An Isomorphic Classification of $C(2^m \times [0, \alpha])$ Spaces

by

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Summary. We present an extension of the classical isomorphic classification of the Banach spaces $C([0, \alpha])$ of all real continuous functions defined on the nondenumerable intervals of ordinals $[0, \alpha]$. As an application, we establish the isomorphic classification of the Banach spaces $C(2^m \times [0, \alpha])$ of all real continuous functions defined on the compact spaces $2^m \times [0, \alpha]$, the topological product of the Cantor cubes $2^m$ with $m$ smaller than the first sequential cardinal, and intervals of ordinal numbers $[0, \alpha]$. Consequently, it is relatively consistent with ZFC that this yields a complete isomorphic classification of $C(2^m \times [0, \alpha])$ spaces.

1. Introduction and statement of the main result. Throughout the paper, we use standard notation and basic concepts in set theory [11] and theory of Banach spaces [12]. However, we want to explain some frequently used terms and fix some notations. For a compact Hausdorff topological space $K$ and $X$ a Banach space, let $C(K, X)$ denote the Banach space of all continuous $X$-valued functions defined on $K$, equipped with the usual supremum norm. When $X = \mathbb{R}$, the set of real numbers, this space will be denote by $C(K)$. As usual, if $K_1$ and $K_2$ are compact spaces, we denote by $K_1 \oplus K_2$ and $K_1 \times K_2$ respectively the topological sum and the topological product of $K_1$ and $K_2$. For a fixed cardinal $m$, $2^m$ denotes the product of $m$ family of copies of the two-point space 2, provided with the product topology. For $\alpha$ an ordinal number, $[0, \alpha]$ denotes the interval of ordinals $\{\xi : 0 \leq \xi \leq \alpha\}$ endowed with the order topology. If $X$ and $Y$ are Banach spaces, then $X \sim Y$ means that $X$ is isomorphic to $Y$. Finally, the symbol $X \oplus Y$ denotes the Cartesian product of $X$ and $Y$.

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Very recently \cite{Galego2020}, it has been shown that is relatively consistent with Zermelo–Fraenkel set theory plus the axiom of choice (ZFC) that for any infinite cardinals $m$ and $n$ and nondenumerable ordinals $\xi$ and $\eta$ we have

$$C(2^m \oplus [0, \xi]) \sim C(2^n \oplus [0, \eta]) \quad \text{if and only if} \quad m = n \text{ and } C([0, \xi]) \sim C([0, \eta]).$$

In other words, the isomorphic classification of $C(2^m \oplus [0, \alpha])$ spaces is reduced to the isomorphic classification of $C([0, \alpha])$ spaces, in the case where $\alpha \geq \omega_1$. Recall that the isomorphic classification of $C([0, \alpha])$ spaces is due to Bessaga and Pełczyński \cite{Bessaga1961} in the case where $\omega \leq \alpha < \omega_1$; Semadeni \cite{Semadeni1963} in the case where $\omega_1 < \alpha \leq \omega_1 \omega$; Labbé \cite{Labbé1965} in the case where $\omega_1 \omega < \alpha < \omega^*\omega$; and independently Kislyakov \cite{Kislyakov1972} and Gul’ko and Os’kin \cite{Gulkko1972} in the general case.

In the present paper, we turn our attention to $C(2^m \times [0, \alpha])$ spaces. In contrast with the isomorphic classification of $C(2^m \oplus [0, \alpha])$ spaces mentioned above, the situation becomes quite different when we consider this new family of $C(K)$ spaces. This happens even in the isometric case. Indeed, it is well known that for every infinite cardinal $m$, $2^m$ is homeomorphic to $2^m \times 2^m$. Consequently, $2^m \times [0, \omega_1]$ is homeomorphic to $2^m \times [0, \omega_1 2]$. Therefore by \cite{Kislyakov1972} Theorem 1] we deduce that

$$C(2^m \times [0, \omega_1]) \text{ is isometric to } C(2^m \times [0, \omega_1 2]) \quad \text{but} \quad C([0, \omega_1]) \sim C([0, \omega_1 2]).$$

This motivates us to study the isomorphic classification of $C(2^m \times [0, \alpha])$ spaces. In order to do this, we will prove Theorem 1.7, which is an extension of the isomorphic classification of $C([0, \alpha])$ spaces, with $\alpha \geq \omega_1$.

