Summary. We prove that if \( f : \mathbb{Z}^d \to \mathbb{R} \) is harmonic and there exists a polynomial \( W : \mathbb{Z}^d \to \mathbb{R} \) such that \( f + W \) is nonnegative, then \( f \) is a polynomial.

1. Introduction. Harmonic functions on the integer lattice are closely related to lattice random walks and have been studied by many authors; an introduction and detailed references can be found in a modern monograph by Woess [8]. Many different methods have been successfully applied, including the extreme point theory [2] and martingale approach [4]. The present paper grew out of the author’s bachelor thesis [7] which extended results and methods of Darkiewicz [3]. A similar result for sublinear functions on compactly generated groups of polynomial growth has been obtained by Hebisch and Saloff-Coste [6, Theorem 6.1] by using Gaussian estimates for iterated kernels of random walks.

2. Preliminaries and main results. Let \( d \in \mathbb{N} \) and let \( (e_i)_{i=1}^d \) be the standard orthonormal basis for \( \mathbb{R}^d \). A function \( f : \mathbb{Z}^d \to \mathbb{R} \) is called harmonic if it has the mean value property,

\[
f(x) = \frac{1}{2d} \sum_{i=1}^d [f(x + e_i) + f(x - e_i)] \quad \text{for all } x \in \mathbb{Z}^d.
\]

We say that \( f : \mathbb{Z}^d \to \mathbb{R} \) is a polynomial if there exists a polynomial \( F : \mathbb{R}^d \to \mathbb{R} \) such that \( f = F|_{\mathbb{Z}^d} \).

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For \( t \geq 0 \) let \( Y_1^{(t)}, \ldots, Y_d^{(t)}, Z_1^{(t)}, \ldots, Z_d^{(t)} \) be independent Poisson random variables with mean \( t \).

We will use the following notation:

- \( \|x\|_p = (\sum_{i=1}^d |x_i|^p)^{1/p} \) for \( p \in [1, \infty) \) and \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \),
- \( X_i^{(t)} = Y_i^{(t)} - Z_i^{(t)} \) for \( i = 1, \ldots, d \), \( X^{(t)} = \sum_{i=1}^d X_i^{(t)} e_i \),
- \( g_t(l) = \mathbb{P}(Y_1^{(t)} - Z_1^{(t)} = l) \) for \( l \in \mathbb{Z} \),
- \( G_t(k) = \prod_{i=1}^m g_t(k_i) \) for \( k = (k_1, \ldots, k_m) \in \mathbb{Z}^m \),
- \( q_t(l) = \mathbb{P}(Y_1^{(t)} = l) = e^{-t} t^l / l! \) for \( l \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \).

Note that if \( t \in \mathbb{N} \) then \( q_t(0) \leq q_t(1) \leq \cdots \leq q_t(t-1) = q_t(t) \geq q_t(t+1) \geq q_t(t+2) \geq \cdots \).

We consider the space of all exponentially bounded functions,

\[ \mathcal{L} = \{ f : \mathbb{Z}^d \to \mathbb{R} \mid \exists c_1, c_2 > 0 \ | f(x) | \leq c_1 e^{c_2 \| x \|_1} \text{ for all } x \in \mathbb{Z}^d \}, \]

and define a family of operators \((\mathcal{P}_t)_{t \geq 0}\), \( \mathcal{P}_t : \mathcal{L} \to \mathcal{L} \), by

\[
\mathcal{P}_t(f)(x) = \mathbb{E} f(x + X^{(t)}). 
\]

**Theorem 2.1.** The family \((\mathcal{P}_t)_{t \geq 0}\) is a well-defined semigroup of operators. Moreover, harmonic functions belonging to \( \mathcal{L} \) lie in the domain \( \mathcal{D}_A \) of the infinitesimal generator \( A \) of the semigroup \((\mathcal{P}_t)_{t \geq 0}\), and for \( f \in \mathcal{D}_A \) we have

\[
(Af)(x) = \frac{d}{dt} \mathcal{P}_t(f)(x) \bigg|_{t=0} = \sum_{k \in \mathbb{Z}^d : \| k \|_1 = 1} f(x + k) - 2d f(x).
\]

In particular, if \( f \in \mathcal{L} \) is harmonic, then \((Af)(x) = 0 \) for all \( x \in \mathbb{Z}^d \), and so

\[
\mathcal{P}_t(f)(x) = \sum_{k \in \mathbb{Z}^d} G_t(k) f(x + k) = f(x) \quad \text{for all } x \in \mathbb{Z}^d.
\]

