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Subadditive Pressure for IFS with Triangular Maps

by

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Summary. We investigate properties of the zero of the subadditive pressure which is a most important tool to estimate the Hausdorff dimension of the attractor of a non-conformal iterated function system (IFS). Our result is a generalization of the main results of Miao and Falconer [Fractals 15 (2007)] and Manning and Simon [Nonlinearity 20 (2007)].

1. Introduction. Since the main goal of this paper is to improve a tool which is used to estimate the Hausdorff dimension, we first define the Hausdorff measure and Hausdorff dimension of a bounded set $A \subset \mathbb{R}^n$. Let

$$\mathcal{H}^s_\delta = \inf\Big\{\sum_i |U_i|^s : A \subset \bigcup_i U_i,\ |U_i| < \delta\Big\} \tag{1.1}$$

where $|U|$ is the diameter of U. Now we define the *s-dimensional Hausdorff measure* of A by

$$\mathcal{H}^s(A) = \lim_{\delta\to\infty} \mathcal{H}^s_\delta(A). \tag{1.2}$$

We call

$$\dim_{\mathrm{H}} A = \inf\{s : \mathcal{H}^s(A) = 0\} \tag{1.3}$$

the *Hausdorff dimension* of A.

We consider the Hausdorff dimension of the attractors of iterated function systems (IFS) which are non-conformal. (We say that a map is *conformal* if the derivative is a similarity at every point.) The dimension theory of non-conformal IFS is difficult and there are only very few results. The most

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important tool in this field is the subadditive pressure, defined by K. Falconer [4] and L. Barreira [1]. Unfortunately, we know very little about the subadditive pressure itself.

In the conformal case, the subadditive pressure coincides with the usual topological pressure (see for example [10, Chapter 9]).

The simplest non-conformal situation is the case of self-affine IFS. To study the dimension of a self-affine attractor we consider the kth approximation of the attractor with the so called kth cylinders which are naturally defined by the k-fold application of the functions of the IFS. To measure the contribution of such a cylinder to the covering sum which appears in the definition of the Hausdorff measure (see (1.1) and (1.2)), for each of these cylinders we consider the singular value functions. These are non-negative valued functions defined in a neighborhood of the attractor. The dimension of the attractor is related to the exponential growth rate of the sum of the values of these exponentially many singular value functions in the self-affine case (see [2]). To verify this it was essential that this exponential growth rate is the same wherever we evaluate these singular value functions, since they are constant in the self-affine case.

Falconer [4] and Barreira [1] considered the more general situation when the IFS is no longer self-affine. In this case, using a similar method, it turns out that under a technical condition (which Barreira called the 1-bunched property) the exponential growth rate of the sum of the values of the singular value functions does not depend on where they are evaluated. We express this phenomenon by saying that "the insensitivity property holds".

This is an important property of the subadditive pressure and in general we do not know if it holds or not. The main goal of this paper is to verify this property in a special case when the 1-bunched property does not hold but the IFS consists of maps with lower triangular derivative matrices. This is a generalization of the result of K. Simon and A. Manning [8]. They proved the same assertion in two dimensions.

Even if the 1-bunched condition is not satisfied, Zhang [11] found that the zero of the subadditive pressure is an upper bound for the Hausdorff dimension. As an application we supply two examples of IFS for which we are able to calculate the Hausdorff dimension using the insensitivity property.

The main theorem is also a generalization of a recent results by K. Falconer and J. Miao [6]. They gave an estimate for the Hausdorff dimension of self-affine fractals generated by upper triangular matrices. We will give an estimate for the subadditive pressure in the non-conformal case and we will prove that the subadditive pressure depends only on the diagonal elements of the derivative matrices in the case when the matrices are triangular. In this paper we use the method of K. Falconer and J. Miao's article [6].

2. Definitions. In this section we define our iterated function system and the subadditive pressure.

Throughout this paper we will always assume the following. Let $M \subset \mathbb{R}^n$ be a non-empty, open and bounded set, and let $F_i : M \to M$ be a contractive maps for every $i = 1, \dots, l$. For $\mathbf{i} = i_1 \dots i_k$, $i_j \in \{1, \dots, l\}$, we write $F_{\mathbf{i}}(\underline{x}) = F_{i_1} \circ \dots \circ F_{i_n}(\underline{x})$. Our principal assumption about the maps F_i, $i = 1, \dots, l$, is that

$$F_i(x_1, \dots, x_n) = (f_i^1(x_1), f_i^2(x_1, x_2), \dots, f_i^n(x_1, \dots, x_n)), \tag{2.1}$$

and $F_i(x_1, \dots, x_n) \in C^{1+\varepsilon}(\overline{M})$ for every $i = 1, \dots, l$. Moreover, we require that $D_{\underline{x}} F_i$ is regular (a non-singular matrix) for every $\underline{x} \in \overline{M}$ and every $i \in \{1, \dots, l\}$. Denote the elements of $D_{\underline{x}} F_{\mathbf{i}}$ by $x_{ij}(\mathbf{i}, \underline{x})$.

Proposition 2.1. *There exists a real constant* $0 < C < \infty$ *such that*

$$C^{-1} < \frac{|x_{ii}(\mathbf{i}, \underline{x})|}{|x_{ii}(\mathbf{i}, \underline{y})|} < C \tag{2.2}$$

for every $\underline{x}, \underline{y} \in \overline{M}$ *and every* $\mathbf{i} \in \{1, \dots, l\}^* = \bigcup_{r \geq 1} \{1, \dots, i\}^r$.

Proof. Let $G_i^{(m)} : \mathbb{R}^m \to \mathbb{R}^m$, for every integer m between 1 and n, be the restriction of F_i to the first m components, i.e.

$$G_i^{(m)}(x_1, \dots, x_m) := (f_i^1(x_1), f_i^2(x_1, x_2), \dots, f_i^m(x_1, \dots, x_m)).$$

From [9, Proposition 20.1(3), p. 198] it follows that for every $\underline{x}, \underline{y} \in \overline{M}$, every finite sequence $\mathbf{i} \in \{1, \dots, l\}^*$, and $1 \leq m \leq n$ there exists a real constant $0 < C_m < \infty$ such that

$$C_m^{-1} < \frac{\operatorname{Jac} G_{\mathbf{i}}^{(m)}(\underline{x})}{\operatorname{Jac} G_{\mathbf{i}}^{(m)}(\underline{y})} < C_m.$$

Since for every m, the matrix $D_{\underline{x}} G_{\mathbf{i}}^{(m)}$ is lower triangular, its Jacobian is

$$\operatorname{Jac} G_{\mathbf{i}}^{(m)}(\underline{x}) = |x_{11}(\mathbf{i}, \underline{x}) \cdots x_{mm}(\mathbf{i}, \underline{x})|.$$

