

# Adaptive Deterministic Dyadic Grids on Spaces of Homogeneous Type

by

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**Summary.** In the context of spaces of homogeneous type, we develop a method to deterministically construct dyadic grids, specifically adapted to a given combinatorial situation. This method is used to estimate vector-valued operators rearranging martingale difference sequences such as the Haar system.

**1. Introduction.** In [5, 6], T. Figiel developed martingale methods to prove a vector-valued  $T(1)$  theorem by decomposing the singular integral operator  $T$  into an absolutely converging series of basic building blocks  $T_m$  and  $U_m$ ,  $m \in \mathbb{Z}$ . Those operators are given by the linear extension of

$$(1.1) \quad T_m h_I = h_{I+m|I|} \quad \text{and} \quad U_m h_I = \mathbb{1}_{I+m|I|} - \mathbb{1}_I,$$

where  $\{h_I\}$  denotes the Haar system on standard dyadic intervals  $I$ , and  $\mathbb{1}_A$  the characteristic function of the set  $A$ . The crucial norm estimates obtained in [5] take the form

$$(1.2) \quad \|T_m : L_E^p \rightarrow L_E^p\| \leq C(\log_2(2 + |m|))^\alpha,$$

$$(1.3) \quad \|U_m : L_E^p \rightarrow L_E^p\| \leq C(\log_2(2 + |m|))^\beta,$$

where  $1 < p < \infty$  and the constant  $C > 0$  depends only on  $p$ , the UMD-constant of the Banach space  $E$  and  $\alpha, \beta < 1$ . Estimates (1.2) and (1.3) are obtained by hard combinatorial arguments, analyzing structure and position of dyadic intervals.

T. Figiel's decomposition method was extended in [13] to spaces of homogeneous type to obtain a vector-valued  $T(1)$  theorem, requiring norm

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estimates for the building blocks  $T_m$  and  $U_m$  in the setting of spaces of homogeneous type. These estimates are proved by hard combinatorial arguments similar to [5].

In [10], an alternative proof for the estimates of  $T_m$  and  $U_m$  is given which eliminates the hard combinatorics of [5] to a great extent. Adapting the dyadic grid by means of an algebraic shift,  $T_m$  and  $U_m$  are decomposed into roughly  $\log_2(2 + |m|)$  martingale transform operators, thereby yielding (1.2) and (1.3). The shift of the dyadic grid is accomplished by the one-third-trick, which originates in the work of [4], [17], [7], and [1].

Adaptive dyadic grids also proved to be a valuable tool for estimating so called stripe operators in [9]. Those stripe operators were used in [11] and [9] to show weak lower semicontinuity of functionals with separately convex integrands on scalar-valued  $L^p$  and vector-valued  $L^p$ , respectively. For the scalar-valued  $L^2$  version of this result, cf. [14].

In this paper we extend the method from [10] to construct adapted dyadic grids in spaces of homogeneous type, which allows us to

- (i) simplify the combinatorial arguments for the estimation of the rearrangement operators  $T_m$  used in the proof of the  $T(1)$  theorem in [13],
- (ii) generalize the vector-valued result in [9] on stripe operators to spaces of homogeneous type.

**Related recent developments.** In [8], T. P. Hytönen presented a proof of T. Figiel's vector-valued  $T(1)$  theorem (cf. [6]), based upon randomized dyadic grids, originating in [15, 16]. By contrast, the method developed in the present paper allows us to adapt a dyadic grid *deterministically* to a given combinatorial situation.

**2. Preliminaries.** In this section we present some basic facts concerning spaces of homogeneous type. For basic facts on UMD-spaces used within this work, the notion of Rademacher type and cotype as well as Kahane's contraction principle and Bourgain's version of Stein's martingale inequality, we refer to [10].

Let  $X$  be a set. A mapping  $d : X \times X \rightarrow \mathbb{R}_0^+$  with the properties that for all  $x, y, z \in X$ ,

- (1)  $d(x, y) = 0 \Leftrightarrow x = y$ ,
- (2)  $d(x, y) = d(y, x)$ ,
- (3)  $d(x, y) \leq K_X(d(x, z) + d(z, y))$  for some constant  $K_X \geq 1$  only depending on  $X$ ,

is called a *quasimetric*, and  $(X, d)$  is called a *quasimetric space*. Given a

quasimetric  $d$ , we define the ball centered at  $x \in X$  with radius  $r > 0$  as

$$B(x, r) := \{y \in X : d(x, y) < r\}.$$

As usual, a set  $A \subset X$  is called *open* if for all  $x \in X$  there exists  $r > 0$  such that  $B(x, r) \subseteq A$ . Furthermore, for an arbitrary set  $A \subset X$  and  $r > 0$ , define

$$B(A, r) := \{y \in X : d(A, y) < r\}.$$

Let  $(X, d)$  be a quasimetric space such that every ball in the quasimetric  $d$  is open and  $|\cdot|$  be a Borel measure. If the *doubling condition* holds, i.e. there is a constant  $C_d > 0$  such that

$$(2.1) \quad 0 < |B(x, 2r)| \leq C_d |B(x, r)| < \infty, \quad x \in X, r > 0,$$

then  $(X, d, |\cdot|)$  is called a *space of homogeneous type*. Since for a given quasimetric space  $(X, d)$ , the balls in  $X$  are not necessarily open, we added this condition to the definition. It holds if for instance one imposes a Hölder condition on  $d$ : There exist  $c < \infty$  and  $0 < \beta < 1$  such that for all  $x, y, z \in X$  we have

$$(2.2) \quad |d(x, z) - d(y, z)| \leq c \cdot d(x, y)^\beta \max\{d(x, z), d(y, z)\}^{1-\beta}.$$

In fact, R. A. Macías and C. Segovia [12] proved that for every space of homogeneous type there exists an equivalent quasimetric with the desired Hölder property. Here, a quasimetric  $d'$  is *equivalent* to a quasimetric  $d$  if there exists a finite constant  $c$  such that

$$\frac{1}{c}d(x, y) \leq d'(x, y) \leq cd(x, y), \quad x, y \in X.$$

Let  $\mathcal{C}$  be a collection of arbitrary sets. It is called *nested* if  $A \cap B \in \{A, B, \emptyset\}$  for all  $A, B \in \mathcal{C}$ . For a given nested collection  $\mathcal{C}$  we define the predecessor  $\pi_{\mathcal{C}}(C)$  of  $C$  with respect to  $\mathcal{C}$  by

$$\pi_{\mathcal{C}}(C) := \bigcap \{D : D \supseteq C, D \in \mathcal{C} \cup \{X\}\}.$$

**Dyadic cubes.** In spaces of homogeneous type, one can construct a collection of subsets that has similar properties to dyadic cubes in  $\mathbb{R}^k$  (cf. M. Christ [2] and G. David [3]).

**THEOREM 2.1.** *Let  $(X, d, |\cdot|)$  be a space of homogeneous type. Then there exists a system  $\mathcal{Q}$  of open subsets of  $X$  with centers  $m_A \in A$  for  $A \in \mathcal{Q}$  and a splitting  $\mathcal{Q} = \bigcup_{n \in \mathbb{Z}} \mathcal{Q}_n$  such that the following properties are satisfied with uniform constants  $q < 1, C_1, C_2, C_3, \eta \in \mathbb{R}^+, N \in \mathbb{N}$ :*

- (1) *For all  $n \in \mathbb{Z}$  we have  $X = \bigcup_{A \in \mathcal{Q}_n} A$  up to  $|\cdot|$ -null sets.*
- (2) *For  $A \in \mathcal{Q}_k$  and  $B \in \mathcal{Q}_n$  with  $k \leq n$ , we have either  $B \subset A$  or  $A \cap B = \emptyset$ .*
- (3) *For each  $B \in \mathcal{Q}_n$  and every  $k \leq n$ , there is exactly one  $A \in \mathcal{Q}_k$  such that  $B \subset A$ .*

(4) For all  $n \in \mathbb{Z}$  and  $A \in \mathcal{Q}_n$  we have  $B(m_A, C_1 q^n) \subseteq A \subseteq B(m_A, C_2 q^n)$ .

(5) Let  $A \in \mathcal{Q}_n$ . The boundary layer of  $A$  having width  $t$  is given by

$$(2.3) \quad \partial_t A := \{x \in A : d(x, X \setminus A) \leq tq^n\},$$

and satisfies the measure estimate

$$(2.4) \quad |\partial_t A| < C_3 t^n |A|.$$

(6) For all  $n \in \mathbb{Z}$ , the collection  $\mathcal{Q}_n$  is countable.