**Proof.** If \( f \in \mathcal{L} \), then there exist \( c_1, c_2, \tilde{c}_1(t) > 0 \) such that

\[
|\mathbb{E} f(x + X^{(t)})| \leq c_1 \mathbb{E} e^{c_2 \| x + X^{(t)} \|_1} \leq c_1 e^{c_2 \| x \|_1} (\mathbb{E} e^{c_2 \| X_1^{(t)} \|_1})^d = \tilde{c}_1(t) e^{c_2 \| x \|_1},
\]

so \( \mathcal{P}_t(f) \in \mathcal{L} \). Observe that \( \mathcal{P}_0(f) = f \). If \( s, t \geq 0 \) and \( \tilde{X}^{(s)} \) is a copy of \( X^{(s)} \) independent of \( X^{(t)} \), then \( X^{(t)} + \tilde{X}^{(s)} \sim X^{(t+s)} \), so one can easily check that \((\mathcal{P}_t)_{t \geq 0}\) is a semigroup. The last part is a simple calculation. \( \blacksquare \)

**Lemma 2.2.** If \((r_i)_{i \in \mathbb{N}}\) are independent \( \pm 1 \) symmetric Bernoulli random variables and \( M \) is a Poisson variable with mean \( 4t \), such that \( M \) and \((r_i)_{i \in \mathbb{N}}\) are independent, then

\[
X_1^{(t)} \sim \frac{1}{2} (r_1 + \cdots + r_{2M}).
\]
Moreover, for $l \in \mathbb{N}_0$,
\[
g_t(l) = g_t(-l) = \sum_{n=0}^{\infty} e^{-4t} \frac{t^n}{n!} \left( \frac{2n}{n + l} \right),
\]
so if $l_1, l_2 \in \mathbb{Z}$ and $0 \leq l_1 \leq l_2$, then
\[
g_t(l_1) \geq g_t(l_2).
\]

**Proof.** To prove the first statement, it is enough to show that the characteristic functions of both random variables are equal. We have
\[
\phi_{X_1^{(t)}}(x) = \phi_{Y_1^{(t)}}(x) \phi_{Z_1^{(t)}}(-x) = e^{t(e^{ix} - 1)} e^{t(e^{-ix} - 1)} = e^{t(2 \cos x - 2)}
\]
and
\[
\phi_{(r_1 + \cdots + r_{2M})/2}(x) = \sum_{n=0}^{\infty} \mathbb{P}(M = n) \phi_{(r_1 + \cdots + r_{2n})/2}(x)
\]
\[
= \sum_{n=0}^{\infty} e^{-4t} \frac{(4t)^n}{n!} \left( \phi_{r_1/2}(x) \right)^2 = e^{4t(-1 + \cos^2(x/2))} = e^{-4t \sin^2(x/2)},
\]
as
\[
\phi_{r_1/2}(x) = \phi_{r_1}(x/2) = \frac{1}{2} (e^{-ix/2} + e^{ix/2}) = \cos(x/2).
\]

To finish the proof observe that for $l \in \mathbb{N}_0$ we have
\[
g_t(l) = \mathbb{P}\left( \frac{1}{2} (r_1 + \cdots + r_{2M}) = l \right) = \sum_{n=0}^{\infty} \mathbb{P}(M = n) \mathbb{P}(r_1 + \cdots + r_{2n} = 2l)
\]
\[
= \sum_{n=0}^{\infty} e^{-4t} \frac{(4t)^n}{n!} \frac{1}{2^{2n}} \left( \frac{2n}{n + l} \right) = \sum_{n=0}^{\infty} e^{-4t} \frac{t^n}{n!} \left( \frac{2n}{n + l} \right)
\]
and $\left( \frac{2n}{n + l_1} \right) \geq \left( \frac{2n}{n + l_2} \right)$ for $0 \leq l_1 \leq l_2$. ■

**Lemma 2.3.** For every $\varepsilon > 0$ and $d \in \mathbb{N}$ we can find $0 < s < t$ such that
\[
g_t(k) \geq (1 - \varepsilon) g_s(k - 1) \quad \text{for } k \in \mathbb{Z}
\]
and
\[
G_t(k) \geq (1 - \varepsilon) G_s(k - e_1) \quad \text{for } k \in \mathbb{Z}^d.
\]

**Proof.** If the first inequality holds for $k = 1, 2, \ldots, m$ then it holds for $k = 0, -1, \ldots, -m$. Indeed, for $k = -1, -2, \ldots, -m$ we have (see Lemma 2.2)
The function $N_{234k}$ satisfies

$$
\mathbb{P}(X_1^{(t)} = k) = \mathbb{P}(X_1^{(t)} = -k) \geq (1 - \varepsilon)\mathbb{P}(X_1^{(s)} = -k - 1)
$$

$$
= (1 - \varepsilon)\mathbb{P}(X_1^{(s)} = k + 1) \geq (1 - \varepsilon)\mathbb{P}(X_1^{(s)} = k - 1)
$$

and

$$
\mathbb{P}(X_1^{(t)} = 0) \geq \mathbb{P}(X_1^{(t)} = 1) \geq (1 - \varepsilon)\mathbb{P}(X_1^{(s)} = 0) \geq (1 - \varepsilon)\mathbb{P}(X_1^{(s)} = -1).
$$