Therefore for every integer $1 \leq m < n$ and every $\underline{x}, \underline{y} \in M$,

$$\frac{C_m^{-1}}{C_{m+1}} < \frac{\operatorname{Jac} G_{\mathbf{i}}^{(m)}(\underline{x}) / \operatorname{Jac} G_{\mathbf{i}}^{(m)}(\underline{y})}{\operatorname{Jac} G_{\mathbf{i}}^{(m+1)}(\underline{x}) / \operatorname{Jac} G_{\mathbf{i}}^{(m+1)}(\underline{y})} < \frac{C_m}{C_{m+1}^{-1}}$$

and

$$\frac{\operatorname{Jac} G_{\mathbf{i}}^{(m)}(\underline{x}) / \operatorname{Jac} G_{\mathbf{i}}^{(m)}(\underline{y})}{\operatorname{Jac} G_{\mathbf{i}}^{(m+1)}(\underline{x}) / \operatorname{Jac} G_{\mathbf{i}}^{(m+1)}(\underline{y})} = \frac{|x_{m+1,m+1}(\mathbf{i}, \underline{y})|}{|x_{m+1,m+1}(\mathbf{i}, \underline{x})|}.$$

Then choosing $C := \max_{1 \leq m < n-1} \{C_m / C_{m+1}^{-1}, C_1\}$ completes the proof. ∎

The singular values of a linear contraction T are the positive square roots of the eigenvalues of TT^*, where T^* is the transpose of T. Let $\alpha_k(D_{\underline{x}}F_{\mathbf{i}})$ be the kth greatest singular value of the matrix $D_{\underline{x}}F_{\mathbf{i}}$. The *singular value function* ϕ^s is defined for $0 \le s \le n$ as

$$\phi^s(D_{\underline{x}}F_{\mathbf{i}}) := \alpha_1(D_{\underline{x}}F_{\mathbf{i}})\cdots\alpha_{k-1}(D_{\underline{x}}F_{\mathbf{i}})\alpha_k(D_{\underline{x}}F_{\mathbf{i}})^{s-k+1} \tag{2.3}$$

where $k-1 < s \le k$ and k is a positive integer. We define the maximum and the minimum of the singular value function as

$$\overline{\phi}^s(\mathbf{i}) := \max_{\underline{x}\in\overline{M}} \phi^s(D_{\underline{x}}F_{\mathbf{i}}), \qquad \underline{\phi}^s(\mathbf{i}) := \min_{\underline{x}\in\overline{M}} \phi^s(D_{\underline{x}}F_{\mathbf{i}}).$$

We define the *subadditive pressure* after K. Falconer [4] and L. Barreira [1]:

$$P(s) := \lim_{k\to\infty} \frac{1}{k}\log\sum_{|\mathbf{i}|=k} \overline{\phi}^s(\mathbf{i}), \tag{2.4}$$

and define the *lower pressure* by

$$\underline{P}(s) := \liminf_{k\to\infty} \frac{1}{k}\log\sum_{|\mathbf{i}|=k} \underline{\phi}^s(\mathbf{i}). \tag{2.5}$$

3. Subadditive pressure for triangular maps. In this section we state and prove the main theorem of the paper, namely that the subadditive pressure is equal to the lower pressure, which implies the insensitivity property. More precisely, it implies that the exponential growth rate of the sum of the values of the singular value functions does not depend on where they are evaluated (see (2.4), (2.5)).

THEOREM 3.1. *Let $0 \le s \le n$. If $F_1,\dots,F_l$ are contractive maps of the form* (2.1) *and $F_i \in C^{1+\varepsilon}$ for every $1 \le i \le l$ then*

$$P(s) = \underline{P}(s).$$

In the following we state some linear algebra definitions and lemmas, the proofs of which can be found in [6].

The m-dimensional exterior algebra Φ^m is a vector space spanned by formal elements $v_1\wedge\cdots\wedge v_m$ with $v_i \in \mathbb{R}^n$ such that $v_1\wedge\cdots\wedge v_m = 0$ if $v_i = v_j$ for some $i \ne j$, and interchanging two different elements reverses the sign, i.e. $v_1\wedge\cdots\wedge v_i\wedge\cdots\wedge v_j\wedge\cdots\wedge v_m = -v_1\wedge\cdots\wedge v_j\cdots\wedge v_i\wedge\cdots\wedge v_m$ if $i \ne j$. Then Φ^m has dimension $\binom{n}{m}$ with basis $\{e_{j_1}\wedge\cdots\wedge e_{j_m} : 1 \le j_1 < \cdots < j_m \le n\}$ where $\{e_1,\dots,e_n\}$ is any orthonormal basis of $\mathbb{R}^n$.

Let us define a scalar product on Φ^m in the following way. Let

$$\langle v_1\wedge\cdots\wedge v_m, u_1\wedge\cdots\wedge u_m\rangle_{\Phi^m} = \det((\langle v_i, u_j\rangle)_{i,j=1\dots m}),$$

where $\langle\cdot,\cdot\rangle$ is the usual scalar product on $\mathbb{R}^n$. One can extend $\langle\cdot,\cdot\rangle_{\Phi^m}$ to all of Φ^m in the natural way. Then Φ^m becomes a Hilbert space. Let $\|\cdot\|$ be the corresponding norm. Then it is easy to see that $\|v_1\wedge\cdots\wedge v_m\|$ is equal to the absolute m-dimensional volume of the parallelepiped spanned by $v_1, \ldots, v_m$ (see [7, p. 44]).

We may also define another norm $\|\cdot\|_\infty$ on Φ^m by

$$\Big\| \sum_{1\le i_1<\cdots<i_m\le m} \lambda_{i_1\cdots i_m}(e_{i_1}\wedge\cdots\wedge e_{i_m})\Big\|_\infty := \max|\lambda_{i_1\ldots i_m}|.$$

If $T:\mathbb{R}^n\to\mathbb{R}^n$ is linear then there is an induced linear mapping $\widetilde{T}:\Phi^m\to\Phi^m$ given by

$$\widetilde{T}(v_1\wedge\cdots\wedge v_m) := (Tv_1)\wedge\cdots\wedge(Tv_m).$$

The norms on Φ^m induce norms on the space $\mathfrak{L}(\Phi^m,\Phi^m)$ of linear mappings in the usual way by

$$\|\widetilde{T}\| = \sup_{w\in\Phi^m,\, w\neq 0} \frac{\|\widetilde{T}w\|}{\|w\|}.$$

