from Proposition 2 by modifying the arguments used in the proof of [3, Lemma 5].

Suppose that $f \in \mathcal{SPE}^1(M)$, and let η, m_0, ε and ϱ be as in Proposition 2. Take $m_1 \ge m_0$ and $0 < \lambda < 1$ such that $\varrho < \lambda$ and $\varepsilon/12 > (\varrho/\lambda)^{m_1}$. To get the conclusion it is enough to prove that for all periodic backward orbits $\mathbf{p} = \{p_{-i}\}_{i=0}^{\infty}$ $(\subset P(f))$,

$$\|D_{\mathbf{p}} f^{-m_1}(v)\| \le \lambda^{m_1}\|v\| \quad \text{for } v \in T_pM. \tag{9}$$

Indeed, if (9) is established, then we can see that

$$\|D_x f^{m_1}(v)\| \ge \mu\|v\| \quad \text{for } v \in T_xM$$

for all $x \in M$, where $\mu = \lambda^{-m_1} > 1$ (see Remark (2)). Thus f is expanding by making use of Mather's trick.

If (9) is false, then there are $p \in P(f)$ $(f^k(p) = p)$ and $v \in T_pM$ $(\|v\| = 1)$ such that $\|D_{\mathbf{p}} f^{-m_1}(v)\| > \lambda^{m_1}$. Thus, examining the Jordan canonical form of $D_p f^k : T_pM \to T_pM$, we can find $w \in T_pM$ $(\|w\| = 1)$ such that

$$\begin{aligned} \|\widehat{f}_{\mathbf{p}}^{-kn}(v) - \widehat{f}_{\mathbf{p}}^{-kn}(w)\| \to 0 \quad &\text{as } n \to \infty, \\ \widehat{f}_{\mathbf{p}}^{-kn_i}(w) \to w \quad &\text{as } i \to \infty \end{aligned} \tag{10}$$

for some increasing subsequence $\{n_i\}_{i=0}^{\infty} \subset \mathbb{N}$ (see [3, p. 171]).

Now, let $u' = w + \varepsilon v/4$ and set $u = u'/\|u'\|$. Then $\lim_{i\to\infty} \widehat{f}_{\mathbf{p}}^{-kn_i}(u) = w$ by (10). Since $\|u - w\| < \varepsilon/2$, both $(u, \mathbf{p})$ and $(w, \mathbf{p})$ are elements of $\Sigma_\varepsilon(f)$. Thus, by Proposition 2 we see that

$$\|D_{\mathbf{p}} f^{-m_1}(u)\| \le \varrho^{m_1}\|u\|, \quad \|D_{\mathbf{p}} f^{-m_1}(w)\| \le \varrho^{m_1}\|w\|.$$

On the other hand, since we may suppose that $0 < \varepsilon < 1/2$,

$$\|D_{\mathbf{p}} f^{-m_1}(u)\| \ge \frac{2}{3}\,\|D_{\mathbf{p}} f^{-m_1}(u')\|.$$

Thus

$$\begin{aligned} \|D_{\mathbf{p}} f^{-m_1}(u)\| &\ge \frac{2}{3}\left\|\frac{\varepsilon}{4} D_{\mathbf{p}} f^{-m_1}(v) + D_{\mathbf{p}} f^{-m_1}(w)\right\| > \frac{2}{3}\left(\frac{\varepsilon}{4}\lambda^{m_1} - \varrho^{m_1}\right) \\ &> \frac{2}{3}\,(3\varrho^{m_1} - \varrho^{m_1}) = \frac{4}{3}\varrho^{m_1} > \varrho^{m_1}. \end{aligned}$$

This is a contradiction, and so (9) is proved. ■

2. Proof of Theorem B. Let $f \in \mathcal{SPE}^1(M)$ with constants ν and c. Recall that f is positively expansive and has the shadowing property; that is, for every $\varepsilon > 0$, there is $\delta > 0$ such that for every δ-pseudo-orbit $\{x_i\}_{i=0}^{\infty}$, there exists $y \in M$ satisfying $d(f^i(y), x_i) < \varepsilon$ for all $i \ge 0$. For any $0 < \varepsilon < c/3$, let $0 < \delta < \min\{\varepsilon, \nu\}$ be as in the shadowing property of f. We may assume $(L+1)\varepsilon + \delta < c$. Here $L \ge 1$ is a constant such that if $\varrho_{C^1}(f, g) < \delta$ $(g \in C^1(M))$, then