For $k \geq 1$ we have

$$
\mathbb{P}(X_t = k) = \sum_{l=0}^{\infty} \mathbb{P}(Y_t = l + k)\mathbb{P}(Z_t = l) = \sum_{l=0}^{\infty} e^{-2t} \frac{t^{2l+k}}{l!(l+k)!},
$$

$$
\mathbb{P}(X_s = k - 1) = \sum_{l=0}^{\infty} e^{-2s} \frac{s^{2l+k-1}}{l!(l+k-1)!}.
$$

Let $s > 1$ be such that $\sqrt{s} \in \mathbb{N}$ and set $t = s + \sqrt{s}$. We then have

$$
\mathbb{P}(X_t = k) \geq \sum_{l=0}^{\infty} e^{-2t} \frac{t^{2l+k}}{l!(l+k)!} = \sum_{l=0}^{\infty} e^{-2s} \frac{s^{2l+k-1}}{l!(l+k-1)!}.
$$

It is enough to prove that

$$
\inf_{k \geq 1, l \geq 0} \left( e^{-2t} \frac{t^{2(l+\sqrt{s})+k}}{(l+\sqrt{s})!(l+\sqrt{s}+k)!} / e^{-2s} \frac{s^{2l+k-1}}{l!(l+k-1)!} \right) \xrightarrow{s \to \infty} 1.
$$

We consider the expression

$$
p_{l,k}(s) := e^{2(s-t)}st^{2\sqrt{s}} \left( \frac{t}{s} \right)^{l+k} \left( \frac{l+k-1}{l+\sqrt{s}+k} \right) \left( \frac{t}{s} \right)^l \frac{l!}{(l+\sqrt{s})!}.
$$

The function $\mathbb{N} \ni n \mapsto (t/s)^n(n-1)!/(n+\sqrt{s})!$ has its minimum at $n = s(1 + \sqrt{s})/(t - s) = t$. Similarly, the function $\mathbb{N}_0 \ni n \mapsto (t/s)^n n!/(n+\sqrt{s})!$ has its minimum at $n = s\sqrt{s}/(t-s) = s$. Therefore

$$
p_{l,k}(s) \geq p_{s,t-s}(s) = e^{2(s-t)}st^{2\sqrt{s}} \left( \frac{t}{s} \right)^{t+s} \left( \frac{t}{s} \right)^{t} \frac{t!}{(t+\sqrt{s})!} \frac{1}{(s+\sqrt{s})!}.
$$

Using Stirling’s formula we get $s!/(s + 2\sqrt{s})! \approx e^{2\sqrt{s}s}/(s + 2\sqrt{s})^{s+2\sqrt{s}}$ as $s \to \infty$, hence we arrive at

$$
\inf_{k \geq 1, l \geq 0} p_{l,k}(s) \approx s^{-s-\sqrt{s}+1}(s + \sqrt{s})^{2s+3\sqrt{s}-1}(s + 2\sqrt{s})^{-s-2\sqrt{s}}
$$

$$
= \sqrt{s^{-2s-2\sqrt{s}+2+2s+3\sqrt{s}-1}}(1 + \sqrt{s})^{-\sqrt{s}-1}(1 + \sqrt{s})^{2s+4\sqrt{s}}(s + 2\sqrt{s})^{-s-2\sqrt{s}}
$$

$$
= \left( \frac{\sqrt{s}}{1 + \sqrt{s}} \right)^{\sqrt{s}+1} \left( \frac{s + 2\sqrt{s} + 1}{s + 2\sqrt{s}} \right)^{s+2\sqrt{s}} \xrightarrow{s \to \infty} e^{-1} e = 1.
$$
To prove the second part observe that the first inequality yields

\[ G_t(k) = g_t(k_1) \cdots g_t(k_d) \geq (1 - \varepsilon) g_s(k_1 - 1) g_t(k_2) \cdots g_t(k_d) \]

\[ \geq (1 - \varepsilon)^d G_s(k - e_1), \]

since

\[ g_t(l) = g_t(\|l\|) \geq g_t(\|l\| + 1) \geq (1 - \varepsilon) g_s(\|l\|) = (1 - \varepsilon) g_s(l). \]

A sequence \((x_i)_{i=0}^n \subset \mathbb{Z}^d\) is called a path in \(\mathbb{Z}^d\) between \(x_0\) and \(x_n\) if \(\|x_i - x_{i+1}\|_1 = 1\) for \(i = 0, \ldots, n - 1\). For \(k \in \mathbb{Z}^d\) let \(L_n(k)\) denote the number of paths in \(\mathbb{Z}^d\) between 0 and \(k\).

