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# Sharp Weak-Type Inequality for the Haar System, Harmonic Functions and Martingales

by

Adam OSĘKOWSKI

*Presented by Stanisław KWAPIEŃ*

**Summary.** Let $(h_k)_{k\geq 0}$ be the Haar system on $[0,1]$. We show that for any vectors $a_k$ from a separable Hilbert space $\mathcal{H}$ and any $\varepsilon_k \in [-1,1]$, $k=0,1,2,\ldots$, we have the sharp inequality

$$\Big\|\sum_{k=0}^n \varepsilon_k a_k h_k\Big\|_{W([0,1])} \leq 2\Big\|\sum_{k=0}^n a_k h_k\Big\|_{L^\infty([0,1])}, \quad n=0,1,2,\ldots,$$

where $W([0,1])$ is the weak-$L^\infty$ space introduced by Bennett, DeVore and Sharpley. The above estimate is generalized to the sharp weak-type bound

$$\|Y\|_{W(\Omega)} \leq 2\|X\|_{L^\infty(\Omega)},$$

where $X$ and $Y$ stand for $\mathcal{H}$-valued martingales such that $Y$ is differentially subordinate to $X$. An application to harmonic functions on Euclidean domains is presented.

**1. Introduction.** Our motivation comes from a certain basic question about the Haar system $(h_k)_{k\geq 0}$, an important basis for $L^p([0,1])$, $1\leq p<\infty$. As shown by Marcinkiewicz [8] (see also Paley [10]), this basis is unconditional if $1<p<\infty$. That is, there exists a universal constant $c_p \in (0,\infty)$ such that

$$c_p^{-1}\Big\|\sum_{k=0}^n a_k h_k\Big\|_p \leq \Big\|\sum_{k=0}^n \varepsilon_k a_k h_k\Big\|_p \leq c_p\Big\|\sum_{k=0}^n a_k h_k\Big\|_p \tag{1.1}$$

for any $n$ and any $a_k\in\mathbb{R}$, $\varepsilon_k\in\{-1,1\}$, $k=0,1,\ldots,n$. This result was extended by Burkholder [3] to the martingale setting. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a

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nonatomic probability space, filtered by $(\mathcal{F}_k)_{k\geq 0}$, a nondecreasing family of sub-$\sigma$-fields of $\mathcal{F}$. Let $f = (f_k)_{k\geq 0}$ be a real-valued martingale with the difference sequence $(df_k)_{k\geq 0}$ given by $df_0 = f_0$ and $df_k = f_k - f_{k-1}$ for $k \geq 1$. Let $g$ be a transform of $f$ by a real predictable sequence $v = (v_k)_{k\geq 0}$ bounded in absolute value by 1: that is, $dg_k = v_k df_k$ for all $k \geq 0$ and by *predictability* we mean that each term $v_k$ is measurable with respect to $\mathcal{F}_{(k-1)\vee 0}$. Then (cf. [3]) for $1 < p < \infty$ there is an absolute constant $c'_p$ for which

$$\|g\|_p \leq c'_p \|f\|_p. \tag{1.2}$$

Here we have used the notation $\|f\|_p = \sup_n \|f_n\|_p$. Let $c_p(1.1)$, $c'_p(1.2)$ denote the optimal constants in (1.1) and (1.2), respectively. The Haar system is a martingale difference sequence with respect to its natural filtration (on the probability space being the interval $(0, 1]$ with its Borel subsets and Lebesgue measure), and hence so is $(a_k h_k)_{k\geq 0}$ for given fixed real numbers $a_0, a_1, a_2, \ldots$. Therefore, $c_p(1.1) \leq c'_p(1.2)$ for all $1 < p < \infty$. It follows from the results of Burkholder [4] and Maurey [9] that in fact the constants coincide: $c_p(1.1) = c'_p(1.2)$ for all $1 < p < \infty$. A celebrated theorem of Burkholder [5] asserts that $c_p(1.1)= \max\{p - 1, (p - 1)^{-1}\}$ for $1 < p < \infty$. Furthermore, the constant does not change if we allow the martingales and the coefficients $a_k$ to take values in a separable Hilbert space $\mathcal{H}$. For $p = 1$ the inequalities (1.1) and (1.2) do not hold with any finite constant, but we have an appropriate weak-type bound. Here is the result of Burkholder [5], valid for a wider range of exponents: if $1 \leq p \leq 2$, then for any $f$ and $g$ as above we have