**Remark 1.1.** First notice that if $0 \leq m < \aleph_0$, then $C(2^m \times [0, \alpha]) \sim C([0, \alpha 2^m])$. Assume now that $m \geq \aleph_0$ and $\alpha < \omega_1$. According to the classical Milyutin theorem \cite{Milyutin1963} Theorem 21.5.10] about the isomorphic classification of $C(K)$ spaces with $K$ compact metric nondenumerable, $C(2^{\aleph_0} \times [0, \alpha]) \sim C(2^{\aleph_0})$. Consequently, by \cite{Milyutin1963} Theorem 20.5.6] we deduce $C(2^m \times [0, \alpha]) \sim C(2^{\aleph_0} \times [0, \alpha]) \sim C(2^m, C(2^{\aleph_0} \times [0, \alpha])) \sim C(2^m \times 2^{\aleph_0}) \sim C(2^m)$. Observe also that for every $m \geq \aleph_0$, $n \geq \aleph_0$ and $\alpha \geq \omega_1$, we have $C(2^m) \sim C(2^n \times [0, \alpha])$. Indeed, otherwise $C([0, \alpha])$ would be isomorphic to a subspace of $C(2^m)$, which is absurd by \cite{Arosio1995} Theorem 4.5] and \cite{Arhangel'skii1983} Theorem 2.3.17]

So it remains to consider the cases $m \geq \aleph_0$ and $\alpha \geq \omega_1$. In order to describe our results, we recall that a cardinal $m$ is called **sequential** if there exists a sequentially continuous but not continuous real-valued function on $2^m$. We also recall that a function $f : 2^m \to \mathbb{R}$ is said to be **sequentially continuous** when $f(k_n)$ converges to $f(k)$ whenever the sequence $(k_n)_{n<\omega}$ converges to $k$ in $2^m$ (see \cite{Arosio1995} and \cite{Arhangel'skii1983}). The cardinality of the ordinal $\xi$ will be denoted by $\xi$. Our first result is as follows.
THEOREM 1.2. Suppose that m and n are nonsequential infinite cardinals and \( \xi \) and \( \eta \) are nondenumerable ordinals. Then

\[
C(2^m \times [0, \xi]) \sim C(2^n \times [0, \eta]) \quad \text{implies that} \quad m = n \quad \text{and} \quad \bar{\xi} = \bar{\eta}.
\]

The following is an analogue of the above mentioned result of \cite{8}.

THEOREM 1.3. Let m be a nonsequential infinite cardinal, \( \alpha \) a nondenumerable initial ordinal and \( \xi \leq \eta \) ordinals with \( \bar{\xi} = \bar{\eta} = \bar{\alpha} \). If \( \alpha \) is singular or \( \alpha^2 \leq \xi \), then

\[
C(2^m \times [0, \xi]) \sim C(2^m \times [0, \eta]) \quad \text{if and only if} \quad C([0, \xi]) \sim C([0, \eta]).
\]

The next theorems complete the isomorphic classification of \( C(2^m \times [0, \alpha]) \) spaces with m a nonsequential cardinal.

THEOREM 1.4. Let m be a nonsequential infinite cardinal, \( \alpha \) a nondenumerable regular ordinal and \( \xi \) and \( \eta \) in \([\alpha, \alpha^2]\). Let \( \xi', \eta', \gamma \) and \( \delta \) be ordinals such that \( \xi = \alpha \xi' + \gamma \), \( \eta = \alpha \eta' + \delta \), \( \xi', \eta' \leq \alpha \) and \( \gamma, \delta < \alpha \). Then

\[
C(2^m \times [0, \xi]) \sim C(2^m \times [0, \eta]) \quad \text{if and only if} \quad \bar{\xi}' \bar{\eta}' \leq \aleph_0 \quad \text{or} \quad \bar{\xi}' = \bar{\eta}'.
\]

THEOREM 1.5. Let m be a nonsequential infinite cardinal, \( \alpha \) a nonde- numerable regular ordinal and \( \xi \) and \( \eta \) with \( \bar{\xi} = \bar{\eta} \) and \( \alpha \leq \xi < \alpha^2 \leq \eta \). Then

\[
C(2^m \times [0, \xi]) \not\sim C(2^m \times [0, \eta]).
\]