**Lemma 2.4.** Let \(f : \mathbb{Z}^d \to \mathbb{R}\) be harmonic. Suppose there exists a polynomial \(W : \mathbb{Z}^d \to \mathbb{R}\) such that \(f(x) \geq -W(x)\). Then \(f \in L\).

**Proof.** Using simple induction we can prove that for \(f\) harmonic and \(n \in \mathbb{N}\) we have

\[ f(0) = \frac{1}{(2d)^n} \sum_{k \in \mathbb{Z}^d} f(k) L_n(k). \]

Let \(l \in \mathbb{Z}^d\). Then \(L_{\|l\|_1}(l) \geq 1\) and

\[ f(0)(2d)^{\|l\|_1} = \sum_{k \in \mathbb{Z}^d} (f(k) + W(k)) L_{\|l\|_1}(k) - \sum_{k \in \mathbb{Z}^d} W(k) L_{\|l\|_1}(k) \]

\[ \geq (f(l) + W(l)) - \max_{k : \|k\|_1 \leq \|l\|_1} |W(k)| \cdot (2d)^{\|l\|_1}, \]

hence

\[ f(l) \leq f(0)(2d)^{\|l\|_1} + (2d)^{\|l\|_1} \max_{k : \|k\|_1 \leq \|l\|_1} |W(k)| - W(l) \leq c_1 e^{c_2 \|l\|_1} \]

for some \(c_1, c_2 > 0\) which depend only on \(f\) and \(W\) but not on \(l\). Since \(f\) is polynomially bounded from below we have \(f \in L\). □

Now we may recover the classical strong Liouville property of harmonic functions on \(\mathbb{Z}^d\). Woess \([8]\) traces back its weak form to Blackwell \([1]\); see also \([2]\) and \([5]\).

**Theorem 2.5.** If \(f : \mathbb{Z}^d \to \mathbb{R}\) is harmonic and \(f \geq 0\) then \(f\) is constant.

**Proof.** By Lemma 2.4 we have \(f \in L\). Let \(x \in \mathbb{Z}^d\). Lemma 2.3 implies that there exist \(t > s > 0\) such that

\[ \cdots \]
\[
f(x) - f(x + e_1) = P_t(f)(x) - P_s(f)(x + e_1)
= \sum_{k \in \mathbb{Z}^d} f(x + k)G_t(k) - \sum_{k \in \mathbb{Z}^d} f(x + k + e_1)G_s(k)
= \sum_{k \in \mathbb{Z}^d} f(x + k)(G_t(k) - G_s(k - e_1))
\geq -\varepsilon \sum_{k \in \mathbb{Z}^d} f(x + k)G_s(k - e_1) = -\varepsilon f(x + e_1).
\]

By letting \(\varepsilon \to 0\) we get \(f(x) \geq f(x + e_1)\). Applying this inequality to the harmonic function \(x \mapsto g(x) = f(-x)\) we get \(f(x) = f(x + e_1)\) and similarly \(f(x) = f(x + e_i)\) for \(i = 1, \ldots, d\). \(\blacksquare\)

We will now prove some auxiliary lemmas.

**Lemma 2.6.** Let \(n \in \mathbb{N}\) and let \(k \in \mathbb{Z}\) satisfy \(|k| \leq n\). Then
\[
\frac{1}{2\sqrt{n}} \left(1 - \frac{k^2}{n}\right) \leq \frac{1}{2^{2n}} \left(\frac{2n}{n + k}\right) \leq \frac{1}{\sqrt{2n+1}} e^{-\frac{k^2}{2n}} \leq \frac{1}{2n + 1} e^{-\frac{k^2}{2n}}.
\]

**Proof.** We can assume \(k \geq 0\). By multiplying the obvious inequalities \((2j - 1)^2 \geq 2j(2j - 2)\) for \(j = 2, 3, \ldots, n\) and \((2j)^2 \geq (2j - 1)(2j + 1)\) for \(j = 1, 2, \ldots, n\) we arrive at \(((2n - 1)!!)^2 \geq \frac{1}{2}(2n)!!(2n - 2)!!\) and \(((2n)!!)^2 \geq (2n - 1)!!(2n + 1)!!\), so that
\[
\frac{1}{4n} \leq \left(\frac{(2n - 1)!!}{(2n)!!}\right)^2 \leq \frac{1}{2n + 1}.
\]