$$d(g(x), g(y)) \le L d(x, y) \quad (x, y \in M).$$

It is easy to see that for any given sequence $\{g_i\}_{i=1}^{\infty} \subset C^1(M)$ and any $\{x_i\}_{i=0}^{\infty} \in \mathcal{O}(\{g_i\}_{i=1}^{\infty})$, if $\varrho_{C^1}(f, g_i) < \delta$ for i, then the $\{g_i\}_{i=1}^{\infty}$-orbit $\{x_i\}_{i=0}^{\infty}$ ($x_0 = x$) of x is a δ-pseudo-orbit of f. Indeed, $d(f(x_i), x_{i+1}) = d(f(x_i), g_{i+1}(x_i)) < \delta$ for all i.

Let $\{g_i\}_{i=1}^{n} \subset C^1(M)$ be a given finite sequence such that $\varrho_{C^1}(f, g_i) < \delta$ for all i. Denote by $\{g_i\}_{i=1}^{\infty}$ the cyclic infinite sequence composed of $\{g_i\}_{i=1}^{n}$; that is, $g_{jn+i} = g_i$ for all $j \geq 0$ and $1 \leq i \leq n$. Then, modifying the method used in [8] we obtain the next lemma.

Lemma 4. *Under the above notations, there exists a map $h : M \to M$ such that*

$$f^n(h(x)) = h(g_n \circ \cdots \circ g_1(x)) \quad \text{and} \quad d(h(x), x) < \varepsilon$$

for all $x \in M$.

Proof. Let $\{g_i\}_{i=1}^{\infty}$ be as above. Then, since any $\{x_i\}_{i=0}^{\infty} \in \mathcal{O}(\{g_i\}_{i=1}^{\infty})$ ($x_0 = x$) is a δ-pseudo-orbit of f, there exists $y \in M$ such that $d(f^i(y), x_i) < \varepsilon$ for all $i \geq 0$. If there exists another point $y' \in M$ ($y \neq y'$) satisfying $d(f^i(y'), x_i) < \varepsilon$ for all i, then

$$d(f^i(y), f^i(y')) \leq d(f^i(y), x_i) + d(x_i, f^i(y')) < 2\varepsilon$$

for all i. Thus $y = y'$ since f is positively expansive with constant c. This is a contradiction.

Denote the point y by $h(x)$. Then we have

$$d(f^{jn+i}(h(x)), g_i \circ \cdots \circ g_1 \circ (g_n \circ \cdots \circ g_1)^j(x)) = d(f^{jn+i}(h(x)), x_{jn+i}) < \varepsilon, \tag{11}$$

and $d(h(x), x) < \varepsilon$ for all $x \in M$ (recall that $d(f^i(h(x)), x_i) = d(f^i(y), x_i) < \varepsilon$ for all i).

If we replace x with $x_n = g_n \circ \cdots \circ g_1(x)$ in (11), then

$$d(f^{jn+i}(h(x_n)), g_i \circ \cdots \circ g_1(x_{n(j+1)})) < \varepsilon. \tag{12}$$

Here $x_{n(j+1)} = (g_n \circ \cdots \circ g_1)^{j+1}(x)$.

On the other hand, we have

$$\begin{aligned} &d(f^{jn+i}(f^n(h(x))), g_i \circ \cdots \circ g_1(x_{n(j+1)})) \\ &\qquad \leq d(f(f^{jn+i-1}(f^n(h(x)))), f(g_{i-1} \circ \cdots \circ g_1(x_{n(j+1)}))) \\ &\qquad\quad + d(f(g_{i-1} \circ \cdots \circ g_1(x_{n(j+1)})), g_i(g_{i-1} \circ \cdots \circ g_1(x_{n(j+1)}))) \\ &\qquad \leq Ld(f^{jn+i-1}(f^n(h(x))), g_{i-1} \circ \cdots \circ g_1(x_{n(j+1)})) + \delta \\ &\qquad \leq Ld(f^{(j+1)n+i-1}(h(x)), x_{n(j+1)+i-1}) + \delta < L\varepsilon + \delta \end{aligned}$$

by (11). Here $x_{n(j+1)+i-1} = g_{i-1} \circ \cdots \circ g_1(x_{n(j+1)})$.