REMARK 1.6. It is well known that it is relatively consistent with ZFC that there exist no sequential cardinals (see \cite{20}). So it is relatively consistent with ZFC that Theorems 1.2–1.5 provide a complete isomorphic classification of \( C(2^m \times [0, \alpha]) \) spaces. Furthermore, since \( \aleph_0 \) is not sequential \cite{17}, the above theorems give a complete isomorphic classification of \( C(2^{\aleph_0} \times [0, \alpha]) \) spaces, without using the continuum hypothesis. Thus, we have got an answer to Question 3.5 raised in \cite{7}.

Although our work is motivated by the search for the isomorphic classification of \( C(2^m \times [0, \alpha]) \) spaces, our main result holds for a more general setting. Indeed, from now on, our task is to prove Theorem 1.7. The preceding theorems are immediate consequences of Theorem 1.7 and Lemma 2.5.

Henceforth following \cite{2}, the \( C([0, \alpha], X) \) spaces will also be denoted by \( X^\alpha \). Theorem 1.7 states that the isomorphic classification of \( X^\alpha \) spaces, with \( \alpha \geq \omega_1 \) and \( X \) having the Mazur property and containing no subspace isomorphic to \( c_0 \), obtained recently in \cite{9} is also true under the weaker hypothesis that \( X \) has the Mazur property and contains no subspace isomorphic to \( c_0(\Gamma) \), where \( \Gamma \) is a set of cardinality \( \aleph_1 \).

We recall that a Banach space \( X \) is said to have the Mazur property if every element of \( X^{**} \), the bidual space of \( X \), which is sequentially weak* continuous is weak* continuous and thus is an element of \( X \). Such spaces were
investigated in [1, 16] and also in [13] and [28] where they were called d-
complete and μB-spaces, respectively. The class of Banach spaces having the
Mazur property includes the $C(2^m)$ spaces for every nonsequential cardinal $m$
[20] (see also [21, Theorem 5.2.c]). Given a set $I$, we denote by $|I|$ the
 cardinality of $I$.

**Theorem 1.7.** Let $X$ be a Banach space having the Mazur property and
containing no subspace isomorphic to $c_0(I)$, where $|I| = \aleph_1$, let $\alpha$ be an
initial ordinal and $\xi \leq \eta$ two infinite ordinals.

1. If $X^\xi \sim X^\eta$ then $\hat{\xi} = \hat{\eta}$.
2. Suppose $\hat{\xi} = \hat{\eta} = \alpha'$ and assume that $\alpha'$ is a singular ordinal, or $\alpha'$ is a
   nondenumerable regular ordinal with $\alpha'^2 \leq \xi$. Then $X^\xi \sim X^\eta$ if and
   only if $\eta < \xi^\omega$.
3. Suppose that $\alpha'$ is a nondenumerable regular ordinal, $\xi, \eta' \in [\alpha, \alpha'^2]$
   and let $\xi', \eta', \gamma$ and $\delta$ be ordinals such that $\xi = \alpha\xi' + \gamma$, $\eta = \alpha\eta' + \delta$,
   $\xi', \eta' \leq \alpha$ and $\gamma, \delta < \alpha$. Then $X^\xi \sim X^\eta$ if and only if either $\xi' \eta' \leq \aleph_0$
   and $c_0(I, X) \sim c_0(J, X)$ where $I$ and $J$ are sets with $|I| = \xi'$ and
   $|J| = \eta'$, or $\xi' = \eta'$.
4. Suppose that $\alpha'$ is a nondenumerable regular ordinal and $\alpha \leq \xi < \alpha'^2 \leq \eta$.
   Then $X^\xi \sim X^\eta$.

2. Preliminary lemmas. In this section we state and prove several lemmas from which Theorem 1.7 follows easily. The first three lemmas provide
sufficient conditions for a Banach space $X$ to contain a subspace isomorphic to
$c_0(I)$, where $|I| = \aleph_1$. If $X$ and $Y$ are Banach spaces, then $X \hookrightarrow Y$
means that $X$ is isomorphic to a subspace of $Y$.