$$\|g\|_{p,\infty} \leq \left(\frac{2}{\Gamma(p+1)}\right)^{1/p} \|f\|_p \tag{1.3}$$

and the bound is sharp. Here $\|g\|_{p,\infty} = \sup_{\lambda>0} \lambda(\mathbb{P}(\sup_n |g_n| \geq \lambda))^{1/p}$ denotes the weak $p$th norm of $g$. For $p > 2$, Suh [11] showed that

$$\|g\|_{p,\infty} \leq \left(\frac{p^{p-1}}{2}\right)^{1/p} \|f\|_p \tag{1.4}$$

and that the constant $(p^{p-1}/2)^{1/p}$ is the best possible. Both (1.3) and (1.4) remain sharp in the special case of the estimates for the Haar system with real coefficients.

In fact, all these bounds are valid under less restrictive assumption of differential subordination, and can further be extended to the continuous-time setting. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is complete, and equip it with a right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$ such that $\mathcal{F}_0$ contains all the events of probability 0. Let $X$, $Y$ be two adapted cadlag martingales taking values in $\mathcal{H}$ which, as we may and do assume from now on, is equal to $\ell^2$. Following Wang [12], we say that $Y$ is *differentially subordinate* to $X$, if the process $([X, X]_t - [Y, Y]_t)_{t\geq 0}$ is nondecreasing and nonnegative as a function of $t$.

Here $[X,Y]=\sum_{j=0}^{\infty}[X^j,Y^j]$, where $X^j$, $Y^j$ stand for the $j$th coordinates of $X$ and $Y$, respectively, and $[X^j,Y^j]$ is the quadratic covariance process of $X^j$ and $Y^j$ (see e.g. Dellacherie and Meyer [7]). If we treat the discrete-time martingales $f=(f_k)_{k=0}^{\infty}$ and $g=(g_k)_{k=0}^{\infty}$ as continuous-time processes (via $X_t=f_{\lfloor t\rfloor}$ and $Y_t=g_{\lfloor t\rfloor}$ for $t\geq 0$), then the above condition reads

$$|dg_k|\leq |df_k| \quad \text{for } k\geq 0,$$

which is the original definition of differential subordination due to Burkholder [5]. Of course, this condition is satisfied by the martingale transforms studied above. Thus the following theorem (see [11] and [12]) generalizes the previous bounds (1.2)–(1.4). We use the notation $\|X\|_p=\sup_t\|X_t\|_p$ and $\|X\|_{p,\infty}=\sup_{\lambda>0}\lambda(\mathbb{P}(\sup_t|X_t|\geq\lambda))^{1/p}$, analogous to that of the discrete-time setting.

THEOREM 1.1. *If $Y$ is differentially subordinate to $X$, then*

$$\|Y\|_p\leq \max\{p-1,(p-1)^{-1}\}\|X\|_p, \quad 1\leq p<\infty, \tag{1.5}$$

$$\begin{aligned} \|Y\|_{p,\infty}&\leq\left(\frac{2}{\Gamma(p+1)}\right)^{1/p}\|X\|_p, && 1\leq p\leq 2,\\ \|Y\|_{p,\infty}&\leq\left(\frac{p^{p-1}}{2}\right)^{1/p}\|X\|_p, && 2\leq p<\infty, \end{aligned}\tag{1.6}$$

*and the inequalities are sharp.*

The purpose of this paper is to study an estimate which can be regarded a version of the weak-type inequality for $p=\infty$. We need some more notation. For a given random variable $\xi$ defined on a nonatomic probability space (real- or vector-valued), we define $\xi^*$, the *decreasing rearrangement* of $\xi$, by

$$\xi^*(t)=\inf\{\lambda\geq 0:\mathbb{P}(|\xi|>\lambda)\leq t\}.$$

Then $\xi^{**}:(0,1]\to[0,\infty)$, the *maximal function* of $\xi^*$, is given by

$$\xi^{**}(t)=\frac{1}{t}\int_0^t\xi^*(s)\,ds, \quad t\in(0,1].$$

One easily verifies that $\xi^{**}$ can alternatively be defined by

$$\xi^{**}(t)=\frac{1}{t}\sup\Big\{\int_E|\xi|\,d\mathbb{P}:E\in\mathcal{F},\,\mathbb{P}(E)=t\Big\}.$$

