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ALGEBRAIC GEOMETRY

A Note on the Rational Cuspidal Curves

by

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Summary. In this short note we give an elementary combinatorial argument, showing that the conjecture of J. Fernández de Bobadilla, I. Luengo-Velasco, A. Melle-Hernández and A. Némethi [Proc. London Math. Soc. 92 (2006), 99–138, Conjecture 1] follows from Theorem 5.4 of Brodzik and Livingston [arXiv:1304.1062] in the case of rational cuspidal curves with two critical points.

1. Introduction. In this short note we deal with irreducible algebraic curves $C \subset \mathbb{C}P^2$. Such a curve has a finite set $\{z_i\}_{i=1}^n$ of singular points such that a neighbourhood of each singular point intersects C in a cone over a link $K_i \subset S^3$. We would like to know what possible configurations $\{K_i\}_{i=1}^n$ of links arise in this way. We consider only the case in which each K_i is connected (in this case K_i is a knot), and thus C is a rational curve, meaning that there is a rational surjective map $\mathbb{C}P^1 \to C$. Such a curve is called *rational cuspidal*. We refer to [M] for a survey on rational cuspidal curves.

Suppose that z is a cuspidal singular point of a curve C, and B is a sufficiently small ball around z. Let $\Psi(t) = (x(t), y(t))$ be a local parametrization of $C \cap B$ near z. For any polynomial $P(x, y)$ we look at the order at 0 of the analytic map $t \mapsto P(x(t), y(t)) \in \mathbb{C}$. Let S be the set of integers which can be realized as the order for some P. Then S is a subsemigroup of $\mathbb{Z}_{\geq 0}$. We call it the *semigroup of the singular point* (see [W] for the details and proofs). The *gap sequence*, $G = \mathbb{Z}_{\geq 0} \setminus S$, has precisely $\mu/2$ elements, where the largest one is $\mu - 1$. Here μ stands for the Milnor number. Assume that

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K is the link of the singular point z. The Alexander polynomial of K can be written in the form

$$\Delta_K(t) = \sum_{i=0}^{2m} (-1)^i t^{n_i},$$

where $(n_i)_{i=0}^{2m}$ is an increasing sequence with $n_0 = 0$ and $n_{2m} = 2g$, for $g = g(K)$ being the genus of K. Writing $t^{2n_i} - t^{2n_{i-1}} = (t-1)(t^{2n_i - 1} + t^{2n_i - 2} + \cdots + t^{2n_{i-1}})$ yields the representation

$$\Delta_K(t) = 1 + (t-1)\sum_{j=1}^{k} t^{g_j} \tag{1}$$

for some finite sequence $0 < g_1 < \cdots < g_k$. We have the following lemma (see [W, Exercise 5.7.7]), which relates the Alexander polynomial to the gap sequence of a singular point.

Lemma 1. *The sequence $g_1, \ldots, g_k$ in* (1) *is the gap sequence of the semigroup of the singular point. In particular, $k = |G| = \mu/2$, where μ is the Milnor number, so $|G|$ is the genus.*

If we write $t^{g_j} = (t-1)(t^{g_j - 1} + t^{g_j - 2} + \cdots + 1) + 1$, we obtain

$$\Delta_K(t) = 1 + (t-1)g(K) + (t-1)^2 \sum_{j=0}^{\mu-2} k_j t^j,$$

where $k_j = |\{m > j : m \notin S\}|$. This motivates the following definition.

Definition. For any finite increasing sequence G of positive integers we define

$$I_G(m) = |\{k \in G \cup \mathbb{Z}_{<0} : k \geq m\}|.$$

We shall call I_G the *gap function*, because in most applications G will be the gap sequence of some semigroup.

Clearly, for $j = 0, 1, \ldots, \mu - 2$ we have $I_G(j+1) = k_j$.

In [FLMN] the following conjecture was proposed.