Then with respect to the norm $\|\cdot\|$,

$$\|\widetilde{T}\| = \phi^m(T), \tag{3.1}$$

and with respect to $\|\cdot\|_\infty$,

$$\|\widetilde{T}\|_\infty = \max\{|T^{(m)}| : T^{(m)} \text{ is an } m\times m \text{ minor of } T\}, \tag{3.2}$$

where $T^{(m)} = T\binom{r_1,\ldots,r_m}{s_1,\ldots,s_m}$ is the determinant of the $m\times m$ minor of the $n\times n$ matrix T which is formed by the elements of T in rows $1\le r_1<\cdots<r_m\le n$ and columns $1\le s_1<\cdots<s_m\le n$. The space $\mathfrak{L}(\Phi^m,\Phi^m)$ is of dimension $\binom{n}{m}^2$. Since any two norms on a finite-dimensional normed space are equivalent, there are constants $0<c_1<c_2<\infty$ depending only on n and m such that

$$c_1\|\widetilde{T}\|_\infty \le \|\widetilde{T}\| \le c_2\|\widetilde{T}\|_\infty. \tag{3.3}$$

Now we will state several lemmas relating to minors of matrices. We will need some well-known facts.

Lemma 3.2. *Let $x_i\ge 0$ for $i=1,\ldots,m$, and $p\in\mathbb{R}^+$.*

1. *If $p>1$, then $(x_1^p+\cdots+x_m^p)\le(x_1+\cdots+x_m)^p\le m^{p-1}(x_1^p+\cdots+x_m^p)$.*
2. *If $0<p\le 1$, then $m^{p-1}(x_1^p+\cdots+x_m^p)\le(x_1+\cdots+x_m)^p\le x_1^p+\cdots+x_m^p$.*

Lemma 3.3. *Let a_n be a sequence of real numbers such that $a_{n+m}\le a_n+a_m$. Then the limit $\lim_{n\to\infty}a_n/n$ exists and equals $\inf_n a_n/n$.*

We first look at the expansion of $m\times m$ minors of the product of k matrices $A=A_1\cdots A_k$, where for $i=1,\ldots,k$,

$$A_i = \begin{bmatrix} a^i_{11} & a^i_{12} & \dots & a^i_{1n} \\ a^i_{21} & a^i_{22} & \dots & a^i_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a^i_{n1} & a^i_{n2} & \dots & a^i_{nn} \end{bmatrix}.$$

LEMMA 3.4. *For $1 \leq m \leq n$, the $m \times m$ minors of $A = A_1 \cdots A_k$ have formal expansions in terms of the entries of the A_i of the form*

$$A\binom{r_1, \dots, r_m}{s_1, \dots, s_m} = \sum_{c_1, \dots, c_k} \pm a^1_{1(c_1)} \cdots a^1_{m(c_1)} a^2_{1(c_2)} \cdots a^2_{m(c_2)} \cdots a^k_{1(c_k)} \cdots a^k_{m(c_k)}$$

such that for each $i = 1, \dots, k$, the $a^i_{1(c_i)}, \dots, a^i_{m(c_i)}$ are distinct entries a^i_{rs} of A_i. In particular, for each i, $1(c_i), \dots, m(c_i)$ denote pairs (r, s) corresponding to entries in m different rows and columns of A_i, and the sum is over all such entry combinations $(c_1, \dots, c_k)$ with appropriate sign $\pm$.

The proof of this lemma can be found in [6, Lemma 2.2]. Now we consider lower triangular matrices. For $i = 1, \dots, k$, let

$$U_i = \begin{bmatrix} u^i_1 & 0 & \dots & 0 \\ u^i_{21} & u^i_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ u^i_{n1} & u^i_{n2} & \dots & u^i_n \end{bmatrix}.$$

We consider the product

$$U = U_1 \cdots U_k = \begin{bmatrix} u_1 & 0 & \dots & 0 \\ u_{21} & u_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \dots & u_n \end{bmatrix}.$$

We note that

$$u_{rs} = \sum_{r \geq r_1 \geq \dots \geq r_{k-1} \geq s} u^1_{rr_1} u^2_{r_1 r_2} \cdots u^k_{r_{k-1}s}, \qquad 1 \leq r \leq s \leq n, \tag{3.4}$$

since all other products are 0.

LEMMA 3.5. *With notations as above, let $U_1, \dots, U_k$ be lower triangular matrices and $U = U_1 \cdots U_k$. Then*

(1) *If $r < s$, then $u_{rs} = 0$.*
(2) *If $r = s$, then $u_{rs} \equiv u_r = u^1_r \cdots u^k_r$.*
(3) *If $r > s$, then the sum (3.4) for u_{rs} has at most $k^{r-s} \leq k^{n-1}$ non-zero terms. Moreover, each non-zero summand $u^1_{rr_1} u^2_{r_1 r_2} \cdots u^k_{r_{k-1}s}$ has at*

most $n-1$ non-diagonal terms in the product, i.e. terms with $r \neq r_1$ or $r_i \neq r_{i+1}$ or $r_{k-1} \neq s$.

The proof can also be found in [6, Lemma 2.3] for upper triangular matrices. Now we extend the estimate of Lemma 3.5 to minors.

LEMMA 3.6. *Let $U_1, \dots, U_k$ and U be lower triangular matrices as above. Then each $m \times m$ minor of U has an expansion of the form*

$$U\begin{pmatrix} r_1, \dots, r_m \\ s_1, \dots, s_m \end{pmatrix} = \sum_{c_1,\dots,c_k} \pm u^1_{1(c_1)} u^2_{1(c_2)} \cdots u^k_{1(c_k)} \cdots u^1_{m(c_1)} u^2_{m(c_2)} \cdots u^k_{m(c_k)}$$

where $1(c_i), \dots, m(c_i)$ are as in Lemma 3.4 and

(1) *there are at most $m!k^{m(n-1)}$ terms in the sum which are non-zero,*
(2) *each summand contains at most $(n-1)^m$ non-diagonal elements in the product.*