(7) For all  $n \in \mathbb{Z}$  and  $A \in \mathcal{Q}_n$  we have  $N(A) := |\{B \in \mathcal{Q}_{n+1} : B \subseteq A\}| \leq N$ .

(8) For all  $n \in \mathbb{Z}$  and  $A \in \mathcal{Q}_n$  there exists a subcollection  $\mathcal{S}$  of  $\mathcal{Q}_{n+1}$  with  $|\mathcal{S}| \leq N$  and

$$A = \bigcup_{B \in \mathcal{S}} B \quad \text{up to } |\cdot| \text{-null sets.}$$

We define the *level* of a cube  $A \in \mathcal{Q}_n$  as  $\text{lev } A := n$ , and furthermore

$$(2.5) \quad r \diamond A := B(A, r q^{\text{lev } A}), \quad A \in \mathcal{Q}, r > 0.$$

In the following,  $(X, d, |\cdot|)$  denotes a space of homogeneous type, equipped with a quasimetric  $d$  and a measure  $|\cdot|$ .

LEMMA 2.2. *Let  $A \in \mathcal{Q}$  and  $r > 0$ . Then*

$$r \diamond A \subset B(m_A, K_X(C_2 + r)q^{\text{lev } A}).$$

*Proof.* Let  $z \in r \diamond A = B(A, r q^{\text{lev } A})$  and estimate

$$\begin{aligned} d(m_A, z) &\leq \inf_{y \in A} K_X(d(m_A, y) + d(y, z)) \\ &\leq K_X(C_2 q^{\text{lev } A} + d(A, z)) \\ &\leq K_X(C_2 q^{\text{lev } A} + r q^{\text{lev } A}). \quad \blacksquare \end{aligned}$$

LEMMA 2.3. *Let  $A_1, A_2 \in \mathcal{Q}$  and assume that*

$$(r_1 \diamond A_1) \cap (r_2 \diamond A_2) \neq \emptyset$$

*for some  $r_1, r_2 > 0$ . Then*

$$r_2 \diamond A_2 \subset r \diamond A_1,$$

*where  $r = 2K_X^3(C_2 + r_2)q^{\text{lev } A_2 - \text{lev } A_1} + K_X r_1$ .*

*Proof.* Let  $y \in (r_1 \diamond A_1) \cap (r_2 \diamond A_2)$  and  $z \in (r_2 \diamond A_2)$ . Then

$$d(z, A_1) \leq K_X(d(z, y) + d(y, A_1)).$$

Note that  $d(y, A_1) \leq r_1 q^{\text{lev } A_1}$  and observe

$$\begin{aligned} d(z, y) &\leq K_X(d(z, m_{A_2}) + d(m_{A_2}, y)) \\ &\leq 2K_X^2(C_2 + r_2)q^{\text{lev } A_2}, \end{aligned}$$

where we have used Lemma 2.2 for the latter estimate. Combining our estimates yields

$$d(z, A_1) \leq K_X (2K_X^2(C_2 + r_2)q^{\text{lev } A_2 - \text{lev } A_1} + r_1)q^{\text{lev } A_1},$$

and the assertion of the lemma follows. ■

**3. Adaptive dyadic grids.** In this section we provide a customizable way to adapt dyadic grids, which is then applied in Section 4 to estimate the rearrangement operators  $T_m$ .

We recall that  $K_X, C_d$  are constants defined by the quasimetric and the space  $X$  of homogeneous type, and  $C_1, C_2$  are determined by the collection of dyadic cubes (cf. Section 2). For a given collection  $\mathcal{A}$  of dyadic cubes in  $X$  and  $\alpha > 0$  we define

$$(3.1) \quad \mathcal{A}^{(\alpha)} := \bigcup_{A \in \mathcal{A}} \{Q \in \mathcal{Q}_{\text{lev } A} : Q \cap \alpha \diamond A \neq \emptyset\}.$$

The following result is a version of the well known one-third-trick in spaces of homogeneous type.

**THEOREM 3.1.** *Let  $C_R > 0$  and  $\mu \in \mathbb{N}$  be such that*

$$(3.2) \quad 4K_X^3(1 + C_2/C_R)q^\mu \leq 1.$$

*Let  $\mathcal{A} \subset \mathcal{Q}$  be a finite collection of cubes satisfying*

(1) *the separation condition*

$$(3.3) \quad (C_R \diamond A_1) \cap (C_R \diamond A_2) = \emptyset$$

*for all  $A_1 \neq A_2$  in  $\mathcal{A}$  with  $\text{lev } A_1 = \text{lev } A_2$ ,*

(2) *the small successor condition*

$$(3.4) \quad \text{lev } A \geq \mu + \text{lev } \pi(A), \quad A \in \mathcal{A}^{(\alpha)},$$

*where  $\alpha = 2K_X^3(C_2 + C_R) + C_R/2$  and  $\pi \equiv \pi_{\mathcal{A}^{(\alpha)}}$ .*

*Let  $\varphi : \mathcal{A} \rightarrow \mathcal{P}(\mathcal{A})$  be a map such that*

$$(3.5) \quad \text{lev } Q > \text{lev } A, \quad A \in \mathcal{A} \text{ and } Q \in \varphi(A),$$

$$(3.6) \quad \varphi(A)^* \subset \frac{C_R}{2K_X} \diamond A, \quad A \in \mathcal{A}.$$

*Then there exist a collection  $\mathcal{B}$  of adapted cubes in  $X$  and a bijective map  $\sigma : \mathcal{A} \rightarrow \mathcal{B}$  satisfying*

$$(3.7) \quad A \cup \sigma(\varphi(A))^* \subset \sigma(A) \subset C_R \diamond A, \quad A \in \mathcal{A},$$

*and the measure estimate*

$$(3.8) \quad |\sigma(A)| \leq C_d \left( \frac{K_X(C_2 + C_R)}{C_1} \right)^{\log_2(C_d)} \cdot |A|, \quad A \in \mathcal{A}.$$

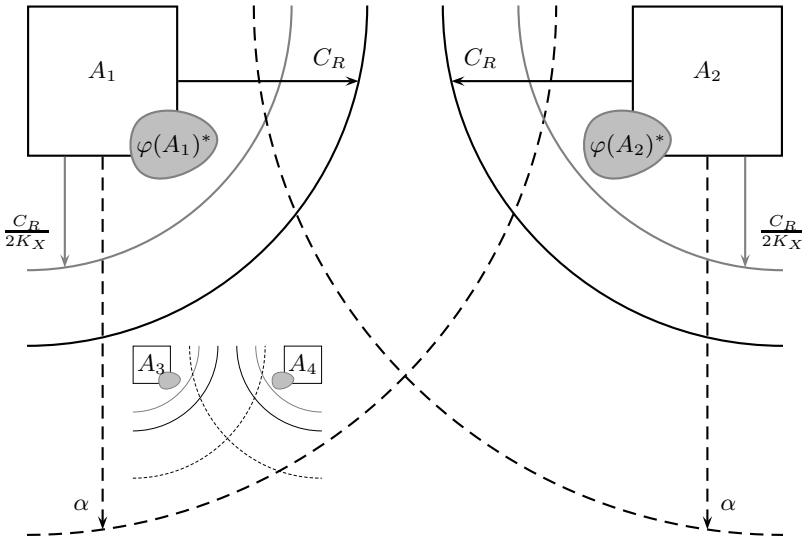


Fig. 1

Moreover, the collection

$$(3.9) \quad \mathcal{B} = \{\sigma(A) : A \in \mathcal{A}\}$$

is nested.

The hypotheses of Theorem 3.1 are visualized in Figure 1.

*Proof.* We set  $\widetilde{\mathcal{A}}_j := \mathcal{A} \cap \mathcal{Q}_j$ ,  $j \in \mathbb{Z}$ . Let the sequence  $j_\ell$  be such that  $\widetilde{\mathcal{A}}_{j_\ell} \neq \emptyset$  and  $\widetilde{\mathcal{A}}_k = \emptyset$  for all  $j_{\ell-1} < k < j_\ell$ ,  $\ell \leq 0$ . Then define  $\mathcal{A}_\ell := \widetilde{\mathcal{A}}_{j_\ell}$ ,  $\ell \leq 0$  and assume without restriction that  $\mathcal{A}_0$  consists of the cubes in  $\mathcal{A}$  with maximal level. The proof proceeds by induction on  $\text{lev } A$  for cubes  $A \in \mathcal{A}$ , starting with cubes in  $\mathcal{A}_0$ .