Conjecture 1. *Suppose that a rational cuspidal curve C of degree d has critical points $z_1, \ldots, z_n$. Let $K_1, \ldots, K_n$ be the corresponding links of singular points and let $\Delta_1, \ldots, \Delta_n$ be their Alexander polynomials. Let $g = \sum g(K_i)$. Let $\Delta = \Delta_1 \cdot \ldots \cdot \Delta_n$, expanded as*

$$\Delta(t) = 1 + \frac{(d-1)(d-2)}{2}(t-1) + (t-1)^2 \sum_{j=0}^{2g-2} k_j t^j.$$

Then for any $j = 0, \ldots, d-3$ we have $k_{d(d-j-3)} \leq (j+1)(j+2)/2$, with equality for $n = 1$.

This conjecture was verified in the case $n = 1$ by Borodzik and Livingston [BL].

We define the infimum convolution of n functions.

Definition. Let $I_1, \dots, I_n : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$. We define

$$(I_1 \diamond \cdots \diamond I_n)(k) = \min_{\substack{k_1,\dots,k_n \in \mathbb{Z} \\ k_1+\cdots+k_n=k}} (I_1(k_1) + \cdots + I_n(k_n)).$$

In [BL] the authors proved the following theorem.

Theorem 1 (see [BL, Theorem 5.4]). *Let C be a rational cuspidal curve of degree d. Let $I_1, \dots, I_n$ be all the gap functions associated to singular points on C. Then for any $j \in \{-1, 0, \dots, d-2\}$ we have*

$$I_1 \diamond \cdots \diamond I_n(jd + 1) = \frac{1}{2}(j - d + 1)(j - d + 2).$$

Note that $|G_1| + \cdots + |G_n| = (d-1)(d-2)/2$. Therefore, one can give an equivalent reformulation of Conjecture 1.

Conjecture 2. *Suppose that a rational cuspidal curve C of degree d has critical points $z_1, \dots, z_n$. Let $K_1, \dots, K_n$ be the corresponding links of singular points and let $\Delta_1, \dots, \Delta_n$ be their Alexander polynomials. Moreover, let $G_1, \dots, G_n$ be the gap sequences of these points. Let $g = |G_1| + \cdots + |G_n|$ be the genus of K. Let $\Delta = \Delta_1 \cdot \ldots \cdot \Delta_n$, expanded as*

$$\Delta(t) = 1 + (t-1)g + (t-1)^2 \sum_{j=0}^{2g-2} k_j t^j,$$

and let $I = I_1 \diamond \cdots \diamond I_n$. Then for any $j = 0, \dots, d-3$ we have $k_{d(d-j-3)} \leq I(d(d-j-3)+1)$, with equality for $n = 1$.

In this note we give an elementary argument, showing that Conjecture 1 follows from Theorem 1 for $n = 2$. The idea of our proof is to forget about the specific structure of the problem coming from the theory of singularities and to prove Conjecture 2 for general sets G_1, G_2. Namely, we have the following theorem.

Theorem 2. *Let G, H be two finite sets of positive integers and let $I_G, I_H : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ be their gap functions. Define the polynomials*

$$\Delta_G(t) = 1 + (t-1)|G| + (t-1)^2 \sum_{j \geq 0} k_j^G t^j,$$

$$\Delta_H(t) = 1 + (t-1)|H| + (t-1)^2 \sum_{j \geq 0} k_j^H t^j,$$

where $k_j^G = I_G(j+1)$, $k_j^H = I_H(j+1)$, $j \geq 0$. *Set* $\Delta = \Delta_G \cdot \Delta_H$ *and* $I = I_G \diamond I_H$. *Then*

$$\Delta(t) = 1 + (t-1)(|G| + |H|) + (t-1)^2 \sum_{j \geq 0} k_j t^j,$$

where $k_j \leq I(j+1)$ *for* $j \geq 0$.

This gives the proof of Conjecture 1 in the case $n = 2$.

It is natural to ask whether the above theorem is valid for arbitrary $n \geq 2$. Recently, after we found our elementary combinatorial argument for $n = 2$, J. Bodnár and A. Némethi [BN] showed that Conjecture 1 is false for $n \geq 3$. We provide their example below. They also found yet another proof of Conjecture 1 in the case of two singularities.