The proof is analogous to the proof of [6, Lemma 2.4]. Before we prove Theorem 3.1, we define two sums:

$$\begin{aligned} (3.5)\qquad & H(s,r) \\ &= \max_{\substack{j_1,\dots,j_{m-1} \\ j'_1,\dots,j'_m}} \sum_{|\mathbf{i}|=r} (d_{j_1j_1}(\mathbf{i}) \cdots d_{j_{m-1}j_{m-1}}(\mathbf{i}))^{m-s} (d_{j'_1j'_1}(\mathbf{i}) \cdots d_{j'_mj'_m}(\mathbf{i}))^{s-m+1} \end{aligned}$$

where $m-1 < s \leq m$ and $d_{jj}(\mathbf{i}) = \inf_{\underline{x}} |x_{jj}(\mathbf{i}, \underline{x})|$, and

$$\begin{aligned} (3.6)\qquad & T(s,r) \\ &= \max_{\substack{j_1,\dots,j_{m-1} \\ j'_1,\dots,j'_m}} \sum_{|\mathbf{i}|=r} (t_{j_1j_1}(\mathbf{i}) \cdots t_{j_{m-1}j_{m-1}}(\mathbf{i}))^{m-s} (t_{j'_1j'_1}(\mathbf{i}) \cdots t_{j'_mj'_m}(\mathbf{i}))^{s-m+1} \end{aligned}$$

where $m-1 < s \leq m$ and $t_{jj}(\mathbf{i}) = \sup_{\underline{x}} |x_{jj}(\mathbf{i}, \underline{x})|$. It is easy to see from Proposition 2.1 and the definition of the two sums that

$$H(s,r) \leq T(s,r) \leq C^s H(s,r). \tag{3.7}$$

LEMMA 3.7. *For any positive integers r, z, $T(s, r+z) \leq T(s,r)T(s,z)$. Moreover, $\lim_{r\to\infty} \log T(s,r)/r$ exists and equals $\inf_r \log T(s,r)/r$.*

Proof. From the definition of $T(s,r)$ it follows that

$$\begin{aligned} & T(s, r+z) \\ &= \max_{\substack{j_1,\dots,j_{m-1} \\ j'_1,\dots,j'_m}} \sum_{|\mathbf{i}|=r+z} (t_{j_1j_1}(\mathbf{i}) \cdots t_{j_{m-1}j_{m-1}}(\mathbf{i}))^{m-s} (t_{j'_1j'_1}(\mathbf{i}) \cdots t_{j'_mj'_m}(\mathbf{i}))^{s-m+1} \end{aligned}$$

$$\le \max_{\substack{j_1,\dots,j_{m-1}\\ j'_1,\dots,j'_m}} \Big(\sum_{|\mathbf{i}|=r} \sum_{|\mathbf{h}|=z} (t_{j_1j_1}(\mathbf{i})t_{j_1j_1}(\mathbf{h}) \cdots t_{j_{m-1}j_{m-1}}(\mathbf{i})t_{j_{m-1}j_{m-1}}(\mathbf{h}))^{m-s}$$

$$\times (t_{j'_1j'_1}(\mathbf{i})t_{j'_1j'_1}(\mathbf{h}) \cdots t_{j'_mj'_m}(\mathbf{i})t_{j'_mj'_m}(\mathbf{h}))^{s-m+1}\Big)$$

$$= \max_{\substack{j_1,\dots,j_{m-1}\\ j'_1,\dots,j'_m}} \Big(\sum_{|\mathbf{i}|=r} (t_{j_1j_1}(\mathbf{i}) \cdots t_{j_{m-1}j_{m-1}}(\mathbf{i}))^{m-s}(t_{j'_1j'_1}(\mathbf{i}) \cdots t_{j'_mj'_m}(\mathbf{i}))^{s-m+1}$$

$$\times \sum_{|\mathbf{h}|=z} (t_{j_1j_1}(\mathbf{h}) \cdots t_{j_{m-1}j_{m-1}}(\mathbf{h}))^{m-s}(t_{j'_1j'_1}(\mathbf{h}) \cdots t_{j'_mj'_m}(\mathbf{h}))^{s-m+1}\Big)$$

$$\le T(s,r)T(s,z).$$

The existence of the limit follows from Lemma 3.3. ■

The proof of Theorem 3.1 follows the lines of the proof of [6, Theorem 2.5], but our theorem is not a consequence of [6, Theorem 2.5]. The most important alteration is that some of the functions in [6] are affine, while the derivatives in our case are not constant matrices. To control the consequences of this phenomenon in our proof, we have to state a lemma.

Lemma 3.8. *Let X be a compact subset of $\mathbb{R}^n$ and let $\{f_i\}$ be finitely many continuous, real-valued functions. Then*

$$\sup_{\underline{x}\in X} \max_i f_i(\underline{x}) = \max_i \sup_{\underline{x}\in X} f_i(\underline{x}).$$

Proof. Since X is compact, there are $\underline{x}_i \in X$ such that $f_i(\underline{x}_i) = \sup_{\underline{x}} f_i(\underline{x})$. Therefore

$$\sup_{\underline{x}} \max_i f_i(\underline{x}) \le \max_i \sup_{\underline{x}} f_i(\underline{x}) = \max_i f_i(\underline{x}_i) = \max_{i,j} f_i(\underline{x}_j)$$
$$= \max_j \max_i f_i(\underline{x}_j) \le \sup_{\underline{x}} \max_i f_i(\underline{x}),$$

which was to be proved. ■

Moreover, in the proof of [6, Theorem 2.5], the singular value functions and the minors of the derivative matrices were compared. During the proof of Theorem 3.1 we will do this as well; however, we have to introduce in the proof a new IFS, which will be the rth iteration of the original IFS, since we have to separate the growth rates of the non-zero and the non-diagonal terms of the minors of the derivative matrices.

Proof of Theorem 3.1. Let

$$\{G_h\}_{h=1}^{l^r} = \{F_{i_1\cdots i_r}\}_{i_1=1,\dots,i_r=1}^{l,\dots,l}, \tag{3.8}$$

so that each h corresponds to a suitable finite sequence $\mathbf{i} \in \{1,\dots,l\}^r$ of length r. Let us define

$$\overline{\phi'}^s(\mathbf{h}) = \sup_{\underline{x}} \phi^s(D_{\underline{x}} G_{\mathbf{h}}), \qquad \underline{\phi'}^s(\mathbf{h}) = \inf_{\underline{x}} \phi^s(D_{\underline{x}} G_{\mathbf{h}})$$

for $\mathbf{h} \in \{1, \dots, l^r\}^*$, corresponding to IFS $\{G_h\}_{h=1}^{l^r}$ (see (2.3)).