**Lemma 2.1.** Let $X$ be a Banach space and $\alpha$ an nondenumerable infinite
initial ordinal. Suppose that $\mathbb{R}^\alpha \hookrightarrow X^\eta$ for some $\eta < \alpha$. Then $c_0(I) \hookrightarrow X$, where $|I| = \aleph_1$.

**Proof.** Assume first that $\alpha$ is a regular ordinal. Let $I$ be the set of isolated
points of $[0, \alpha]$. Then $c_0(I) \hookrightarrow \mathbb{R}^\alpha$. So there exists an isomorphism $T$
from $c_0(I)$ onto a subspace of $X^\eta$. Let $M \in [0, +\infty[$ be such that $M \leq \|T(x)\|$ for
all $x \in c_0(I)$, $\|x\| = 1$. Denote by $(e_i)_{i \in I}$ the unit-vectors basis of $c_0(I)$,
that is, $e_i(j) = 1$ if $i = j$, $e_i(j) = 0$ if $i \neq j$, for all $i, j \in I$. Let $K$ be
the set of isolated points of $[0, \eta]$. Thus, by hypothesis $|K| < \alpha$. For fixed
$k \in K$, we define $I_k = \{i \in I : M/2 \leq \|T(e_i)(k)\|\}$. Therefore $I = \bigcup_{k \in K} I_k$.
Hence there is a $k \in K$ satisfying $|I_k| = |I|$. We identify $c_0(I_k)$ with the
subspace of $c_0(I)$ consisting of those elements $f$ such that $f(\gamma) = 0$ for every
$\gamma \notin I_k$. Let $P_k : X^\eta \rightarrow X$ be the natural projection, that is, $P_k(f) = f(k)$
for all $f \in X^\eta$. Next, consider the operator $L = P_k T|_{c_0(I_k)} : c_0(I_k) \rightarrow X$.
Then $\inf \{\|L(e_i)\| : i \in I_k\} > 0$. So, according to Remark 1 which follows
[22, Theorem 3.4], there exists $\Gamma \subset I_k$ with $|\Gamma| = |I_k|$ such that $L|_{c_0(\Gamma)}$ is an isomorphism onto its image. So we are done.

Let us now suppose that $\alpha$ is a singular ordinal. Then there exists an ordinal limit $\lambda$ such that $\alpha = \omega \lambda$. Let $\gamma$ be an ordinal satisfying $\eta < \omega \gamma + 1 < \omega \lambda$. It is known that $\omega \gamma + 1$ is regular. Moreover, by hypothesis $\mathbb{R}^{\omega \gamma + 1} \hookrightarrow \mathbb{R}^{\omega \lambda} \hookrightarrow X^\eta$. Hence by what we have just proved, we conclude that $c_0(\Gamma) \hookrightarrow X$, where $|\Gamma| = \aleph_1$. ■

In the same fashion we can prove:

**Lemma 2.2.** Let $X$ be a Banach space such that $c_0(I) \hookrightarrow c_0(J, X)$ for some sets $I$ and $J$ with $|J| < |I|$ and $|I| \geq \aleph_1$. Then $c_0(\Gamma) \hookrightarrow X$, where $|\Gamma| = \aleph_1$.

Before stating the next lemma, we recall some definitions from [6] and [9]. Let $\gamma$ be an ordinal. A $\gamma$-sequence in a set $A$ is a function $f : [1, \gamma] \to A$ and will be denoted by $(x_\theta)_{\theta < \gamma}$. If $A$ is a topological space and $\beta$ is an ordinal, we will say that the $\gamma$-sequence $(x_\theta)_{\theta < \gamma}$ is $\beta$-continuous if for every $\beta$-sequence of ordinals $(\theta_\xi)_{\xi < \beta}$ of $[0, \gamma]$ which converges to $\theta_\beta$ when $\xi$ converges to $\beta$, the $\beta$-sequence $x_{\theta_\xi}$ converges to $x_{\theta_\beta}$.