To finish the proof it suffices to observe that
\[
\frac{1}{2^{2n}} \left(\frac{2n}{n + k}\right) = \frac{(2n - 1)!!}{(2n)!!} \cdot \prod_{j=1}^{k} \left(1 - \frac{k}{n + j}\right)
\]
and
\[
1 - \frac{k^2}{n} \leq \left(1 - \frac{k}{n}\right)^k \leq \prod_{j=1}^{k} \left(1 - \frac{k}{n + j}\right) \leq \left(1 - \frac{k}{2n}\right)^k \leq e^{-\frac{k^2}{2n}}. \quad \blacksquare
\]

**Lemma 2.7.** There exists a constant \(C > 0\) such that for \(k \in \mathbb{Z}^d \setminus \{0\},\)
\[
G_{\|k\|_1^2}(k) \geq C^d \|k\|_{1}^{-2d}.
\]

**Proof.** Let \(t > 0\) and \(k = (k_1, \ldots, k_d) \in \mathbb{Z}^d\). We have (see Lemma 2.2)
\[
g_t(k_i) \geq e^{-4t} \frac{t^n}{n!} \left(\frac{2n}{n + k_i}\right) \geq e^{-4t} \frac{t^n}{n!} \left(\frac{2n}{n + \|k\|_1}\right) \quad (i = 1, \ldots, d, n \in \mathbb{N}).
\]

We set \(t = \|k\|_1^2\) and \(n = 4t\). Then \(e^{-4t} t^n = e^{-n} n^n / 4^n\), so that
\[
g_t(k_i) \geq q_n(n) \cdot \frac{1}{2^{2n}} \left(\frac{2n}{n + \|k\|_1}\right) \geq q_n(n) \cdot \frac{1}{2\sqrt{n}} \left(1 - \frac{\|k\|_1^2}{n}\right) = \frac{3}{16} q_n(n) / \|k\|_1,
\]
where we have used Lemma 2.6. Note that by Chebyshev’s inequality,
\[ P(|Y_1^{(n)} - n| \geq 2\sqrt{n}) = P(|Y_1^{(n)} - \mathbb{E}Y_1^{(n)}| \geq 2\sqrt{n}) \leq \frac{D^2Y_1^{(n)}}{4n} = 1/4, \]
so that
\[ 3/4 \leq P(|Y_1^{(n)} - n| < 2\sqrt{n}) = \sum_{m \in \mathbb{N}_0: |m-n| < 2\sqrt{n}} q_n(m) \leq \text{card}\{m \in \mathbb{N}_0: |m-n| < 2\sqrt{n}\} \cdot q_n(n) \leq 8\|k\|_1 \cdot q_n(n). \]

Hence
\[ g_t(k_i) \geq \frac{3}{32\|k\|_1} \cdot \frac{3}{16\|k\|_1} = \frac{C}{\|k\|_1^2} \]
and therefore
\[ G_{\|k\|_1^2}(k) = \prod_{i=1}^{d} g_t(k_i) \geq C^d\|k\|_1^{-2d}. \]

**Lemma 2.8.** Let \( W : \mathbb{R}^d \to \mathbb{R} \) be a polynomial. Define \( H_W : \mathbb{R} \to \mathbb{R} \) by
\[ H_W(t) = \mathbb{P}_t(W)(0) = \sum_{k \in \mathbb{Z}^d} G_t(k)W(k). \]
Then \( H_W \) is a polynomial.

**Proof.** \( H_W \) is well defined since \( W|_{\mathbb{Z}^d} \in \mathcal{L} \). Because of the product structure of \( G_t \) it is enough to consider the case \( d = 1 \) and \( W(z) = z^l \) for \( l \in \mathbb{N} \). The characteristic function
\[ \phi_X^{(t)}(z) = e^{-4t \sin^2(z/2)} \]
is smooth, so that
\[ H_W(t) = \mathbb{E}[(X_1^{(t)})^l] = (-i)^l \frac{d^l\phi_X^{(t)}}{dz^l}(0), \]
which is clearly a polynomial in \( t \). □

**Lemma 2.9.** Let \( f : \mathbb{Z}^d \to \mathbb{R} \) be harmonic. Suppose there exists a polynomial \( W : \mathbb{Z}^d \to \mathbb{R} \) such that \( f \geq -W \). Then \( |f| \leq R \) for some polynomial \( R : \mathbb{Z}^d \to \mathbb{R} \).