STEP 1. We begin the induction by defining

$$\sigma(A) := A \text{ for } A \in \mathcal{A}_0 \quad \text{and} \quad \mathcal{B}_0 := \{\sigma(A) : A \in \mathcal{A}_0\}.$$

Observe that (3.7) holds for all  $A \in \mathcal{A}_0$ . Now, let  $k < 0$  and assume that all the cubes  $\sigma(A)$ ,  $A \in \mathcal{A}_j$ , and the collections  $\mathcal{B}_j := \{\sigma(A) : A \in \mathcal{A}_j\}$  are already defined for all  $j > k$ . In order to construct  $\sigma(A)$ , let  $A \in \mathcal{A}_k$  and define

$$(3.10) \quad \sigma(A) := A \cup \sigma(\varphi(A))^* \cup \bigcup \{B \in \mathcal{B}_j : j > k, B \cap (A \cup \sigma(\varphi(A))^*) \neq \emptyset\}.$$

We collect all those cubes in

$$\mathcal{B}_k := \{\sigma(A) : A \in \mathcal{A}_k\}.$$

Finally, the set  $\mathcal{B}$  of all adapted cubes is defined as

$$\mathcal{B} := \bigcup_j \mathcal{B}_j.$$

In the next two steps we will inductively verify the nestedness of  $\mathcal{B}$  and the localization property (3.7).

STEP 2. Here we prove the nestedness of  $\mathcal{B}$ . To this end, define the level of an adapted cube  $B = \sigma(A)$  by  $\text{lev } B = \text{lev } A$ . Let  $B_1, B_2 \in \mathcal{B}$  be such that  $B_1 \cap B_2 \neq \emptyset$  and assume  $\text{lev } B_1 \leq \text{lev } B_2$ . If  $\text{lev } B_1 = \text{lev } B_2$ , then (3.3) and (3.7) yield  $B_1 = B_2$ . So we may now assume that  $\text{lev } B_1 < \text{lev } B_2$ . Choose  $A_1 \in \mathcal{A}$  such that  $\sigma(A_1) = B_1$ . If  $B_2 \cap (A_1 \cup \sigma(\varphi(A_1)))^* = \emptyset$  we get  $B_1 \cap B_2 = \emptyset$  by definition of  $B_1$ , cf. (3.10). This contradicts the assumption  $B_1 \cap B_2 \neq \emptyset$ . Thus,  $B_2 \cap (A_1 \cup \sigma(\varphi(A_1)))^* \neq \emptyset$  and, by (3.10) again, we infer  $B_2 \subset B_1$ , proving the nestedness of  $\mathcal{B}$ .

STEP 3. In this step we will verify (3.7). Assume that (3.7) is true for all  $A \in \mathcal{A}_j$ ,  $j > k$ . Recall that  $\mathcal{B}$  is nested by Step 2 of this proof. Now, let  $A \in \mathcal{A}_k$  be fixed. First, note that  $A \cup \sigma(\varphi(A))^* \subset \sigma(A)$  by the definition of  $\sigma(A)$  (cf. (3.10)). Secondly, we show that  $\sigma(A) \subset C_R \diamond A$ . Let  $B \in \mathcal{B}_j$ ,  $j > k$ , be such that  $B \cap (A \cup \sigma(\varphi(A))^*) \neq \emptyset$ . The condition  $B \cap (A \cup \sigma(\varphi(A))^*) \neq \emptyset$  is covered by the cases

- (1)  $B \cap \frac{C_R}{2K_X} \diamond A \neq \emptyset$ ,
- (2) there exists a  $Q \in \varphi(A)$  such that  $B \cap \sigma(Q) \neq \emptyset$ , and so by (3.9) either
  - (a)  $\sigma(Q) \subset B$ , or
  - (b)  $B \subset \sigma(Q)$ .

First, let us consider case (1). Due to the induction hypothesis, (3.7) is true for  $\sigma^{-1}(B)$ , that is,  $B \subset C_R \diamond \sigma^{-1}(B)$ . Thus, Lemma 2.3 implies

$$(3.11) \quad B \subset C_R \diamond \sigma^{-1}(B) \subset r \diamond A,$$

where  $r = 2K_X^3(C_2 + C_R) \cdot q^{\text{lev } \sigma^{-1}(B) - \text{lev } A} + C_R/2$ . Observe that since  $r \leq \alpha$ , we can find a cube  $\tilde{A} \in \mathcal{A}_k^{(\alpha)}$  such that  $\sigma^{-1}(B) \subsetneq \tilde{A}$ . Hence  $\text{lev } \sigma^{-1}(B) \geq \mu + \text{lev } \tilde{A} = \mu + \text{lev } A$  and so  $r \leq 2K_X^3(C_2 + C_R)q^\mu + C_R/2$ . Since  $r \leq C_R$  by (3.2), the inclusion  $B \subset C_R \diamond A$  follows.

In case (2a), the first inclusion in (3.7) yields  $Q \subset \sigma(Q) \subset B$ . Since  $Q \subset \varphi(A)^* \subset \frac{C_R}{2K_X} \diamond A$  by (3.6), in particular  $B \cap \frac{C_R}{2K_X} \diamond A \neq \emptyset$ . Hence, case (2a) is covered by case (1). In case (2b), condition (3.6) implies that  $\sigma(Q) \cap \frac{C_R}{2K_X} \diamond A \neq \emptyset$ . Applying the proof of case (1) to  $\sigma(Q)$  instead of  $B$ , we obtain  $\sigma(Q) \subset C_R \diamond A$ , and thus  $B \subset C_R \diamond A$ .

To summarize, in any of the cases (1), (2a) and (2b), the condition  $B \cap (A \cup \sigma(\varphi(A))^*) \neq \emptyset$  yields  $B \subset C_R \diamond A$ , which proves (3.7), i.e.,  $\sigma(A) \subset C_R \diamond A$ .

Finally, the measure estimate (3.8) is an immediate consequence of the doubling condition (2.1) and

$$B(m_A, C_1 q^{\text{lev } A}) \subset A \subset \sigma(A) \subset C_R \diamond A \subset B(m_A, K_X(C_2 + C_R)q^{\text{lev } A}),$$

where the latter inclusion follows from Lemma 2.2. ■

**4. Rearrangement operators.** Following [13], we define and analyze rearrangement operators on spaces of homogeneous type, thereby extending the rearrangement operators  $T_m$  introduced in [5], which act on the standard Haar system.

**The shift relation  $\tau$ .** Let  $m \in \mathbb{R}$ ,  $m > 0$  and  $\tau \subset \bigcup_{j \in \mathbb{Z}} \mathcal{Q}_j \times \mathcal{Q}_j$  have the properties

- (P1)  $Q \subset m \diamond P$  for all  $(P, Q) \in \tau$  (cf. Figure 2),
- (P2) there exists a finite partition  $\tau_1, \dots, \tau_M$  of  $\tau$  such that  $\tau_k$  is a bijective function for all  $1 \leq k \leq M$ .

The relation  $\tau$  generalizes the classical shift  $I \mapsto I + m|I|$  on  $\mathbb{R}$  (cf. [5]).

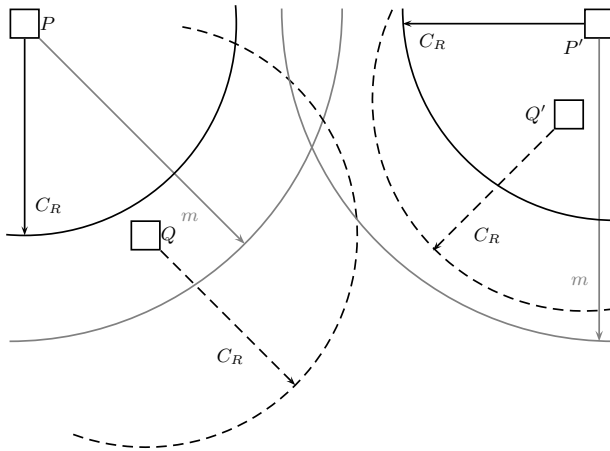


Fig. 2

In order to apply Theorem 3.1 to our shift  $\tau$ , we decompose  $\tau_k$  into suitable subcollections in the following way.