Let $X$ be a Banach space, $\alpha$ an ordinal number and $\varphi$ a cardinal number. By $X_\varphi^\alpha$ we will denote the space of all $x^{**} \in X^{**}$ having the following property: for every set $B$ with $|B| = \varphi$, $\beta < \alpha$ and $B$-family $x^b = (x_\xi^b)_{\xi < \beta}$, $b \in B$, of $\beta$-sequences of $X^*$ such that there exists $M \in \mathbb{R}$ with $\|x_\xi^b(b)\| \leq M$ for every $b \in B$ and $\xi < \beta$ and such that $x_\xi^b(b(x)) \xrightarrow{\xi \to \beta} 0$ for all $x \in X$, uniformly in $b$, we have $x^{**}(x_\xi^b(b)) \xrightarrow{\xi \to \beta} 0$ uniformly in $b$.

Clearly $X_\varphi^\alpha$ is a closed subspace of $X^{**}$ and $cX \subset X_\varphi^\alpha$, where $cX$ is the canonical image of $X$ in $X^{**}$. Observe that if $X$ has the Mazur property, then $X_\varphi^\alpha = cX$.

Let $X$ be a Banach space and $\alpha$ a nondenumerable regular ordinal. Following [9, Definition 2.2], we set $[X]_\alpha = \bigcap_{\varphi < \alpha} X_\varphi^\alpha$.

We are ready to generalize [9, Lemma 2.8].

**Lemma 2.3.** Let $X$ be a Banach space having the Mazur property and $\alpha$ a nondenumerable regular ordinal. If $\mathbb{R}^{\alpha^2} \hookrightarrow X^\eta$ for some $\eta < \alpha^2$, then $c_0(\Gamma) \hookrightarrow X$, where $|\Gamma| = \aleph_1$.

**Proof.** We distinguish two cases:

**Case 1:** $\eta < \alpha$. In this case, $\mathbb{R}^{\alpha} \hookrightarrow X^\eta$. Hence by Lemma 2.1, $c_0(\Gamma) \hookrightarrow X$, where $|\Gamma| = \aleph_1$, and we are done.

**Case 2:** $\alpha \leq \eta < \alpha^2$. Thus $\eta = \alpha \xi + \theta$ for some ordinals $\xi < \alpha$ and $\theta < \alpha$. Since then $\mathbb{R}^\eta \sim \mathbb{R}^{\alpha \xi}$ [14, Theorem 2], we have $\mathbb{R}^{\alpha^2} \hookrightarrow \mathbb{R}^\eta \hookrightarrow C([0, \eta], X) \sim C([0, \alpha \xi], X)$. 

\
Let $I$ and $J$ be two sets with $|I| = \bar{1}$ and $|J| = \bar{2}$. According to [9, Lemma 2.10], we have
\[ c_0(I) \sim \frac{[\mathbb{R}^{\alpha_1}]_{\alpha} \hookrightarrow [X^{\alpha_2}]_{\alpha} \sim c_0(J, X). \]
Thus by Lemma 2.2 we infer that $c_0(I) \hookrightarrow X$, where $|I| = \aleph_1$. So we are also done. ■

The main step in proving Theorem 1.7 is the following result. It is a generalization of part of [9, Lemma 2.10] (see also [25, Theorem 3.2]).

**Lemma 2.4.** Let $\alpha$ be a nondenumerable initial ordinal and $\xi \leq \eta$ ordinals with $\bar{\xi} = \bar{\eta} = \bar{\alpha}$. Put $\alpha_0 = \alpha$ if $\alpha$ is a singular ordinal and $\alpha_0 = \alpha^2$ if $\alpha$ is a regular ordinal. Suppose that $X$ is a Banach space having the Mazur property and containing no subspace isomorphic to $c_0(I)$, where $|I| = \aleph_1$.

If $\mathbb{R}^\eta \hookrightarrow X^\xi$ with $\alpha_0 \leq \xi$, then $\mathbb{R}^\eta \hookrightarrow \mathbb{R}^\xi$.