**Proof.** We have \( f \in \mathcal{L} \) (see Lemma 2.4). Proposition 2.1 yields
\[ f(0) = \sum_{k \in \mathbb{Z}^d} G_t(k)f(k), \]
hence for all \( l \in \mathbb{Z}^d \),
\[ f(0) = \sum_{k \in \mathbb{Z}^d} G_t(k)(f(k) + W(k)) - \sum_{k \in \mathbb{Z}^d} G_t(k)W(k) \geq G_t(l)(f(l) + W(l)) - H_W(t). \]
Therefore

\[ f(0) + H_W(t) \geq G_t(l)(f(l) + W(l)). \]

There exists a constant \( c = c(d) > 0 \) such that (see Lemma 2.7) for all \( l \neq 0 \),

\[ G_{\|l\|_1^2}(l) \geq c\|l\|_1^{-2d}. \]

Hence for \( l \neq 0 \),

\[ f(0) + H_W(\|l\|_1^2) \geq c(f(l) + W(l))\|l\|_1^{-2d} \]

and therefore

\[ f(l) \leq c^{-1}\|l\|_1^{2d} f(0) + H_W(\|l\|_1^2) - W(l). \]

Since the right-hand side is polynomially bounded from above in \( l \), we have

\[ f(l) \leq P(l) \]

for some polynomial \( P: \mathbb{R}^d \to \mathbb{R} \) and all \( l \in \mathbb{Z}^d \). One can easily check that \( |f(l)| \leq 1 + P(l)^2 + [W(l)]^2 \).

**Lemma 2.10.** For all \( x \in \mathbb{Z}, n \in \mathbb{N}, a, b \in \mathbb{R} \) and \( p \geq 0 \) we have

\[ |a + b|^p \leq 2^p(|a|^p + |b|^p) \]

and

\[ |x|^n - |x + 1|^n | \leq 1 + 2^n |x|^{n-1}. \]

**Proof.** Without loss of generality we may assume that \( |a| \leq |b| \). Then \( |a + b|^p \leq (2|b|)^p \leq 2^p(|a|^p + |b|^p) \).

To prove the second inequality note that

\[ |(x + 1)^n| - |x^n| \leq |(x + 1)^n - x^n| = \left| \sum_{k=0}^{n-1} \binom{n}{k} x^k \right| \leq 1 + \sum_{k=1}^{n-1} \binom{n}{k} |x|^{n-1} \leq 1 + 2^n |x|^{n-1}. \]

**Lemma 2.11.** If \( t > 0 \) then

\[ g_t(0) \leq \frac{1}{2\sqrt{t}} \]

and

\[ \mathbb{E}|X_1^{(t)}|^m \leq b(m)t^{m/2} + c(m) \]

for some constants \( b(m), c(m) > 0 \) and \( m \in \mathbb{N} \).

**Proof.** Let \( M \) be the Poisson variable with mean \( 4t \). By Lemma 2.2, Lemma 2.6 and Jensen’s inequality we have

\[ g_t(0) = \sum_{n=0}^{\infty} e^{-4t} \frac{(4t)^n}{n!} \frac{1}{2^{2n}} \binom{2n}{n} \leq \sum_{n=0}^{\infty} e^{-4t} \frac{(4t)^n}{n!} \frac{1}{\sqrt{n+1}} = \mathbb{E} \frac{1}{\sqrt{M + 1}} \leq \left( \mathbb{E} \frac{1}{M + 1} \right)^{1/2} \]
and 
\[ \mathbb{E} \frac{1}{M+1} = \sum_{n=0}^{\infty} e^{-4t} \frac{(4t)^n}{(n+1)!} = \frac{1}{4t} \sum_{n=0}^{\infty} e^{-4t} \frac{(4t)^{n+1}}{(n+1)!} \leq \frac{1}{4t}. \]

To prove the second part, let \( M, r_1, r_2, \ldots \) be as in Lemma 2.2. For fixed \( k \in \mathbb{N} \) and all \( i \leq k \) we have \( \mathbb{E} \exp \left( \frac{r_i}{\sqrt{k}} \right) = 1 + \sum_{s=1}^{\infty} k^{-s} / (2s)! \leq 1 + ek^{-1} \leq e^e/k \), so that

\[
\frac{1}{m!} \mathbb{E} \left( \frac{r_1 + \cdots + r_k}{\sqrt{k}} \right)^m \leq \mathbb{E} \exp \left( \frac{r_1 + \cdots + r_k}{\sqrt{k}} \right) = \prod_{i=1}^{k} \mathbb{E} \exp \left( \frac{r_i}{\sqrt{k}} \right) \leq e^e.
\]

Hence

\[ \mathbb{E} |r_1 + \cdots + r_k|^m = 2 \mathbb{E} (r_1 + \cdots + r_k)^m \leq 2e^e m! \cdot k^{m/2} \]

and therefore, by Lemma 2.2

\[ \mathbb{E} |X_1^{(t)}|^m \leq 2e^e m! \cdot 2^{-m} \cdot (\mathbb{E} (2M))^{m/2} \leq 2e^e m! \cdot (\mathbb{E} M)^{1/2}. \]