It is easy to see that

$$\sum_{|\mathbf{i}|=kr} \phi^s(D_{\underline{x}} F_{\mathbf{i}}) = \sum_{|\mathbf{h}|=k} \phi^s(D_{\underline{x}} G_{\mathbf{h}}). \tag{3.9}$$

where $\mathbf{i} \in \{1, \dots, l\}^{kr}$ and $\mathbf{h} \in \{1, \dots, l^r\}^k$. The elements of $D_{\underline{x}} G_{\mathbf{h}}$, denoted by $y_{ij}(h, \underline{x})$, are equal to $x_{ij}(\mathbf{i}, \underline{x})$ for a suitable finite sequence $\mathbf{i}$ of length r. It is easy to see that

$$\phi^s(D_{\underline{x}} G_{\mathbf{h}}) = (\phi^{m-1}(D_{\underline{x}} G_{\mathbf{h}}))^{m-s} (\phi^m(D_{\underline{x}} G_{\mathbf{h}}))^{s-m+1}, \text{ where } m-1 < s \le m.$$

By using relations (3.1), (3.2) and (3.3) it follows that

$$\phi^m(D_{\underline{x}} G_{\mathbf{h}}) \ge c_2 \max\{|D_{\underline{x}} G_{\mathbf{h}}^{(m)}| : D_{\underline{x}} G_{\mathbf{h}}^{(m)} \text{ is an } m \times m \text{ minor of } D_{\underline{x}} G_{\mathbf{h}}\}.$$

The maximum $m \times m$ minor of $D_{\underline{x}} G_{\mathbf{h}}$ is at least equal to the largest product of m distinct diagonal elements of $D_{\underline{x}} G_{\mathbf{h}}$, since such products are themselves minors of triangular matrices. Therefore

$$\begin{aligned} \underline{\phi'}^s(\mathbf{h}) \ge c_2^s (\inf_{\underline{x}} |y_{j_1 j_1}(\mathbf{h}, \underline{x}) \cdots y_{j_{m-1} j_{m-1}}(\mathbf{h}, \underline{x})|)^{m-s} \\ \times (\inf_{\underline{x}} |y_{j'_1 j'_1}(\mathbf{h}, \underline{x}) \cdots y_{j'_m j'_m}(\mathbf{h}, \underline{x})|)^{s-m+1} \end{aligned}$$

for every $j_1, \dots, j_{m-1}, j'_1, \dots, j'_m$.

By the chain rule

$$\begin{aligned} D_{\underline{x}} G_{\mathbf{h}} &= D_{G_{h_2 \cdots h_k}(\underline{x})} G_{h_1} D_{G_{h_3 \cdots h_k}(\underline{x})} G_{h_2} \cdots D_{\underline{x}} G_{h_k}, \\ y_{jj}(\mathbf{h}, \underline{x}) &= y_{jj}(h_1, G_{h_2 \cdots h_k}(\underline{x})) y_{jj}(h_2, G_{h_3 \cdots h_k}(\underline{x})) \cdots y_{jj}(h_k, \underline{x}). \end{aligned}$$

It follows with the notation $\inf_{\underline{x}} |y_{jj}(h, \underline{x})| = d'_{jj}(h)$ that

$$\begin{aligned} &\inf_{\underline{x}} |y_{j_1 j_1}(\mathbf{h}, \underline{x}) \cdots y_{j_{m-1} j_{m-1}}(\mathbf{h}, \underline{x})|^{m-s} \inf_{\underline{x}} |y_{j'_1 j'_1}(\mathbf{h}, \underline{x}) \cdots y_{j'_m j'_m}(\mathbf{h}, \underline{x})|^{s-m+1} \\ &\quad \ge (d'_{j_1 j_1}(h_1) \cdots d'_{j_1 j_1}(h_k) d'_{j_2 j_2}(h_1) \cdots d'_{j_{m-1} j_{m-1}}(h_1) \cdots d'_{j_{m-1} j_{m-1}}(h_k))^{m-s} \\ &\qquad \times (d'_{j'_1 j'_1}(h_1) \cdots d'_{j'_1 j'_1}(h_k) d'_{j'_2 j'_2}(h_1) \cdots d'_{j'_m j'_m}(h_1) \cdots d'_{j'_m j'_m}(h_k))^{s-m+1}. \end{aligned}$$

The next inequality follows from the rearrangement of the product:

$$\begin{aligned} &\sum_{|\mathbf{h}|=k} \underline{\phi'}^s(\mathbf{h}) \\ &\quad \ge c_2^s \sum_{|\mathbf{h}|=k} (d'_{j_1 j_1}(h_1) \cdots d'_{j_{m-1} j_{m-1}}(h_1))^{m-s} (d'_{j'_1 j'_1}(h_1) \cdots d'_{j'_m j'_m}(h_1))^{s-m+1} \\ &\qquad \cdots (d'_{j_1 j_1}(h_k) \cdots d'_{j_{m-1} j_{m-1}}(h_k))^{m-s} (d'_{j'_1 j'_1}(h_k) \cdots d'_{j'_m j'_m}(h_k))^{s-m+1} \end{aligned}$$

$$= c_2^s((d'_{j_1j_1}(1)\cdots d'_{j_{m-1}j_{m-1}}(1))^{m-s}(d'_{j'_1j'_1}(1)\cdots d'_{j'_mj'_m}(1))^{s-m+1}$$
$$+\cdots+(d'_{j_1j_1}(l^r)\cdots d'_{j_{m-1}j_{m-1}}(l^r))^{m-s}(d'_{j'_1j'_1}(l^r)\cdots d'_{j'_mj'_m}(l^r))^{s-m+1})^k.$$

The inequality above is true for every $j_1,\dots,j_{m-1},j'_1,\dots,j'_m$, therefore we obtain the maximum. From the definition of $\{G_h\}_{h=1}^{l^r}$ and $H(s,r)$ (see (3.5) and (3.8)), it follows that

$$\sum_{|\mathbf{h}|=k}\underline{\phi'}^s(\mathbf{h})\geq c_2^sH(s,r)^k. \tag{3.10}$$

By using relations (3.1), (3.2) and (3.3) it follows similarly that

$$\phi^m(D_{\underline{x}}G_{\mathbf{h}})\leq c_1\max\{|D_{\underline{x}}G_{\mathbf{h}}^{(m)}| : D_{\underline{x}}G_{\mathbf{h}}^{(m)}\text{ is an } m\times m\text{ minor of } D_{\underline{x}}G_{\mathbf{h}}\}.$$

Therefore

$$\sum_{|\mathbf{h}|=k}\overline{\phi'}^s(\mathbf{i})$$
$$\leq c_1^2\sum_{|\mathbf{h}|=k}(\sup_{\underline{x}}\max_{m-1\times m-1\text{ minor}}|D_{\underline{x}}G_{\mathbf{h}}^{(m-1)}|)^{m-s}(\sup_{\underline{x}}\max_{m\times m\text{ minor}}|D_{\underline{x}}G_{\mathbf{h}}^{(m)}|)^{s-m+1}.$$