**Proof.** We introduce two sets of ordinals
\[ I_1 = \{ \theta : \bar{\theta} = \bar{\alpha}, \alpha_0 \leq \theta, \mathbb{R}^\theta \hookrightarrow \mathbb{R}^\gamma, \forall \gamma < \theta \}, \]
\[ I_2 = \{ \theta : \bar{\theta} = \bar{\alpha}, \alpha_0 \leq \theta, \mathbb{R}^\theta \hookrightarrow X^\gamma, \forall \gamma < \theta \}. \]

First of all we will prove that $I_1 = I_2$. Clearly $I_2 \subseteq I_1$. Observe that by Lemmas 2.1 and 2.3 we deduce that $\alpha_0 \notin I_2$. Now, assume that $I_2$ is a proper subset of $I_1$. Let $\alpha_1$ be the least element of $I_1 \setminus I_2$. We have $\alpha_0 < \alpha_1$. Since $\alpha_1 \notin I_2$, there exists an ordinal $\gamma_1 < \alpha_1$ such that $\mathbb{R}^{\alpha_1} \hookrightarrow X^{\gamma_1}$.

Let $\alpha_2 = \min \{ \gamma : \alpha_0 \leq \gamma < \alpha_1, \mathbb{R}^{\alpha_1} \hookrightarrow X^{\gamma} \}$. We have $\alpha_2 \leq \gamma_1$. Now, we will show that $\alpha_2 \in I_2$. If this is not the case, there exists an ordinal $\gamma_2 < \alpha_2$ such that $\mathbb{R}^{\alpha_2} \hookrightarrow \mathbb{R}^{\gamma_2}$. Therefore $C([0, \alpha_2], X) \hookrightarrow C([0, \gamma_2], X)$. Consequently, $\mathbb{R}^{\alpha_1} \hookrightarrow X^{\gamma_2}$, in contradiction with the definition of $\alpha_2$.

So $\alpha_2 \in I_1$ and since $\alpha_2 < \alpha_1$, it follows from the definition of $\alpha_1$ that $\alpha_2 \in I_2$. That is, $\mathbb{R}^{\alpha_2} \hookrightarrow X^{\gamma}$ for all $\gamma < \alpha_2$. Thus by [7, Lemma 3.3], we conclude that $\mathbb{R}^{\alpha_2} \hookrightarrow X^{\alpha_2}$.

On the other hand, note that if $\alpha_1 < \alpha_2^\eta$, then by [14, Theorems 1 and 2], $\mathbb{R}^{\alpha_1} \sim \mathbb{R}^{\alpha_2}$, which is absurd by the definition of $\alpha_1$. Consequently, $\alpha_2^\eta \leq \alpha_1$ and $\mathbb{R}^{\alpha_1} \hookrightarrow X^{\alpha_2}$. Therefore $\mathbb{R}^{\alpha_2} \hookrightarrow X^{\alpha_2}$, in contradiction with what we have just proved above. Hence $I_1 = I_2$.

Next, to complete the proof of the lemma, suppose that $\mathbb{R}^\eta \hookrightarrow \mathbb{R}^\xi$ and let $\xi_1 = \min \{ \theta : \mathbb{R}^\eta \hookrightarrow \mathbb{R}^\theta \}$. Hence $\xi < \xi_1 \leq \eta$ and $\mathbb{R}^{\xi_1} \hookrightarrow \mathbb{R}^\eta$ for all $\gamma < \xi_1$. In particular, $\xi_1 \in I_1 = I_2$, which is absurd, because $\mathbb{R}^{\xi_1} \hookrightarrow \mathbb{R}^\eta \hookrightarrow X^\xi$.

We conclude this section by proving part of Theorem 1.2.

**Lemma 2.5.** If $C(2^m \times [0, \xi]) \sim C(2^n \times [0, \eta])$ for some infinite cardinals $m$ and $n$ and ordinals $\xi$ and $\eta$, then $m = n$. 
Proof. Assume that \( m < n \) and let \( I \) and \( A \) be two sets of the same cardinality of \( \xi \) and \( \eta \), respectively. Therefore \( l_1(I, C(2^m)^*) \sim l_1(A, C(2^n)^*) \). According to \( \text{[28]} \) Proposition 5.2 we infer that

\[
l_1(I, \left( \sum_{2^m} \oplus L^1[0, 1]^m \right)_1) \sim l_1(A, \left( \sum_{2^n} \oplus L^1[0, 1]^n \right)_1).
\]

Recall that given a Banach space \( X \), the dimension of \( X \) is the smallest cardinal \( \delta \) for which there exists a subset of cardinality \( \delta \) with linear span norm-dense in \( X \). Pick a subspace \( H \) of \( L^1[0, 1]^m \) which is isomorphic to a Hilbert space of dimension \( n \) \( \text{[24]} \) Proposition 1.5. Hence

\[
H \hookrightarrow l_1(I, \left( \sum_{2^m} \oplus L^1[0, 1]^m \right)_1).
\]