Now,

\[ \mathbb{E} M^m = \mathbb{E} M^m I_{M<m} + \mathbb{E} M^m I_{M \geq m} \leq m^m + m^m \mathbb{E} (M - m + 1)^m \]

\[ \leq m^m \left( 1 + \sum_{k=m}^{\infty} e^{-4t} \frac{(4t)^k}{k!} k(k-1) \cdots (k-m+1) \right) \]

\[ = m^m (1 + (4t)^m) \]

and it is obvious (see Lemma 2.10) that

\[ \mathbb{E} |X_1^{(t)}|^m \leq b(m) t^{m/2} + c(m) \]

for some constants \( b(m), c(m) > 0 \).

Now we state the key lemma of this paper. Similar estimates for sublinear harmonic functions have been obtained in a more general setting in [6, Theorem 6.1].

**Lemma 2.12.** Let \( n \in \mathbb{N} \) and let \( f : \mathbb{Z}^d \rightarrow \mathbb{R} \) be harmonic. Suppose that there exists a constant \( a_n \) such that

\[ |f(x)| \leq a_n (1 + \|x\|_n^n) \]

for all \( x \in \mathbb{Z}^d \). Then there exists a constant \( a_{n-1} \) such that for all \( x \in \mathbb{Z}^d \),

\[ |f(x + e_1) - f(x)| \leq a_{n-1} (1 + \|x\|_{n-1}^{n-1}). \]

**Proof.** For \( x \in \mathbb{Z}^d \) and any \( t > 0 \) we have

\[ f(x) = \sum_{k \in \mathbb{Z}^d} G_t(k) f(x + k) \]

and

\[ f(x + e_1) = \sum_{k \in \mathbb{Z}^d} G_t(k) f(x + e_1 + k) = \sum_{k \in \mathbb{Z}^d} G_t(k - e_1) f(x + k), \]

hence
\begin{equation*}
|f(x + e_1) - f(x)| \leq \sum_{k \in \mathbb{Z}^d} |G_t(k - e_1) - G_t(k)| |f(x + k)|
\end{equation*}

\begin{equation*}
\leq \sum_{k \in \mathbb{Z}^d} |G_t(k - e_1) - G_t(k)| a_n(1 + \|x + k\|^n_n)
\end{equation*}

\begin{equation*}
= \sum_{k \in \mathbb{Z}^d : k_1 \leq 0} (G_t(k) - G_t(k - e_1)) a_n(1 + \|x + k\|^n_n)
+ \sum_{k \in \mathbb{Z}^d : k_1 > 0} (G_t(k - e_1) - G_t(k)) a_n(1 + \|x + k\|^n_n)
\end{equation*}

\begin{equation*}
= \sum_{k \in \mathbb{Z}^d : k_1 \leq -1} G_t(k)a_n(|x_1 + k_1|^n - |x_1 + k_1 + 1|^n)
+ \sum_{k \in \mathbb{Z}^d : k_1 \geq 1} G_t(k)a_n(|x_1 + k_1 + 1|^n - |x_1 + k_1|^n)
+ \sum_{k \in \{0\} \times \mathbb{Z}^{d-1}} G_t(k)a_n(1 + \|x + k\|^n_n)
+ \sum_{k \in \{0\} \times \mathbb{Z}^{d-1}} G_t(k)a_n(1 + \|x + k + e_1\|^n_n).
\end{equation*}

We have used the product structure of \(G_t\) and Lemma \ref{lem:product}. By using Lemma \ref{lem:product} we get

\begin{equation*}
\sum_{k_1 \leq -1, k \in \mathbb{Z}^d} G_t(k)(|x_1 + k_1|^n - |x_1 + k_1 + 1|^n)
\end{equation*}

\begin{equation*}
+ \sum_{k_1 \geq 1, k \in \mathbb{Z}^d} G_t(k)(|x_1 + k_1 + 1|^n - |x_1 + k_1|^n)
\end{equation*}

\begin{equation*}
\leq \sum_{k \in \mathbb{Z}^d} G_t(k)(2^n|x_1 + k_1|^{n-1} + 1) = 1 + 2^n \sum_{k_1 \in \mathbb{Z}} g_t(k_1)|x_1 + k_1|^{n-1}
\end{equation*}

\begin{equation*}
\leq 1 + 2^{2n-1} \sum_{k_1 \in \mathbb{Z}} g_t(k_1)(|x_1|^{n-1} + |k_1|^{n-1})
\end{equation*}

\begin{equation*}
= 1 + 2^{2n-1}(|x_1|^{n-1} + \mathbb{E}|X_1^{(t)}|^{n-1}).
\end{equation*}