By Lemma 3.8, the order of the supremum and the maximum can be reversed in this situation and we can estimate the sum by

$$C\max_{\left\{{r_1,\dots,r_{m-1}\atop s_1,\dots,s_{m-1}}\right\}}\max_{\left\{{r'_1,\dots,r'_m\atop s'_1,\dots,s'_m}\right\}}\sum_{|\mathbf{h}|=k}(\sup_{\underline{x}}|D_{\underline{x}}G_{\mathbf{h}}^{(m-1)}|)^{m-s}(\sup_{\underline{x}}|D_{\underline{x}}G_{\mathbf{h}}^{(m)}|)^{s-m+1}$$

where $r_1,\dots,r_{m-1}$ are the rows and $s_1,\dots,s_{m-1}$ the columns of the $(m-1)\times(m-1)$ minor, and $r'_1,\dots,r'_m$ are the rows and $s'_1,\dots,s'_m$ the columns of the $m\times m$ minor; moreover $C=c_1^2\binom{n}{m}^2\binom{n}{m-1}^2$. By the chain rule

$$D_{\underline{x}}G_{\mathbf{h}}=D_{G_{h_2\cdots h_k}(\underline{x})}G_{h_1}D_{G_{h_3\cdots h_k}(\underline{x})}G_{h_2}\cdots D_{\underline{x}}G_{h_k},$$

we obtain

$$D_{\underline{x}}G_{\mathbf{h}}\begin{pmatrix}r_1,\dots,r_m\\ s_1,\dots,s_m\end{pmatrix} \tag{3.11}$$
$$=\sum_{c_1,\dots,c_k}\pm y_{1(c_1)}(h_1,G_{h_2\cdots h_k}(\underline{x}))\cdots y_{1(c_k)}(h_k,\underline{x})\cdots y_{m(c_1)}(h_1,G_{h_2\cdots h_k}(\underline{x}))$$
$$\times y_{m(c_2)}(h_2,G_{h_3\cdots h_k}(\underline{x}))\cdots y_{m(c_k)}(h_k,\underline{x}).$$

Therefore

$$(3.12)\qquad \sup_{\underline{x}} |D_{\underline{x}} G_{\mathbf{h}}^{(m)}| \le \sum_{c_1,\dots,c_k} \sup_{\underline{x}} |y_{1(c_1)}(h_1,\underline{x})| \cdots \sup_{\underline{x}} |y_{1(c_k)}(h_k,\underline{x})| \cdots \sup_{\underline{x}} |y_{m(c_1)}(h_1,\underline{x})| \times \sup_{\underline{x}} |y_{m(c_2)}(h_2,\underline{x})| \cdots \sup_{\underline{x}} |y_{m(c_k)}(h_k,\underline{x})|.$$

Denote by $t'_{kl}(h) := \sup_{\underline{x}} |y_{kl}(h,\underline{x})|$ the suprema. It follows from the inequality (3.12) and Lemma 3.2 that

$$(3.13)\qquad \sum_{|\mathbf{h}|=k} \sup_{\underline{x}} |D_{\underline{x}} G_{\mathbf{h}}^{(m-1)}|^{m-s} \sup_{\underline{x}} |D_{\underline{x}} G_{\mathbf{h}}^{(m)}|^{s-m+1} \le \sum_{\substack{c_1,\dots,c_k\\ c'_1,\dots,c'_k}} \Big((t'_{1(c_1)}(1)\cdots t'_{m-1(c_1)}(1))^{m-s} (t'_{1(c'_1)}(1)\cdots t'_{m(c'_1)}(1))^{s-m+1} + \cdots + (t'_{1(c_1)}(l^r)\cdots t'_{m-1(c_1)}(l^r))^{m-s} (t'_{1(c'_1)}(l^r)\cdots t'_{m(c'_1)}(l^r))^{s-m+1} \Big) \times \cdots \times \Big((t'_{1(c_k)}(1)\cdots t'_{m-1(c_k)}(1))^{m-s} (t'_{1(c'_k)}(1)\cdots t'_{m(c'_k)}(1))^{s-m+1} + \cdots + (t'_{1(c_k)}(l^r)\cdots t'_{m-1(c_k)}(l^r))^{m-s} (t'_{1(c'_k)}(l^r)\cdots t'_{m(c'_k)}(l^r))^{s-m+1} \Big).$$

Lemma 3.6 implies that each non-zero term of the above sum has at most $2(n-1)^m = b$ indices $1(c_1),\dots,m-1(c_1),\dots,1(c_k),\dots,m-1(c_k),1(c'_1),\dots,m(c'_1),\dots,1(c'_k),\dots,m(c'_k)$ that are non-diagonal terms. Thus, for each set of indices $(c_1,\dots,c_k,c'_1,\dots,c'_k)$, we have at least $k-b$ of these indices such that $1(c_r),\dots,m-1(c_r),1(c'_r),\dots,m(c'_r)$ are all diagonal entries. For such c_r and c'_r,

$$\Big((t'_{1(c_r)}(1)\cdots t'_{m-1(c_r)}(1))^{m-s} (t'_{1(c'_r)}(1)\cdots t'_{m(c'_r)}(1))^{s-m+1} + \cdots + (t'_{1(c_r)}(l^r)\cdots t'_{m-1(c_1)}(l))^{m-s} (t'_{1(c'_r)}(l^r)\cdots t'_{m(c'_r)}(l^r))^{s-m+1} \Big)$$
$$\le \max_{\{j_1,\dots,j_{m-1}\},\{j'_1,\dots,j'_m\}} \Big((t'_{j_1j_1}(1)\cdots t'_{j_{m-1}j_{m-1}}(1))^{m-s} (t'_{j'_1}(1)\cdots t'_{j'_m}(1))^{s-m+1} + \cdots + (t'_{j_1j_1}(l^r)\cdots t'_{j_{m-1}j_{m-1}}(l^r))^{m-s} (t'_{j'_1}(l^r)\cdots t'_{j'_mj'_m}(l^r))^{s-m+1} \Big)$$
$$= T(s,r).$$

The last equality follows from the definition of $\{G_h\}_{h=1}^{l^r}$ and $T(s,r)$. Hence from (3.13),

$$(3.14)\qquad \sum_{|\mathbf{h}|=k} \sup_{\underline{x}} |D_{\underline{x}} G_{\mathbf{h}}^{(m-1)}|^{m-s} \sup_{\underline{x}} |D_{\underline{x}} G_{\mathbf{h}}^{(m)}|^{s-m+1} \le \sum_{\substack{c_1,\dots,c_k\\ c'_1,\dots,c'_k}} T(s,r)^{k-b}(l^r)^b \le c'' k^q l^{rb} T(s,r)^{k-b},$$

where, by Lemma 3.6, $c'' = m!(m-1)!$ and $q = (2m-1)(n-1)$.