- (C1) First, let us choose a constant  $C_R > 0$  and split  $\tau_k$  into collections  $\mathcal{G}_{k,1}, \dots, \mathcal{G}_{k,M_k}$ , for all  $1 \leq k \leq M$ , such that

$$(4.1) \quad (C_R \diamond \tau_k^i(A_1)) \cap (C_R \diamond \tau_k^n(A_2)) = \emptyset$$

for all  $A_1, A_2 \in \text{pr}_1(\mathcal{G}_{k,j})$ ,  $1 \leq j \leq M_k$ ,  $i, n \in \{0, 1\}$ , where  $\tau_k^i(A)$  is defined to be  $A$  for  $i = 0$  and  $\tau_k(A)$  for  $i = 1$ . The projections onto the first and second coordinates of a relation are denoted by  $\text{pr}_1$



and  $\text{pr}_2$ , respectively. Observe that the constants  $M_k, 1 \leq k \leq M$ , depend only on  $X$  (cf. [13]). We refer to Figure 2 for a picture of the separation condition (4.1).

(C2) Secondly, let  $\ell$  be a positive integer and define

$$\mathcal{H}_{k,j,i}^{(\ell)} = \mathcal{G}_{k,j} \cap \bigcup_{r \in \mathbb{Z}} (\mathcal{Q}_{i+r\ell} \times \mathcal{Q}_{i+r\ell})$$

for all  $1 \leq k \leq M, 1 \leq j \leq M_k, 0 \leq i \leq \ell - 1$ . The parameter  $\ell$  will later be chosen to be approximately  $\log(2 + m)$  with  $m$  being the parameter from (P1).

(C3) Finally, define

$$\psi_{k,j,i}^{(\ell)}(A) = \{(P, Q) \in \mathcal{H}_{k,j,i}^{(\ell)} : P \subsetneq A \text{ or } Q \subsetneq A\}$$

for all  $A \in \text{pr}_1(\mathcal{H}_{k,j,i}^{(\ell)}) \cup \text{pr}_2(\mathcal{H}_{k,j,i}^{(\ell)})$ .

The collections  $\psi_{k,j,i}^{(\ell)}(A)$  are well localized around  $A$ , which is discussed in

LEMMA 4.1. *Let  $m \in \mathbb{R}, m > 0$ , and let  $\ell$  be a positive integer. Then*

$$(\text{pr}_1(\psi_{k,j,i}^{(\ell)}(A)) \cup \text{pr}_2(\psi_{k,j,i}^{(\ell)}(A)))^* \subset (c_1(1 + m)q^\ell) \diamond A$$

for all  $A \in \mathcal{H}_{k,j,i}^{(\ell)}, 1 \leq k \leq M, 1 \leq j \leq M_k, 0 \leq i \leq \ell - 1$ . The constant  $c_1$  depends only on the space  $X$  of homogeneous type.

*Proof.* Let  $(P, Q) \in \psi_{k,j,i}^{(\ell)}(A)$ . Then,  $P \subsetneq A$  or  $Q \subsetneq A$  by definition of  $\psi_{k,j,i}^{(\ell)}$ . We know from (P1) that  $P \cup Q \subset m \diamond P$ , hence Lemma 2.3 yields

$$P \cup Q \subset 2K_X^3(C_2 + m)q^{\text{lev } P - \text{lev } A} \diamond A.$$

Noting that  $\text{lev } P \geq \ell + \text{lev } A$  by (C2) concludes the proof. ■

**The shift operator  $T$ .** In order to define analogues of  $T_m$  on spaces of homogeneous type, we need a substitute  $\{h_Q\}$  for the standard Haar system. We require the system of functions  $\{h_Q\}_{Q \in \mathcal{Q}}$  to satisfy the conditions

(H1)  $\text{supp } h_Q \subset Q$ , for all  $Q \in \mathcal{Q}$ ,

(H2)  $\|h_Q\|_\infty \leq C_h \frac{1}{|P|+|Q|} \int |h_P|$  for all  $(P, Q) \in \tau$ ,

(H3) for every  $k$ , each of the collections  $\{h_P : P \in \text{pr}_1(\tau_k)\}$  and  $\{h_Q : Q \in \text{pr}_2(\tau_k)\}$  is a martingale difference sequence.

The constant  $C_h > 0$  is independent of  $(P, Q)$ . The collections  $\mathcal{H}_{k,j,i}^{(\ell)}$ , defined in (C2), naturally induce the subspaces  $H_{k,j,i}^{(\ell)}$  of  $L_E^p(X)$  given by

$$H_{k,j,i}^{(\ell)} = \left\{ f \in L_E^p(X) : f = \sum_{P \in \text{pr}_1(\mathcal{H}_{k,j,i}^{(\ell)})} \langle f, h_P \rangle h_P \right\}.$$

We now define the shift operator  $T_k$  induced by  $\tau_k$ ,  $1 \leq k \leq M$ , as the linear extension of the map

$$(4.2) \quad h_P \mapsto \begin{cases} h_Q & \text{if } (P, Q) \in \tau_k, \\ 0 & \text{otherwise.} \end{cases}$$

If the collections  $\psi_{k,j,i}^{(\ell)}$  are sufficiently localized, then the operators  $T_k$  are bounded on the subspace  $H_{k,j,i}^{(\ell)}$ . The details are given in the theorem below.

**THEOREM 4.2.** *Let  $X$  be a space of homogeneous type,  $E$  a UMD-space and  $1 < p < \infty$ . Let  $m \in \mathbb{R}$ ,  $m > 0$ . Then there exists a constant  $\beta > 0$  such that for all integers  $\ell$  satisfying*

$$(4.3) \quad (1 + m)q^\ell \leq \beta,$$

we have

$$(4.4) \quad \|T_k f\|_{L_E^p(X)} \leq C \|f\|_{L_E^p(X)}, \quad f \in H_{k,j,i}^{(\ell)},$$

for all  $1 \leq k \leq M$ ,  $1 \leq j \leq M_k$ ,  $0 \leq i \leq \ell - 1$ . The constant  $C$  depends only on  $p$ ,  $X$  and  $E$ , and the constant  $\beta$  only on  $X$ .

*Proof.* Let  $\ell$  satisfying (4.3) be fixed throughout the proof. Conditions on the constant  $\beta$  will be imposed within the proof.

Our goal is to apply Theorem 3.1 to each of the collections  $\mathcal{H}_{k,j,i}^{(\ell)}$ . With  $k, j, i$  fixed, define

$$\mathcal{C} = \mathcal{C}^{(1)} \cup \mathcal{C}^{(2)} = \text{pr}_1(\mathcal{H}_{k,j,i}^{(\ell)}) \cup \text{pr}_2(\mathcal{H}_{k,j,i}^{(\ell)})$$

and let  $\mathcal{A} \subset \mathcal{C}$  be a finite set. The function  $\varphi : \mathcal{A} \rightarrow \mathcal{P}(\mathcal{A})$  is given by

$$\varphi(A) := \text{pr}_1(\psi_{k,j,i}^{(\ell)}(A)) \cup \text{pr}_2(\psi_{k,j,i}^{(\ell)}(A)), \quad A \in \mathcal{A},$$

where  $\psi_{k,j,i}^{(\ell)}$  is defined in (C3). We shall now verify that  $\mathcal{A}$  and  $\varphi$  satisfy the hypotheses of Theorem 3.1.

First, observe that the separation condition (3.3) is satisfied due to (C1). Secondly, let  $\mu = \ell$ ; then (3.2) holds for sufficiently small  $\beta$ , where the constraint for  $\beta$  depends only on  $X$ . Additionally, observe that (C2) implies (3.4). From Lemma 4.1 and (4.3) it follows that  $\varphi(A)^* \subset \frac{C_R}{2K_X} \diamond A$  if  $\beta$  is sufficiently small. Having verified all the hypotheses of Theorem 3.1, we obtain a nested collection of sets  $\mathcal{B}$  and a bijective map  $\sigma : \mathcal{A} \rightarrow \mathcal{B}$  such that

$$A \cup \sigma(\varphi(A))^* \subset \sigma(A) \subset C_R \diamond A \quad \text{and} \quad |\sigma(A)| \leq c_2(1 + C_R)^{\log_2(C_d)} \cdot |A|$$

for all  $A \in \mathcal{A}$ . The constant  $c_2$  depends only on  $X$ .