Example. Consider the semigroups $S_1 = \{6k+7l\colon k, l \geq 0\}$, $S_2 = \{2k+9l\colon k, l \geq 0\}$ and $S_3 = \{2k+5l\colon k, l \geq 0\}$. The corresponding gap sequences are $G_1 = \{1,2,3,4,5,8,9,10,11,15,16,17,22,23,29\}$, $G_2 = \{1,3,5,7\}$ and $G_3 = \{1,3\}$. Then

$$\begin{aligned}\Delta_1(t) &= 1 + (t-1)(t + t^2 + t^3 + t^4 + t^5 + t^8 + t^9 + t^{10} + t^{11} \\ &\qquad + t^{15} + t^{16} + t^{17} + t^{22} + t^{23} + t^{29}),\\ \Delta_2(t) &= 1 + (t-1)(t + t^3 + t^5 + t^7),\\ \Delta_3(t) &= 1 + (t-1)(t + t^3).\end{aligned}$$

We write $\Delta = \Delta_1 \cdot \Delta_2 \cdot \Delta_3$ in the form

$$\Delta(t) = 1 + (|G_1| + |G_2| + |G_3|)(t-1) + (t-1)^2 \sum_{j \geq 0} k_j t^j.$$

One can check that

$$\begin{aligned}(k_j)_{j=0}^{\infty} = (&21,18,20,15,19,13,18,11,16,10,13,10,11,9,10,7,9,5,9,3,9,2,\\ &7,2,5,2,4,2,3,1,3,-1,4,-2,4,-2,3,-2,2,-1,1,0,0,0,\ldots).\end{aligned}$$

We can see that $k_8 = 16$. From Theorem 1 we have $I(9) = 15$. Consequently, $k_8 > I(9)$.

2. Proof of the main result. We begin with a simple lemma.

Lemma 2. *Take* $j \geq 1$. *Then the minimum of the function* $J(l) = I_G(j-l) + I_H(l)$ *is attained for some* $0 \leq l \leq j$.

Proof. Let $l \leq 0$. Then $I_H(l) = |H| - l$ and $I_G(j-l) \geq I_G(j) + l$. Thus,

$$J(l) = I_G(j-l) + I_H(l) \geq I_G(j) + |H| = J(0).$$

In the case when $l \geq j$ we can take $l' = j - l$ and use the above inequality, exchanging the roles of G and H, to get $J(l) \geq J(j)$. ∎

Proof. Our goal is to express the numbers k_j in terms of k_j^G and k_j^H. We have

$$\begin{aligned}\Delta(t) &= \Delta_G(t)\Delta_H(t)\\ &= 1 + (t-1)(|G|+|H|) + (t-1)^2\Big[|G|\cdot|H|\\ &\quad + \sum_{j\geq 0}(k_j^G + k_j^H)t^j + (t-1)\Big(|G|\sum_{j\geq 0}k_j^H t^j + |H|\sum_{j\geq 0}k_j^G t^j\Big)\\ &\quad + (t-1)^2\Big(\sum_{j\geq 0}k_j^G t^j\Big)\Big(\sum_{j\geq 0}k_j^H t^j\Big)\Big]\\ &= 1 + (t-1)(|G|+|H|) + (t-1)^2\Theta(t),\end{aligned}$$

with

$$\Theta(t) = |G|\cdot|H| + k_0^G(1-|H|) + k_0^H(1-|G|) + k_0^G k_0^H + \sum_{j\geq 1} t^j k_j,$$

where

$$k_j = k_j^G(1-|H|) + |H|k_{j-1}^G + k_j^H(1-|G|) + |G|k_{j-1}^H + l_j$$

and

$$l_j = \sum_{u+v=j,\, u,v\geq 0} k_u^G k_v^H - 2\sum_{u+v=j-1,\, u,v\geq 0} k_u^G k_v^H + \sum_{u+v=j-2,\, u,v\geq 0} k_u^G k_v^H.$$

Note that $k_0^G = |G|$ and $k_0^H = |H|$. Therefore,

$$\begin{aligned}k_0 &= |G|\cdot|H| + k_0^G(1-|H|) + k_0^H(1-|G|) + k_0^G k_0^H\\ &= |G|\cdot|H| + |G|(1-|H|) + |H|(1-|G|) + |G|\cdot|H| = |G|+|H|.\end{aligned}$$