Thus, combining this with (12) we have

$$d(f^{jn+i}(h(g_n \circ \cdots \circ g_1(x))), f^{jn+i}(f^n(h(x)))) < (L+1)\varepsilon + \delta < c$$

for all $j \geq 0$ and $1 \leq i \leq n$. Therefore, $f^n(h(x)) = h(g_n \circ \cdots \circ g_1(x))$ is obtained for all $x \in M$ since f is positively expansive. ∎

To complete the proof of Theorem B, it only remains to show that $h : M \to M$ is a homeomorphism. Since f is positively expansive, it is easy to see that for any $\alpha > 0$ there exists an integer $N > 0$ such that $d(f^i(x), f^i(y)) \leq c$ $(x, y \in M)$ for all $0 \leq i \leq N$ implies $d(x, y) < \alpha$.

Recall that there is a constant $L \geq 1$ such that

$$d(g_i(v), g_i(w)) \leq Ld(v, w) \qquad (v, w \in M)$$

since $\varrho_{C^1}(f, g_i) < \delta$ for i. Thus, for the above N, if we choose $\beta > 0$ small enough, then $d(x, y) < \beta$ $(x, y \in M)$ implies $d(x_i, y_i) < c/3$ for $0 \leq i \leq N$. Here $\{x_i\}_{i=0}^{\infty}$ $(x_0 = x)$ and $\{y_i\}_{i=0}^{\infty}$ $(y_0 = y)$ are $\{g_i\}_{i=1}^{\infty}$-orbits. Hence, we have

$$\begin{aligned} d(f^i(x), f^i(y)) &\leq d(f^i(x), x_i) + d(x_i, y_i) + d(y_i, f^i(y)) \\ &< c/3 + c/3 + c/3 = c \end{aligned}$$

for all $0 \leq i \leq N$, and so $d(h(x), h(y)) < \alpha$. Thus the map h is continuous.

To show the injectivity of h, we assume $h(x) = h(y)$ $(x, y \in M)$. Then $d(x_i, y_i) \leq c$ for any i since

$$d(f^i(h(x)), x_i) < \varepsilon \quad \text{and} \quad d(f^i(h(y)), y_i) < \varepsilon$$

for any i. Here $\{x_i\}_{i=0}^{\infty}$ $(x_0 = x)$ and $\{y_i\}_{i=0}^{\infty}$ $(y_0 = y)$ are $\{g_i\}_{i=1}^{\infty}$-orbits. Thus $x = y$ since f is C^1-stably positively expansive. ∎

Acknowledgements. The author gratefully acknowledges the School of Mathematics, Peking University for their warm hospitality during the preparation of this paper, and also thanks Dr. Feliks Przytycki for his kind assistance and suggestions.

References

[1] E. M. Coven and W. L. Reddy, *Positively expansive maps of compact manifolds*, in: Lecture Notes in Math. 819, Springer, Berlin, 1980, 96–110.

[2] J. Franks, *Necessary conditions for stability of diffeomorphisms*, Trans. Amer. Math. Soc. 158 (1971), 301–308.

[3] —, *Time dependent stable diffeomorphisms*, Invent. Math. 24 (1974), 163–172.

[4] D. Ruelle, *Thermodynamic Formalism*, Encyclopedia Math. Appl. 5, Addison-Wesley, Reading, MA, 1978.

[5] K. Sakai, *Periodic points of positively expansive maps*, Proc. Amer. Math. Soc. 94 (1985), 531–534.

[6] —, *Various shadowing properties for positively expansive maps*, Topology Appl. 131 (2003), 15–31.

[7] M. Shub, *Endomorphisms of compact differentiable manifolds*, Amer. J. Math. 91 (1969), 175–199.

[8] P. Walters, *On the pseudo orbit tracing property and its relationship to stability*, in: Lecture Notes in Math. 668, Springer, Berlin, 1978, 231–244.

Kazuhiro Sakai
Department of Mathematics
Utsunomiya University
Utsunomiya 321-8505, Japan
E-mail: kazsakai@cc.utsunomiya-u.ac.jp

Received November 4, 2003;
received in final form March 1, 2004 (7356)