By using (3.7), (3.9), (3.10) and (3.14), we obtain

$$\tag{3.15} \begin{aligned} \sum_{|\mathbf{i}|=kr} \overline{\phi}^s(\mathbf{i}) &= \sum_{|\mathbf{h}|=k} \overline{\phi'}^s(\mathbf{h}) \leq c'' k^q l^{rb} T(s,r)^{k-b} \\ &\leq c''(C^s)^k k^q l^{rb} T(s,r)^{-b} H(s,r)^k \\ &\leq c'''(C^s)^k k^q l^{rb} T(s,r)^{-b} \sum_{|\mathbf{h}|=k} \underline{\phi'}^s(\mathbf{h}) \\ &= c''' k^q l^{rb} T(s,r)^{-b} \sum_{|\mathbf{i}|=kr} \underline{\phi}^s(\mathbf{i}). \end{aligned}$$

We take the logarithm of both sides and divide by kr to obtain

$$\tag{3.16} \begin{aligned} \frac{\log \sum_{|\mathbf{i}|=kr} \overline{\phi}^s(\mathbf{i})}{kr} \leq \frac{\log c'''}{kr} + \frac{q \log k}{kr} + \frac{rb \log l}{kr} + \frac{(kb)\log(C^s)}{kr} \\ + \frac{-b \log T(s,r)}{kr} + \frac{\log \sum_{|\mathbf{i}|=kr} \underline{\phi}^s(\mathbf{i})}{kr} \end{aligned}$$

for any positive integers k, r. We take the limit inferior of both sides as $k \to \infty$ and $r \to \infty$. The limit on the left-hand side of the inequality exists, and on the right-hand side the limit of every term exists and equals zero except the last term. Therefore

$$P(s) \leq \underline{P}(s).$$

As the opposite relation is trivial this completes the proof. ■

The next corollary is a consequence of the previous proof.

Corollary 3.9. *Let $0 \leq s \leq n$. If $F_1, \ldots, F_l$ are contractive maps of the form* (2.1) *and $F_i \in C^{1+\varepsilon}$ for every $1 \leq i \leq l$ then*

$$\tag{3.17} \begin{aligned} P(s) = \lim_{r\to\infty} \frac{1}{r} \log\Big(\max_{\substack{j_1,\ldots,j_{m-1} \\ j'_1,\ldots,j'_m}} \sum_{|\mathbf{i}|=r} (|x_{j_1 j_1}(\mathbf{i},\underline{x})| \cdots |x_{j_{m-1} j_{m-1}}(\mathbf{i},\underline{x})|)^{m-s} \\ \times (|x_{j'_1 j'_1}(\mathbf{i},\underline{x})| \cdots |x_{j'_m j'_m}(\mathbf{i},\underline{x})|)^{s-m+1} \Big) \end{aligned}$$

for every $\underline{x} \in M$.

Proof. It follows from inequality (3.7) that $\lim_{r\to\infty} \log H(s,r)/r$ exists and

$$\lim_{r\to\infty} \frac{\log H(s,r)}{r} = \lim_{r\to\infty} \frac{\log T(s,r)}{r}.$$

It is clear by (3.15) that $\lim_{r\to\infty} (\log T(s,r))/r = P(s)$. Because of the definition of $H(s,r), T(s,r)$, this is exactly what we want to prove. ■

4. Some applications. In this section we compute the Hausdorff dimension of some non-conformal IFS by using Corollary 3.9. It follows from

[11] that the Hausdorff dimension is less than or equal to s_0 where $P(s_0) = 0$. We will show some examples where the root is exactly the dimension.

EXAMPLE 1. The easiest example is the non-linear modified Sierpiński triangle. Let

$$T = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix}$$

and $T_i\underline{x} = T\underline{x} + \underline{v}_i$ for $i = 1, 2, 3$, where $\underline{v}_1 = \binom{0}{0}, \underline{v}_2 = \binom{2/3}{0}, \underline{v}_3 = \binom{1/3}{2/3}$. We call the attractor of this IFS a *modified Sierpiński triangle*. Clearly, its Hausdorff and box dimension is $\frac{\ln 3}{\ln 3} = 1$.

Let $f_i : [0, 1] \to [0, 1]$ for $i = 1, 2, 3$ be functions in $C^{1+\varepsilon}$ such that

$$F_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x/3 + v_i \\ y/3 + f_i(x) + w_i \end{pmatrix}$$

are contractions where $\binom{v_1}{w_1} = \binom{0}{0}, \binom{v_2}{w_2} = \binom{2/3}{0}, \binom{v_3}{w_3} = \binom{1/3}{1/2}$. We can consider the attractor as a non-linear Sierpiński triangle.

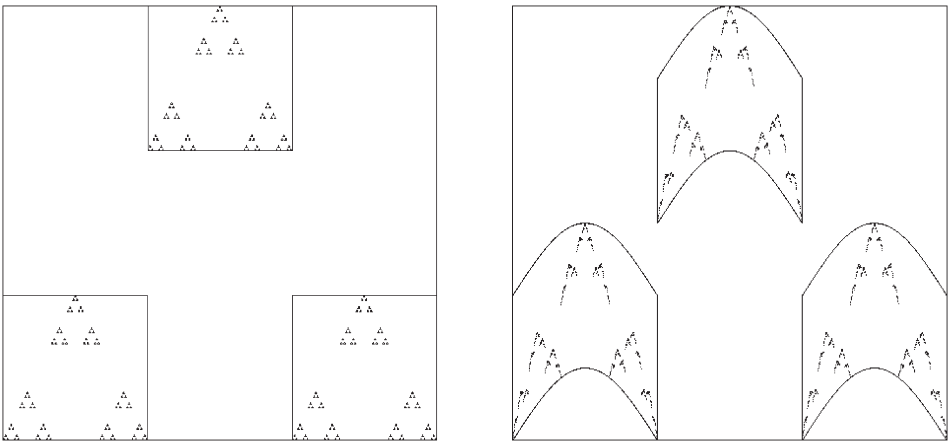

Fig. 1. The image of the modified and the non-linear modified Sierpiński triangle for $f_i(x) = \sin(\pi x)/6$ for every i.

We prove that the Hausdorff dimension of the non-linear modified Sierpiński triangle is equal to 1, assuming that for $i = 1, 2, 3$ we have $f_i \in C^{1+\varepsilon}$ and

$$(f_i'(x))^2 + |f_i'(x)|\sqrt{(f_i'(x))^2 + 4/9} < 16/9.$$

We need this assumption to ensure that $\{F_1, F_2, F_3\}$ is contracting.