Since \( H \) contains no subspace isomorphic to \( l_1 \), by a standard gliding hump argument (see \( \text{[3]} \)), we infer that there exist a finite sum of \( L^1[0, 1]^m \) and \( 1 \leq p < \omega \) such that

\[
H \hookrightarrow L^1[0, 1]^m \oplus L^1[0, 1]^m \oplus \cdots \oplus L^1[0, 1]^m \oplus \mathbb{R}^p,
\]

which is absurd, because it is easy to see that the dimension of \( L^1[0, 1]^m \) is \( m \). \( \blacksquare \)

3. Proof of Theorem 1.7. (1) Assume that \( X^\xi \sim X^\eta \) and \( \bar{\xi} < \bar{\eta} \). Let \( \alpha \) be the initial ordinal of cardinality \( \bar{\eta} \). Then \( \mathbb{R}^\alpha \hookrightarrow X^\eta \sim X^\xi \) and by Lemma 2.1, \( c_0(I) \hookrightarrow X \), where \( |I| = \aleph_1 \), which is absurd.

To prove the sufficiency of the statements (2) and (3) it is enough to keep in mind \( \text{[14]} \) Theorems 1 and 2] and observe that if \( \mathbb{R}^\xi \sim \mathbb{R}^\eta \) then \( X^\xi \sim X^\eta \).

Next we prove the necessity of the statements (2) and (3).

(2) Suppose that \( X^\eta \sim X^\xi \). If \( \eta > \xi^\omega \), then \( \mathbb{R}^\eta \hookrightarrow X^\eta \sim X^\xi \). According to Lemma 2.4 we obtain \( \mathbb{R}^\xi \hookrightarrow \mathbb{R}^\xi \), which is absurd by \( \text{[14]} \) Theorem 1].

(3) Let \( I \) and \( J \) be two sets with \( |I| = \bar{\xi}' \) and \( |J| = \bar{\eta}' \). Since \( X \) has the Mazur property, by \( \text{[9]} \) Remark 2.3 and \( \text{[9]} \) Proposition 2.8, we infer that

\[
c_0(I, X) \sim \frac{X^{\alpha \xi'}}{cX^{\alpha \xi'}} \sim \frac{X^{\alpha \eta'}}{cX^{\alpha \eta'}} \sim c_0(J, X).
\]

Therefore if \( \bar{\xi}' \bar{\eta}' \leq \aleph_0 \) we are done. Suppose now that \( \bar{\xi}' \bar{\eta}' > \aleph_0 \) and \( \bar{\xi}' \neq \bar{\eta}' \). We can assume without loss of generality that \( \bar{\xi}' > \bar{\eta}' \) and \( \bar{\xi}' \geq \aleph_1 \). Since \( c_0(I) \hookrightarrow c_0(J, X) \), it follows by Lemma 2.2 that \( c_0(I) \hookrightarrow X \), where \( |I| = \aleph_1 \), a contradiction.

(4) Suppose that \( X^\xi \sim X^\eta \) with \( \alpha \leq \xi < \alpha^2 \leq \eta \). Then \( \mathbb{R}^{\alpha^2} \hookrightarrow X^\xi \). Hence by Lemma 2.3, \( c_0(I) \hookrightarrow X \), where \( |I| = \aleph_1 \). This contradiction finishes the proof.
4. Some questions. If we are working only in ZFC theory, then the following question arises naturally.

**QUESTION 4.1.** *Is the assumption that the cardinal \( m \) is not sequential in Theorems 1.2–1.5 necessary?*

We close the paper by recalling that the isometric classification of \( C([0, \alpha]) \) spaces is a direct consequence of the homeomorphic classification of \([0, \alpha]\) spaces accomplished by Mazurkiewicz and Sierpiński [18] and the classical Banach–Stone Theorem [27, Theorem 7.8.4]. Therefore our result also leads naturally to the following question:

**QUESTION 4.2.** *Give an isometric classification of \( C(2^m \times [0, \alpha]) \) spaces.*

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