We also have, again by using Lemma \ref{lem:product} several times,

\begin{equation*}
\sum_{k \in \{0\} \times \mathbb{Z}^{d-1}} G_t(k)(1 + \|x + k\|^n_n) + \sum_{k \in \{0\} \times \mathbb{Z}^{d-1}} G_t(k)(1 + \|x + k + e_1\|^n_n)
\end{equation*}

\begin{equation*}
\leq \sum_{k \in \{0\} \times \mathbb{Z}^{d-1}} G_t(k)(2 + 2^n\|x\|^n_n + 2^n\|x + e_1\|^n_n + 2^{n+1}\|k\|^n_n)
\end{equation*}

\begin{equation*}
\leq g_t(0)(2 + 2^n\|x\|^n_n + 2^n\|x + e_1\|^n_n + d 2^{n+1}\mathbb{E}|X_1^{(t)}|^{n})
\end{equation*}

\begin{equation*}
\leq 4^{n+1} g_t(0)(1 + \|x\|^n_n + d \mathbb{E}|X_1^{(t)}|^n),
\end{equation*}
so we arrive at
\[ |f(x + e_1) - f(x)| \]
\[ \leq a_n[1 + 2^{2n-1}(|x_1|^{n-1} + \|X_1^{(t)}\|^{n-1}) + 4^{n+1}g_k(0)(1 + \|x\|_n^n + d \|X_1^{(t)}\|^{n})] \]
\[ \leq 4^{n+2}a_n d[(1 + \|x\|_n^{n-1} + \|X_1^{(t)}\|^{n-1}) + g_k(0)(\|x\|_n^n + \|X_1^{(d)}\|^{n})]. \]

From Lemma 2.11 we infer that there exists a constant \( C = C(n, d) \) such
that for every \( t > 0 \) and every \( x \in \mathbb{Z}^d \),
\[ |f(x + e_1) - f(x)| \leq C a_n[1 + \|x\|_n^{n-1} + t^{(n-1)/2} + t^{-1/2}(\|x\|_n^n + t^{n/2})]. \]

By setting \( t = 1 + \|x\|_1^n \) we complete the proof. ■

**Lemma 2.13.** Let \( f : \mathbb{Z}^d \to \mathbb{R} \) be such that \( f_i(x) = f(x + e_i) - f(x) \) are polynomials for \( i = 1, \ldots, d \). Then \( f \) is a polynomial.

**Proof.** First we consider the case \( d = 1 \). Note that \( f(x) = f(0) \) is determined by values of \( f_1 \). Define a sequence of polynomials \((W_k)_{k=0}^\infty\) by
\[ x^m = \sum_{k=0}^{m-1} \binom{m}{k} W_k(x), \quad m = 1, 2, \ldots. \]

A simple induction yields \( W_k(x+1) - W_k(x) = x^k \) and \( W_k(0) = 0 \). It follows that if \( f_1(x) = \sum_{i=0}^d a_i x^i \) then \( f(x) = f(0) + \sum_{i=0}^d a_i W_i(x) \). If \( d > 1 \) then
\[ f(x_1, \ldots, x_d) = f(x_1, x_2, \ldots, x_d) - f(0, x_2, \ldots, x_d) + f(0, x_2, \ldots, x_d) - f(0, 0, x_3, \ldots, x_d) + \cdots + f(0, \ldots, 0, x_1) - f(0, \ldots, 0) + f(0). \]

By using the same argument as in the case \( d = 1 \) we see that
\[ f(0, \ldots, x_i, \ldots, x_d) - f(0, \ldots, x_{i+1}, \ldots, x_d) \quad (i = 1, \ldots, d) \]
are polynomials. ■

**Main Theorem 2.14.** Let \( f : \mathbb{Z}^d \to \mathbb{R} \) be harmonic. Suppose there exists a polynomial \( W : \mathbb{Z}^d \to \mathbb{R} \) such that \( f(k) \geq -W(k) \) for \( k \in \mathbb{Z}^d \). Then \( f \) is a polynomial.

**Proof.** There exists \( n \in \mathbb{N} \) such that \( |f(x)| \leq a_n(1 + \|x\|_n^n) \) (see Lemma 2.9). We claim that together with the harmonicity of \( f \) this already implies that \( f \) is a polynomial. We prove this by induction on \( n \). For \( n = 0 \) the claim is a consequence of Proposition 2.5. For \( n > 1 \) let \( f_i(x) = f_i(x + e_1) - f(x) \). Note that \( f_i, i = 1, \ldots, d, \) are also harmonic. By Lemma 2.12 and induction hypothesis, \( f_i \) are polynomials, hence by Lemma 2.13 we conclude that \( f \) is a polynomial as well. ■
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