From the definition in this case it is easy to see that $x_{11}(\mathbf{i}, \underline{x}) = x_{22}(\mathbf{i}, \underline{x}) = \frac{1}{3}^{|\mathbf{i}|}$. We can suppose that $1 \leq s < 2$. Then by Corollary 3.9,

$$
\begin{aligned}
P(s) &= \lim_{r\to\infty}\frac{1}{r}\log\Big(\max_{j_1,j_1',j_2'}\sum_{|\mathbf{i}|=r}(|x_{j_1j_1}(\mathbf{i},\underline{x})|)^{2-s}\cdot(|x_{j_1'j_1'}(\mathbf{i},\underline{x})|\,|x_{j_2'j_2'}(\mathbf{i},\underline{x})|)^{s-2+1}\Big)\\
&= \lim_{r\to\infty}\frac{1}{r}\log\left(\sum_{|\mathbf{i}|=r}\left(\frac{1}{3}^{|\mathbf{i}|}\right)^{2-s}\left(\frac{1}{3}^{|\mathbf{i}|}\frac{1}{3}^{|\mathbf{i}|}\right)^{s-1}\right)=\lim_{r\to\infty}\frac{1}{r}\log\left(3^r\frac{1}{3}^{sr}\right)\\
&= \log 3 - s\log 3.
\end{aligned}
$$

It is easy to see that $P(s)=0$ if and only if $s=1$, which is the upper bound of the Hausdorff dimension of the modified non-linear attractor, as follows from [11]. To get a lower bound it is enough to project it onto the x axis and we get the interval $[0,1]$.

Example 2. The next example is a non-linear perturbation of a self-affine IFS. Let $c_1,c_2\in(0,1)$. Consider the following self-affine IFS:

$$
g_0(\underline{x})=\begin{bmatrix} c_1 & 0\\ 0 & c_2\end{bmatrix}\underline{x},\qquad g_1(\underline{x})=\begin{bmatrix} 1-c_1 & 0\\ 0 & 1-c_2\end{bmatrix}\underline{x}+\begin{bmatrix} c_1\\ c_2\end{bmatrix}.
$$

It is easy to see that the attractor of this IFS has Hausdorff dimension 1 since it is the graph of a strictly monotone function. We perturb this IFS as follows. Let

$$
\widetilde{g}_0(x,y)=\begin{bmatrix} c_1x\\ c_2y+f_0(x)\end{bmatrix},\qquad \widetilde{g}_1(x,y)=\begin{bmatrix} (1-c_1)x+c_1\\ (1-c_2)y+c_2+f_1(x)\end{bmatrix}.
$$

where $f_0,f_1\in C^{1+\varepsilon}$ and f_i are periodic with period 1. Moreover, we suppose

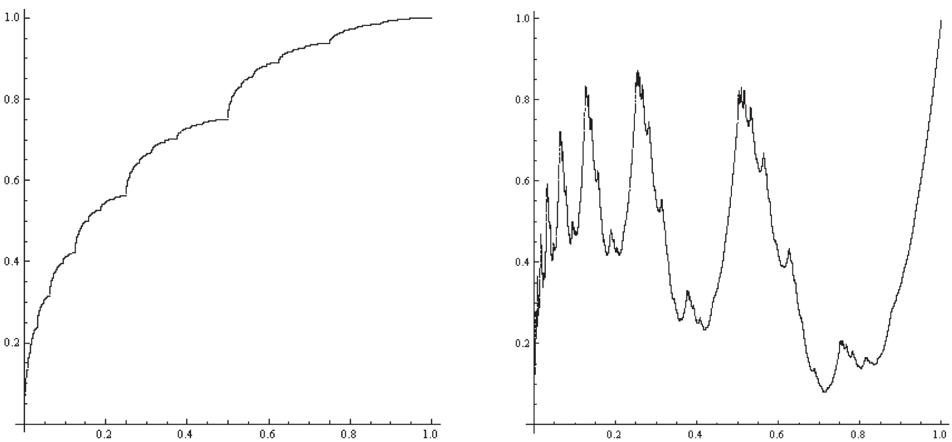

Fig. 2. The images of the attractors in case $c_1=1/2$, $c_2=1/4$, $f_0(x)=(1-c_2)\sin(\pi x)$, $f_1(x)=-c_2\sin(\pi x)$.

that $\widetilde{g}_0, \widetilde{g}_1$ are contractions, namely the following inequalities hold:

$$c_1^2 + (f_0'(x))^2 + c_2^2 + \sqrt{(c_1^2 + (f_0'(x))^2 + c_2^2)^2 - 4c_1^2c_2^2} < 2,$$

$$(1-c_1)^2 + (f_1'(x))^2 + (1-c_2)^2 + \sqrt{((1-c_1)^2 + (f_1'(x))^2 + (1-c_2)^2)^2 - 4(1-c_1)^2(1-c_2)^2} < 2.$$

In this case the Hausdorff dimension of the modified attractor is greater than or equal to 1 since the projection to the x axis is the interval $[0,1]$. To get an upper bound we have to use the subadditive pressure and Corollary 3.9. For every $\mathbf{i} \in \{0,1\}^*$ we have $x_{11}(\mathbf{i}, \underline{x}) = c_1^{\sharp_0 \mathbf{i}}(1-c_1)^{\sharp_1 \mathbf{i}}$ and $x_{22}(\mathbf{i}, \underline{x}) = c_2^{\sharp_0 \mathbf{i}}(1-c_2)^{\sharp_1 \mathbf{i}}$ where $\sharp_j \mathbf{i}$ is the number of js in $\mathbf{i}$. Then

$$\begin{aligned}
&\max_j \sum_{|\mathbf{i}|=r} x_{jj}(\mathbf{i}, \underline{x})^{2-s}(x_{11}(\mathbf{i}, \underline{x}) x_{22}(\mathbf{i}, \underline{x}))^{s-2+1} \\
&= \max_j \sum_{|\mathbf{i}|=r} c_j^{(2-s)\sharp_0 \mathbf{i}}(1-c_j)^{(2-s)\sharp_1 \mathbf{i}} c_1^{(s-1)\sharp_0 \mathbf{i}}(1-c_1)^{(s-1)\sharp_1 \mathbf{i}} \\
&\qquad\qquad \times c_2^{(s-1)\sharp_0 \mathbf{i}}(1-c_2)^{(s-1)\sharp_1 \mathbf{i}} \\
&= \max\{(c_1 c_2^{s-1} + (1-c_1)(1-c_2)^{s-1})^r, (c_2 c_1^{s-1} + (1-c_2)(1-c_1)^{s-1})^r\}.
\end{aligned}$$

Therefore by formula (3.17) we have $P(1) = 0$, and by [11], 1 is an upper bound for the Hausdorff dimension, so the Hausdorff dimension is exactly 1.

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