Let us now define by induction a nested collection of sets supporting the shifts  $\tau$ , beginning with the smallest cubes. Set  $n_{\max} = \max\{\text{lev}(A) : A \in \mathcal{A}\}$  and define

$$\theta(P) := \theta(Q) := \sigma(P) \cup \sigma(Q)$$

for all  $(P, Q) \in \mathcal{H}_{k,j,i}^{(\ell)}$  such that  $\text{lev}(P) = n_{\max}$ . With  $n < n_{\max}$  fixed, assume that  $\theta(A)$  is already defined for all cubes  $A$  satisfying  $\text{lev}(A) > n$ . We specify  $\theta(P) = \theta(Q)$  to be

$$\sigma(P) \cup \sigma(Q) \cup \{\theta(R) : \text{lev } R > \text{lev } P, \theta(R) \cap (\sigma(P) \cup \sigma(Q)) \neq \emptyset\}^*,$$

for all  $(P, Q) \in \mathcal{H}_{k,j,i}^{(\ell)}$  with  $\text{lev}(P) = n$ . As an immediate consequence of the principle of construction, the collection  $\{\theta(A) : A \in \mathcal{A}\}$  is nested and

$$P \cup Q \subset \theta(P) = \theta(Q), \quad (P, Q) \in \mathcal{H}_{k,j,i}^{(\ell)}, P \in \mathcal{A}.$$

Furthermore, a straightforward calculation using Lemma 2.3 and (4.3) shows that there exists a constant  $c_3$  depending only on  $X$  such that

$$\theta(P) \subset (c_3 \diamond P) \cup (c_3 \diamond Q), \quad (P, Q) \in \mathcal{H}_{k,j,i}^{(\ell)}, P \in \mathcal{A},$$

if  $\beta$  is sufficiently small. From the latter inclusion we obtain

$$(4.5) \quad \theta(P) \leq c_4(|P| + |Q|), \quad (P, Q) \in \mathcal{H}_{k,j,i}^{(\ell)}, P \in \mathcal{A},$$

where  $c_4$  depends only on  $X$ . Let us define the filtration  $\{\mathcal{F}_n\}$  by

$$\mathcal{F}_n = \sigma\text{-algebra}(\{\theta(A) : A \in \mathcal{A}, \text{lev } A \leq n\}), \quad n \in \mathbb{Z}.$$

Observe that  $\theta(A)$  is an atom in  $\mathcal{F}_{\text{lev } A}$  for all  $A \in \mathcal{A}$ , since  $\{\theta(A) : A \in \mathcal{A}\}$  is a nested collection. Thus, (H2) and (4.5) imply

$$(4.6) \quad |h_{\tau_k(A)}| \leq c_5 \mathbb{E}(|h_A| \mid \mathcal{F}_n), \quad A \in \mathcal{E}_n,$$

where  $\mathcal{E}_n = \mathcal{Q}_n \cap \mathcal{A} \cap \mathcal{C}^{(1)}$  and  $c_5$  depends only on  $X$  and  $C_h$ .

We will now estimate  $Tf$  for all  $f \in H_{k,j,i}^{(\ell)}$ . Note that (H3) and the UMD-property of  $E$  allow us to assume that  $f$  is of the form

$$f = \sum_n \sum_{A \in \mathcal{E}_n} \langle f, h_A \rangle h_A.$$

Moreover,  $T|_{H_{k,j,i}^{(\ell)}} = T_k|_{H_{k,j,i}^{(\ell)}}$  is a function due to (P2). By employing (H3) again, we introduce Rademacher means in  $\|T_k f\|$  and obtain

$$\begin{aligned} \|T_k f\| &= \left\| \sum_n \sum_{A \in \mathcal{E}_n} \langle f, h_A \rangle h_{\tau_k(A)} \right\| \\ &\approx \int_0^1 \left\| \sum_n r_n(t) \sum_{A \in \mathcal{E}_n} \langle f, h_A \rangle h_{\tau_k(A)} \right\| dt. \end{aligned}$$

Furthermore, estimate (4.6) yields

$$\begin{aligned} \|T_k f\| &\approx \int_0^1 \left\| \sum_n r_n(t) \sum_{A \in \mathcal{E}_n} \langle f, h_A \rangle |h_{\tau_k(A)}| \right\| dt \\ &\lesssim \int_0^1 \left\| \sum_n r_n(t) \mathbb{E} \left( \sum_{A \in \mathcal{E}_n} \langle f, h_A \rangle |h_A| \mid \mathcal{F}_n \right) \right\| dt, \end{aligned}$$

by means of Kahane’s contraction principle. Applying Bourgain’s version of Stein’s martingale inequality gives us

$$\|T_k f\| \lesssim \int_0^1 \left\| \sum_n r_n(t) \sum_{A \in \mathcal{E}_n} \langle f, h_A \rangle |h_A| \right\| dt.$$

Using Kahane’s contraction principle and the UMD-property (cf. (H3)) concludes the proof. ■

Combining the estimates of Theorem 4.2 on the subspaces  $H_{k,j,i}^{(\ell)}$ , we obtain estimates for  $T_k$  on  $\overline{\text{span}}\{h_P : P \in \mathcal{Q}\}$  in the subsequent theorem (cf. [5]).

**THEOREM 4.3.** *Let  $X$  be a space of homogeneous type,  $E$  a UMD-space,  $1 < p < \infty$  and  $m \in \mathbb{R}$ ,  $m > 0$ . Then for all  $1 \leq k \leq M$  the linear operator  $T_k$  satisfies*

$$(4.7) \quad \|T_k f\|_{L_E^p(X)} \leq C \log(2+m)^\alpha \|f\|_{L_E^p(X)}, \quad f \in \overline{\text{span}}\{h_P : P \in \mathcal{Q}\}.$$

If  $L_E^p(X)$  has type  $\mathcal{T}$  and cotype  $\mathcal{C}$ , then  $\alpha < 1$  is given by  $1/\mathcal{T} - 1/\mathcal{C}$ . The constant  $C$  depends only on  $p$ ,  $X$ ,  $E$  and  $\alpha$ .

*Proof.* Within this proof we shall abbreviate  $\|\cdot\|_{L_E^p(X)}$  by  $\|\cdot\|$ . Let  $m > 0$  and choose  $\ell$  as the minimal integer satisfying (4.3), i.e., there exists a constant  $c_1$  only depending on  $X$  with

$$(4.8) \quad \ell \geq c_1 \log(2+m).$$

Assume that  $f$  is a finite sum of the form

$$f = \sum_{j=1}^{M_k} \sum_{i=0}^{\ell-1} \sum_{P \in \mathcal{H}_{k,j,i}^{(\ell)}} f_P h_P.$$

Then, by definition of  $T_k$  and the UMD-property of  $L_E^p(X)$  applied to (H3), we obtain

$$\|T_k f\| \lesssim \int_0^1 \left\| \sum_{j,i} r_{j,i}(t) T_k d_{j,i} \right\| dt,$$

where  $d_{j,i} = \sum_{P \in \mathcal{H}_{k,j,i}^{(\ell)}} f_P h_P$ . The type inequality yields

$$\|T_k f\| \lesssim \left( \sum_{j,i} \|T_k d_{j,i}\|^{\mathcal{T}} \right)^{1/\mathcal{T}},$$

where  $L_E^p(X)$  is of type  $\mathcal{T}$ . Theorem 4.2 implies  $\|T_k d_{j,i}\| \lesssim \|d_{j,i}\|$ , hence

$$\|T_k f\| \lesssim (M_k \ell)^{1/\mathcal{T}-1/c} \left( \sum_{j,i} \|d_{j,i}\|^c \right)^{1/c},$$

where  $L_E^p(X)$  is of cotype  $\mathcal{C}$ . The cotype inequality and the UMD-property show

$$\|T_k f\| \lesssim (M_k \ell)^{1/\mathcal{T}-1/c} \|f\|.$$

Since  $M_k$  depends only on  $X$ , using (4.8) gives (4.7) for finite sums  $f$  in  $\text{span}\{h_P : P \in \mathcal{Q}\}$ , thus concluding the proof by unique extension. ■

**5. Stripe operator.** In this section we define stripe operators on spaces of homogeneous type and provide vector-valued  $L^p$  estimates. Our notion of stripe operators generalizes those on  $\mathbb{R}^k$  analyzed in [9], which will now be briefly reviewed.