From Lemma 2 we get

$$I(1) = \min_{k\in\mathbb{Z}}(I_G(1-k) + I_H(k)) = \min_{k=0,1}(I_G(1-k) + I_H(k)) = |G|+|H| = k_0.$$

From now on, our goal is to prove that $k_j \leq I(j+1)$ for $j\geq 1$. Note that

$$\begin{aligned}l_j &= \sum_{u+v=j,\, u,v\geq 0} k_u^G k_v^H - \sum_{u+v=j,\, u\geq 0, v\geq 1} k_u^G k_{v-1}^H - \sum_{u+v=j,\, u\geq 1, v\geq 0} k_{u-1}^G k_v^H\\ &\quad + \sum_{u+v=j,\, u,v\geq 1} k_{u-1}^G k_{v-1}^H = \sum_{u+v=j,\, u,v\geq 1}(k_u^G - k_{u-1}^G)(k_v^H - k_{v-1}^H)\\ &\quad + k_0^G k_j^H + k_j^G k_0^H - k_0^G k_{j-1}^H - k_{j-1}^G k_0^H.\end{aligned}$$

Thus,

$$k_j = \sum_{u+v=j, u,v\geq 1}(k_u^G - k_{u-1}^G)(k_v^H - k_{v-1}^H) + m_j,$$

where, miraculously,

$$\begin{aligned} m_j &= k_0^G k_j^H + k_j^G k_0^H - k_0^G k_{j-1}^H - k_{j-1}^G k_0^H \\ &\quad + k_j^G(1-|H|) + |H|k_{j-1}^G + k_j^H(1-|G|) + |G|k_{j-1}^H \\ &= |G|k_j^H + k_j^G|H| - |G|k_{j-1}^H - k_{j-1}^G|H| \\ &\quad + k_j^G(1-|H|) + |H|k_{j-1}^G + k_j^H(1-|G|) + |G|k_{j-1}^H \\ &= k_j^G + k_j^H. \end{aligned}$$

We get

$$k_j = k_j^G + k_j^H + \sum_{u+v=j,\, u,v\geq 1} (k_{u-1}^G - k_u^G)(k_{v-1}^H - k_v^H).$$

We are to prove that $k_j \leq (I_G \diamond I_H)(j+1)$. It suffices to prove that $k_j \leq I_G(j+1-l) + I_H(l)$ for every $l \in \mathbb{Z}$. Thus, we have to deal with the inequality

$$k_j^G + k_j^H + \sum_{u+v=j,\, u,v\geq 1} (k_u^G - k_{u-1}^G)(k_v^H - k_{v-1}^H) \leq I_G(j+1-l) + I_H(l), \quad j \geq 1, l \in \mathbb{Z}.$$

By Lemma 2 it suffices to consider $0 \leq l \leq j+1$. Note that if $u+v=j$ then either $u \geq j-l+1$ or $v \geq l$. Thus,

$$\mathbf{1}_{u\in G}\mathbf{1}_{v\in H}\mathbf{1}_{u+v=j} \leq \mathbf{1}_{u\in G\cap[j-l+1,j]} + \mathbf{1}_{v\in H\cap[l,j]},$$

where the indicator functions in the above expression are functions of two variables u and v. We have also used the convention $[a,b] = \emptyset$ for $a > b$. We obtain

$$\begin{aligned} \sum_{u+v=j,u,v\geq 1} (k_{u-1}^G - k_u^G)(k_{v-1}^H - k_v^H) &= \sum_{u+v=j,u,v\geq 0} (k_{u-1}^G - k_u^G)(k_{v-1}^H - k_v^H) \\ &= \sum_{u+v=j,u,v\geq 0} \mathbf{1}_{u\in G}\mathbf{1}_{v\in H} \\ &\leq \sum_{u+v=j,\, u,v\geq 0} (\mathbf{1}_{u\in G\cap[j-l+1,j]} + \mathbf{1}_{v\in H\cap[l,j]}) \\ &= (I_G(j-l+1) - I_G(j+1)) + (I_H(l) - I_H(j+1)) \\ &\quad - (k_j^G + k_j^H) + I_G(j+1-l) + I_H(l). \end{aligned}$$

This concludes the proof. ■

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