For a positive integer  $\lambda$ , the stripes  $\mathcal{S}_\lambda^{(m)}$  of the dyadic cube  $[0, 1]^n$  are given by

$$(5.1) \quad \mathcal{S}_\lambda^{(m)}([0, 1]^k) = \left\{ Q : \begin{array}{l} Q \text{ is a dyadic cube with } |Q| = 2^{-\lambda k}, \\ Q \subset [(m-1)/2^\lambda, m/2^\lambda] \times [0, 1]^{k-1} \end{array} \right\},$$

where  $1 \leq m \leq 2^\lambda$ . For an arbitrary dyadic cube  $A$ , the stripes  $\mathcal{S}_\lambda^{(m)}(A)$  are obtained by scaling and translating  $\mathcal{S}_\lambda^{(m)}([0, 1]^k)$  to the position of  $A$  in the dyadic grid. The stripe operators  $S_\lambda^{(m)}$  are defined by

$$(5.2) \quad S_\lambda^{(m)} h_A := g_{A,\lambda}^{(m)} := \sum_{R \in \mathcal{S}_\lambda^{(m)}(A)} h_R,$$

where  $h_A$  and  $h_R$  denote canonical Haar functions supported on the dyadic cubes  $A$  and  $R$ . Estimates for  $S_\lambda^{(m)}$  on  $L^p$  were used in [11] as well as in [9] to show weak lower semicontinuity for functionals with separately convex integrands on scalar and vector-valued  $L^p$ , respectively.

We will now extend the operators  $S_\lambda^{(m)}$  and their vector-valued estimates to spaces of homogeneous type.

**The stripes  $\mathcal{S}_\lambda^{(m)}$ .** Let  $\lambda$  and  $M$  be positive integers and define the stripes  $\mathcal{S}_\lambda^{(m)}(A)$ ,  $A \in \mathcal{Q}$ ,  $1 \leq m \leq M$  as arbitrary subsets of  $\{B \subset A : \text{lev } B = \text{lev } A + \lambda\}$  satisfying the conditions

- (S1)  $A = \bigcup_{m=1}^M \mathcal{S}_\lambda^{(m)}(A)^*$  is a disjoint union,  
 (S2) there exists an absolute constant  $K_1$  such that

$$|\mathcal{S}_\lambda^{(m)}(A)^*| \leq K_1 |\mathcal{S}_\lambda^{(n)}(A)^*|, \quad 1 \leq m, n \leq M,$$

- (S3)  $\{\mathcal{S}_\lambda^{(m)}(A)^* : A \in \mathcal{Q}\}$  is nested, with  $1 \leq m \leq M$  being fixed,  
 (S4) there exist constants  $\varepsilon > 0$  and  $K_2$  depending only on  $X$  such that for all  $1 \leq m \leq M$  we have

$$|\mathcal{E}_j^{(m)}(A)^*| \leq K_2 q^{j\varepsilon} |A|, \quad 0 \leq j \leq \lambda - 1,$$

where

$$\mathcal{E}_j^{(m)}(A) := \{B \in \mathcal{Q}_{\text{lev } A+j} : B \cap \mathcal{S}_\lambda^{(m)}(A)^* \neq \emptyset\}.$$

The classical stripe (5.1) defined in  $\mathbb{R}^k$  equipped with the Euclidean metric satisfies conditions (S1) to (S4) with parameters  $M = 2^\lambda$ ,  $K_1 = 1$ ,  $K_2 = 1$ ,  $q = 1/2$  and  $\varepsilon = 1$ .

**The stripe operators**  $S_\lambda^{(m)}$ . Let the collection  $\{h_A : A \in \mathcal{Q}\}$  of functions satisfy

- (M1)  $\text{supp } h_A \subset A$ , for all  $A \in \mathcal{Q}$ ,  
 (M2)  $\{h_A : A \in \mathcal{Q}\}$  is a martingale difference sequence.

Moreover, let  $\{g_{A,\lambda}^{(m)} : A \in \mathcal{Q}\}$ ,  $1 \leq m \leq M$ , be collections of functions that satisfy

- (G1)  $\text{supp } g_{A,\lambda}^{(m)} \subset \mathcal{S}_\lambda^{(m)}(A)$  for all  $A \in \mathcal{Q}$  and  $1 \leq m \leq M$ ,  
 (G2)  $\{g_{A,\lambda}^{(m)} : 1 \leq m \leq M, A \in \mathcal{Q}\}$  is a martingale difference sequence,  
 (G3)  $\|g_{A,\lambda}^{(m)}\|_\infty \leq C_g \frac{1}{|\mathcal{S}_\lambda^{(n)}(A)^*|} \int |g_{A,\lambda}^{(n)}|$  for all  $A \in \mathcal{Q}$ ,  $1 \leq m, n \leq M$ ,  
 $m \neq n$  and some constant  $C_g \geq 1$ .

We define the stripe operator  $S_\lambda^{(m)}$ ,  $1 \leq m \leq M$ , as the linear extension of

$$(5.3) \quad S_\lambda^{(m)} h_A := g_{A,\lambda}^{(m)}, \quad A \in \mathcal{Q}.$$

Note that the classical stripe operator (5.2) satisfies all of the above conditions.

LEMMA 5.1. *Let  $g_{A,\lambda}^{(m)}$  and  $g_{A,\lambda}^{(n)}$  be stripe functions satisfying (G1) and (G3). Then*

$$\left| \left\{ |g_{A,\lambda}^{(n)}| \geq \frac{\|g_{A,\lambda}^{(m)}\|_\infty}{2C_g} \right\} \right| \geq \frac{1}{2C_g^2} |\mathcal{S}_\lambda^{(n)}(A)^*|.$$

*Proof.* We shall abbreviate  $g^{(m)} = g_{A,\lambda}^{(m)}$ ,  $g^{(n)} = g_{A,\lambda}^{(n)}$  and  $\mathcal{S} = \mathcal{S}_\lambda^{(n)}(A)$ . Assume the contrary, that is,

$$(5.4) \quad \left| \left\{ |g^{(n)}| \geq \frac{\|g^{(m)}\|_\infty}{2C_g} \right\} \right| < \frac{1}{2C_g^2} |\mathcal{S}^*|.$$

Then (G3) implies

$$\begin{aligned} \frac{|\mathcal{S}^*|}{C_g} \|g^{(m)}\|_\infty &\leq \int_{\mathcal{S}^*} |g^{(n)}| \leq \left| \left\{ |g^{(n)}| < \frac{\|g^{(m)}\|_\infty}{2C_g} \right\} \right| \frac{\|g^{(m)}\|_\infty}{2C_g} \\ &\quad + \left| \left\{ |g^{(n)}| \geq \frac{\|g^{(m)}\|_\infty}{2C_g} \right\} \right| \|g^{(n)}\|_\infty. \end{aligned}$$

Observe that (G3) and (G1) give us  $\|g^{(n)}\|_\infty \leq C_g \|g^{(m)}\|_\infty$ , thus inserting (G1) and (5.4) in the latter display yields a contradiction, proving the lemma. ■

The subsequent results, i.e., the combinatorial Lemma 5.2 and the estimates on stripe operators in Theorems 5.3 and 5.4, are proved in much the same way as their Euclidean counterparts in [9].

LEMMA 5.2. *Let  $\lambda$  and  $k$  be positive integers. Then there exists a constant  $K_3$  depending only on  $X$  such that*

$$\left| \mathcal{S}_\lambda^{(m)}(A)^* \cap \bigcup_{B \in \mathcal{E}^{(m)}(A)} (\mathcal{S}_\lambda^{(m)}(B)^* \cup \mathcal{S}_\lambda^{(n)}(B)^*) \right| \leq K_3 q^{k\varepsilon} |\mathcal{S}_\lambda^{(m)}(A)^*|$$

for all  $1 \leq m, n \leq M$  and  $A \in \mathcal{Q}$ , where

$$\mathcal{E}^{(m)}(A) := \bigcup \{ \mathcal{E}_{dk}^{(m)}(A) : d \in \mathbb{N}, 1 \leq dk \leq \lambda - 1 \}.$$

*Proof.* First, observe that

$$\begin{aligned} \left| \mathcal{S}_\lambda^{(m)}(A)^* \cap \bigcup_{B \in \mathcal{E}^{(m)}(A)} (\mathcal{S}_\lambda^{(m)}(B)^* \cup \mathcal{S}_\lambda^{(n)}(B)^*) \right| \\ \leq \sum_{B \in \mathcal{E}^{(m)}(A)} (|\mathcal{S}_\lambda^{(m)}(B)^*| + |\mathcal{S}_\lambda^{(n)}(B)^*|). \end{aligned}$$

Now we use (S2) to dominate this expression by

$$(5.5) \quad (1 + K_1) \sum_{B \in \mathcal{E}^{(m)}(A)} |\mathcal{S}_\lambda^{(m)}(B)^*|.$$

Note that (S1) and (S2) also give us

$$|\mathcal{S}_\lambda^{(m)}(B)^*| \leq \frac{K_1}{M} |B|.$$

The latter inequality implies that (5.5) is bounded from above by

$$(5.6) \quad \frac{(1 + K_1)K_1}{M} \left( \sum_{d: 1 \leq dk < \lambda} \sum_{B \in \mathcal{E}_{dk}^{(m)}(A)} |B| \right).$$

Employing (S4) we estimate (5.6) by

$$\frac{(1 + K_1)K_1K_2}{M} \left( \sum_{d: 1 \leq dk < \lambda} q^{dk\varepsilon} |A| \right).$$

Finally, applying (S2) concludes the proof of the lemma. ■

**THEOREM 5.3.** *Let  $X$  be a space of homogeneous type,  $E$  a UMD-space and  $1 < p < \infty$ . Let  $\lambda$  be a positive integer. Then there exists a constant  $C$  such that*

$$\|S_\lambda^{(m)} f\|_{L_E^p(X)} \leq C \|S_\lambda^{(n)} f\|_{L_E^p(X)}, \quad f \in \overline{\text{span}}\{h_Q : Q \in \mathcal{Q}\},$$

for all  $1 \leq m, n \leq M$ . The constant  $C$  depends only on  $p, X$  and  $E$ .

*Proof.* Let  $\lambda \geq 1$  and  $m \neq n$  be fixed throughout the proof. Define  $k$  as the smallest positive integer such that  $K_3 q^{k\varepsilon} \leq 1/(4C_g^2)$ , where  $K_3, \varepsilon$  and  $C_g$  are the constants appearing in Lemma 5.2, (S4) and (G3), respectively. Moreover, define the collections

$$\begin{aligned} \mathcal{C}_{j,\nu}^{(\delta)} &:= \bigcup_{\substack{0 \leq i \leq \lambda-1 \\ i \bmod k = \nu}} \mathcal{Q}_{(2j+\delta)\lambda+i}, \quad j \in \mathbb{Z}, \delta \in \{0, 1\}, 0 \leq \nu \leq k-1, \\ \mathcal{C}_\nu^{(\delta)} &:= \bigcup_{j \in \mathbb{Z}} \mathcal{C}_{j,\nu}^{(\delta)}, \quad \delta \in \{0, 1\}, 0 \leq \nu \leq k-1. \end{aligned}$$

With  $\nu$  and  $\delta$  fixed, set

$$A(Q) := (\mathcal{S}_\lambda^{(m)}(Q)^* \cup \mathcal{S}_\lambda^{(n)}(Q)^*) \setminus \bigcup_{\substack{P \in \mathcal{C}_{j,\nu}^{(\delta)} \\ \text{lev } P > \text{lev } Q}} A(P), \quad Q \in \mathcal{C}_{j,\nu}^{(\delta)},$$

for each  $j \in \mathbb{Z}$ . This definition is understood to be by induction on  $\text{lev } Q$ , starting with the maximal level in  $\mathcal{C}_{j,\nu}^{(\delta)}$ . Note that the above union is empty if  $\text{lev } Q$  is maximal in  $\mathcal{C}_{j,\nu}^{(\delta)}$ . Now, Lemma 5.2 and our choice of  $k$  imply

$$(5.7) \quad |A(Q) \cap \mathcal{S}_\lambda^{(n)}(Q)^*| \geq \left(1 - \frac{1}{4C_g^2}\right) |\mathcal{S}_\lambda^{(n)}(Q)^*|.$$

We collect all the sets  $A(Q)$  in  $\mathcal{A}$ , to be more precise

$$\mathcal{A} := \{A(Q) : Q \in \mathcal{C}_{j,\nu}^{(\delta)}\}.$$

The inductive construction of  $A(Q)$  is performed in such a way that  $\mathcal{A}$  is nested. Indeed, if  $P, Q \in \mathcal{C}_{j,\nu}^{(\delta)}$ , then  $A(P) \cap A(Q) = \emptyset$ . Moreover, if  $Q \in \mathcal{C}_{j,\nu}^{(\delta)}$ ,



then  $A(Q)$  consists of cubes in  $\mathcal{Q}_{\text{lev } Q+2\lambda-1}$ . Thus, if  $P \in \mathcal{C}_{i,\nu}^{(\delta)}$  and  $Q \in \mathcal{C}_{j,\nu}^{(\delta)}$  with  $i < j$ , then  $A(Q) \subset Q \subset A(P)$  provided  $A(P) \cap A(Q) \neq \emptyset$ . Hence,  $\mathcal{A}$  is a nested collection.

Let us define

$$\mathcal{A}_j := \{A(Q) \in \mathcal{A} : Q \in \mathcal{Q}_j\}, \quad j \in \mathbb{Z},$$

and the filtrations  $\{\mathcal{F}_j\}$  and  $\{\mathcal{G}_j\}$  by

$$\mathcal{F}_j := \sigma\text{-algebra}\left(\bigcup_{i \leq j} \mathcal{A}_i\right),$$

$$\mathcal{G}_j := \sigma\text{-algebra}(\{\mathcal{S}_\lambda^{(m)}(Q)^* : Q \in \mathcal{Q}_i, i \leq j\}),$$

for all  $j \in \mathbb{Z}$ . Note that some of the sets  $\mathcal{A}_j$  are empty; if  $\mathcal{A}_j = \emptyset$ , we delete the  $\sigma$ -algebras  $\mathcal{F}_j$  and  $\mathcal{G}_j$  from their respective filtrations.

Let  $f \in L_E^p(X)$  have the representation  $f = \sum_{Q \in \mathcal{C}_\nu^{(\delta)}} f_Q h_Q$ . Due to (G2), the UMD-property and Kahane's contraction principle yield

$$(5.8) \quad \|S_\lambda^{(m)} f\| = \left\| \sum_{Q \in \mathcal{C}_\nu^{(\delta)}} f_Q g_{Q,\lambda}^{(m)} \right\| \lesssim \int_0^1 \left\| \sum_{Q \in \mathcal{C}_\nu^{(\delta)}} r_Q(t) f_Q |g_{Q,\lambda}^{(m)}| \right\| dt.$$

First, observe that

$$\mathbb{1}_{\mathcal{S}_\lambda^{(m)}(Q)^*} \leq \frac{|\mathcal{S}_\lambda^{(m)}(Q)^*|}{|A(Q)|} \mathbb{E}(\mathbb{1}_{A(Q)} \mid \mathcal{G}_{\text{lev } Q}), \quad Q \in \mathcal{C}_\nu^{(\delta)}.$$

Secondly, due to our choice of  $k$ , we deduce from Lemma 5.2 that

$$|\mathcal{S}_\lambda^{(m)}(Q)^*| \leq \frac{4}{3} |A(Q)|, \quad Q \in \mathcal{C}_\nu^{(\delta)}.$$

The latter two inequalities imply that for all  $Q \in \mathcal{C}_\nu^{(\delta)}$

$$(5.9) \quad \begin{aligned} |g_{Q,\lambda}^{(m)}| &\leq \|g_{Q,\lambda}^{(m)}\|_\infty \mathbb{1}_{\mathcal{S}_\lambda^{(m)}(Q)^*} \\ &\leq \frac{4}{3} \|g_{Q,\lambda}^{(m)}\|_\infty \mathbb{E}(\mathbb{1}_{A(Q)} \mid \mathcal{G}_{\text{lev } Q}). \end{aligned}$$

Combining (5.8) and (5.9), and applying Kahane's contraction principle, yields

$$\|S_\lambda^{(m)} f\| \lesssim \int_0^1 \left\| \sum_{Q \in \mathcal{C}_\nu^{(\delta)}} r_Q(t) f_Q \|g_{Q,\lambda}^{(m)}\|_\infty \mathbb{E}(\mathbb{1}_{A(Q)} \mid \mathcal{G}_{\text{lev } Q}) \right\| dt.$$

Applying Bourgain's version of Stein's martingale gives

$$(5.10) \quad \|S_\lambda^{(m)} f\| \lesssim \int_0^1 \left\| \sum_{Q \in \mathcal{C}_\nu^{(\delta)}} r_Q(t) f_Q \|g_{Q,\lambda}^{(m)}\|_\infty \mathbb{1}_{A(Q)} \right\| dt.$$

By (G1), the support of  $g_{Q,\lambda}^{(n)}$  is a subset of  $\mathcal{S}_\lambda^{(n)}(Q)^*$ . If we define

$$V := \left\{ |g_{Q,\lambda}^{(n)}| \geq \frac{\|g_{Q,\lambda}^{(m)}\|_\infty}{2C_g} \right\} \cap A(Q) \cap \mathcal{S}_\lambda^{(n)}(Q)^*,$$

then (5.7) and Lemma 5.1 imply

$$(5.11) \quad |V| \geq \frac{1}{4C_g^2} |\mathcal{S}_\lambda^{(n)}(Q)^*| \geq \frac{1}{4C_g^2(1+K_1)} |A(Q)|.$$

As a consequence of the definition of  $V$  and (5.11),

$$\begin{aligned} \|g_{Q,\lambda}^{(m)}\|_\infty \mathbb{1}_{A(Q)} &\leq \left( \frac{2C_g}{|V|} \int_V |g_{Q,\lambda}^{(n)}| \right) \mathbb{1}_{A(Q)} \\ &\leq \left( \frac{8C_g^3(1+K_1)}{|A(Q)|} \int_{A(Q)} |g_{Q,\lambda}^{(n)}| \right) \mathbb{1}_{A(Q)} \\ &\leq 8C_g^3(1+K_1) \mathbb{E}(|g_{Q,\lambda}^{(n)}| \mid \mathcal{F}_{\text{lev } Q}) \end{aligned}$$

for all  $Q \in C_\nu^{(\delta)}$ . Plugging the latter estimate into (5.10) and using Kahane’s contraction principle yields

$$\|S_\lambda^{(m)} f\| \lesssim \left\| \int_0^1 \sum_{Q \in \mathcal{C}_\nu^{(\delta)}} r_Q(t) f_Q \mathbb{E}(|g_{Q,\lambda}^{(n)}| \mid \mathcal{F}_{\text{lev } Q}) dt \right\|.$$

Subsequently, applying Stein’s martingale inequality, Kahane’s contraction principle to pass from  $|g_{Q,\lambda}^{(n)}|$  to  $g_{Q,\lambda}^{(n)}$  and finally using the UMD-property to dispose of the Rademacher functions, concludes the proof. ■

Applying the estimate in Theorem 5.3, i.e., the uniform equivalence of stripe operators, we obtain upper and lower estimates for  $S_\lambda^{(m)}$  via the cotype and type inequalities, respectively.

**THEOREM 5.4.** *Let  $X$  be a space of homogeneous type,  $E$  a UMD-space and  $1 < p < \infty$ . Moreover, let  $\lambda$  be a positive integer and  $1 \leq m \leq M$ . If we assume*

$$(5.12) \quad \left\| \sum_{n=1}^M S_\lambda^{(n)} h_Q \right\|_\infty \leq C_S \frac{1}{|Q|} \int |h_Q|, \quad Q \in \mathcal{Q},$$

then

$$(5.13) \quad \|S_\lambda^{(m)} f\|_{L_E^p(X)} \leq CC_S M^{-1/C} \|f\|_{L_E^p(X)}, \quad f \in \overline{\text{span}}\{h_P : P \in \mathcal{Q}\},$$

where  $L_E^p(X)$  has cotype  $\mathcal{C}$  and the constant  $C$  depends only on  $p$ ,  $X$  and  $E$ .

On the other hand, if we assume

$$(5.14) \quad \|h_Q\|_\infty \leq C_S \sum_{n=1}^M \frac{1}{|Q|} \int |S_\lambda^{(n)} h_Q|, \quad Q \in \mathcal{Q},$$

then

$$(5.15) \quad \|S_\lambda^{(m)} f\|_{L_E^p(X)} \geq C C_S^{-1} M^{-1/\mathcal{T}} \|f\|_{L_E^p(X)}, \quad f \in \overline{\text{span}}\{h_P : P \in \mathcal{Q}\},$$

where  $L_E^p(X)$  has type  $\mathcal{T}$  and the constant  $C$  depends only on  $p$ ,  $X$  and  $E$ .

*Proof.* First, we prove inequality (5.13) under the hypothesis (5.12). Let  $f = \sum_Q f_Q h_Q$  be a finite sum and  $m$  be an integer in the range  $1 \leq m \leq M$ . By (M2),  $\{h_Q\}$  is a martingale difference sequence, thus

$$\|f\| \gtrsim \int_0^1 \left\| \sum_Q r_Q(t) f_Q |h_Q| \right\| dt$$

as a consequence of the UMD-property of  $E$  and Kahane's contraction principle. Define the filtration  $\{\mathcal{F}_j\}$  by

$$\mathcal{F}_j = \sigma\text{-algebra} \left( \bigcup_{i \leq j} \mathcal{Q}_i \right).$$

Then Bourgain's version of Stein's martingale inequality yields

$$(5.16) \quad \|f\| \gtrsim \int_0^1 \left\| \sum_Q r_Q(t) f_Q \mathbb{E}(|h_Q| \mid \mathcal{F}_{\text{lev } Q}) \right\| dt.$$

Observe that  $Q$  is an atom in the  $\sigma$ -algebra  $\mathcal{F}_{\text{lev } Q}$  for all  $Q \in \mathcal{Q}$ , and thus

$$\mathbb{E}(|h_Q| \mid \mathcal{F}_{\text{lev } Q}) = \left( \frac{1}{|Q|} \int |h_Q| \right) \mathbb{1}_Q \geq C_S^{-1} \left\| \sum_{n=1}^M S_\lambda^{(n)} h_Q \right\|_\infty \mathbb{1}_Q,$$

where we used (M1) and (5.12). Plugging the latter inequality into (5.16) and using Kahane's contraction principle implies

$$\|f\| \gtrsim C_S^{-1} \int_0^1 \left\| \sum_Q r_Q(t) f_Q \sum_{n=1}^M S_\lambda^{(n)} h_Q \right\| dt.$$

Condition (G2) and the UMD-property of  $L_E^p(X)$  yield

$$(5.17) \quad \|f\| \gtrsim C_S^{-1} \left\| \sum_{j \in \mathbb{Z}} \sum_{n=1}^M S_\lambda^{(n)} \left( \sum_{Q \in \mathcal{Q}_j} f_Q h_Q \right) \right\|.$$

Now let

$$d_{j,n} := S_\lambda^{(n)} \left( \sum_{Q \in \mathcal{Q}_j} f_Q h_Q \right)$$

and observe that  $(d_{j,n})$  is a martingale difference sequence with respect to the lexicographic ordering on the index pairs  $(j, n)$ . Thus, (5.17) implies

$$\|f\| \gtrsim C_S^{-1} \int_0^1 \left\| \sum_{n=1}^M r_n(t) \sum_{j \in \mathbb{Z}} d_{j,n} \right\| dt.$$

Since  $L_E^p(X)$  has cotype  $\mathcal{C}$ , we employ the cotype inequality to obtain

$$\|f\| \gtrsim C_S^{-1} \left( \sum_{n=1}^M \left\| \sum_{j \in \mathbb{Z}} d_{j,n} \right\|^{\mathcal{C}} \right)^{1/\mathcal{C}} = C_S^{-1} \left( \sum_{n=1}^M \|S_\lambda^{(n)} f\|^{\mathcal{C}} \right)^{1/\mathcal{C}}.$$

By Theorem 5.3,  $\|S_\lambda^{(n)} f\| \gtrsim \|S_\lambda^{(m)} f\|$  for all  $1 \leq n \leq M$ , and therefore

$$\|f\| \gtrsim C_S^{-1} M^{1/\mathcal{C}} \|S_\lambda^{(m)} f\|,$$

proving (5.13).

A similar argument replacing the cotype inequality by the type inequality proves (5.15) under the condition (5.14). ■

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