We are ready to introduce the weak-$L^\infty$ space. Following Bennett, DeVore and Sharpley [1], we let

$$\|\xi\|_{W(\Omega)}=\sup_{t\in(0,1]}(\xi^{**}(t)-\xi^*(t))$$

and define $W(\Omega)=\{\xi:\|\xi\|_{W(\Omega)}<\infty\}$. Let us describe the motivation behind the definition of this class. Note that for each $1\leq p<\infty$, the usual weak space $L^{p,\infty}$ properly contains $L^p$, but for $p=\infty$, the two spaces coincide. Thus, there is no Marcinkiewicz interpolation theorem between $L^1$ and $L^\infty$ for operators which are unbounded on $L^\infty$. The space $W$ was invented to fill this gap. It contains $L^\infty$, can be understood as an appropriate limit of $L^{p,\infty}$ as $p \to \infty$, and has an appropriate interpolation property: if an operator $T$ is bounded from $L^1$ to $L^{1,\infty}$ and from $L^\infty$ to $W$, then it can be extended to a bounded operator on all $L^p$ spaces, $1 < p < \infty$. See [1] for details. There are also some deep connections between $W$ and BMO: consult [1] or the monograph [2] by Bennett and Sharpley for an explanation and much more on the subject.

There is a natural question about weak-$L^\infty$ estimates (in the sense of the above space $W$) for differentially subordinated martingales and the Haar system. The answer is contained in the theorem below. In analogy to the previous notation, the weak-$L^\infty$ norm of a martingale $X$ is given by $\|X\|_{W(\Omega)} = \sup_{t\geq 0} \|X_t\|_{W(\Omega)}$.

THEOREM 1.2. *Let $X$ and $Y$ be $\mathcal{H}$-valued martingales such that $Y$ is differentially subordinate to $X$. Then*

$$\|Y\|_{W(\Omega)} \leq 2\|X\|_\infty \tag{1.7}$$

*and the constant* 2 *is the best possible.*

The constant 2 is already the best possible in the corresponding bound for the Haar system. As we will see, there are nonzero real constants $a_0$, $a_1$, $a_2$ and signs $\varepsilon_0$, $\varepsilon_1$, $\varepsilon_2$ such that

$$\|\varepsilon_0 a_0 h_0 + \varepsilon_1 a_1 h_1 + \varepsilon_2 a_2 h_2\|_{W([0,1])} = 2\|a_0 h_0 + a_1 h_1 + a_2 h_2\|_{L^\infty([0,1])}.$$

The paper is organized as follows. The next section contains the proof of Theorem 1.2. We will transform the inequality (1.7) into a more convenient form, to study which we will exploit Burkholder's technique: the inequality will be extracted from the existence of certain special functions, with appropriate majorization and concavity. Section 3 is devoted to applications: we obtain a related weak-type bound for harmonic functions given on Euclidean domains.

**2. Proof of Theorem 1.2.** Let $\mathcal{B}$ denote the closed unit ball of $\mathcal{H}$, and consider the sets

$$\mathcal{D}_1 = \{(x,y) \in \mathcal{B} \times \mathcal{H} : |x| + |y| \leq \lambda + 1\},$$
$$\mathcal{D}_2 = \{(x,y) \in \mathcal{B} \times \mathcal{H} : |x| + |y| > \lambda + 1\}.$$

The key object in the proof of (1.7) is the family $(u_\lambda)_{\lambda\geq 0}$ of functions on $\mathcal{B} \times \mathcal{H}$, given by

$$u_\lambda(x,y) = \begin{cases} 0 & \text{if } (x,y) \in \mathcal{D}_1, \\ \frac{1}{2}(|y| - 1 - \lambda)^2 - \frac{1}{2}|x|^2 & \text{if } (x,y) \in \mathcal{D}_2. \end{cases}$$

In the lemmas below, we study some crucial properties of these functions.

LEMMA 2.1. *For any $x \in \mathcal{B}$ and $y \in \mathcal{H}$ we have the estimates*

$$u_\lambda(x, y) \geq (|y| - \lambda - 2)1_{\{|y|>\lambda\}}, \tag{2.1}$$

$$u_\lambda(x, y) \leq \frac{1}{2}(|y| - 1 - \lambda)^2 - \frac{1}{2}|x|^2. \tag{2.2}$$

*Proof.* We start with (2.1). If $|y| \leq \lambda$, then $|x| + |y| \leq \lambda + 1$, and hence both sides of (2.1) are equal. Next, if $|y| > \lambda$ and $|x| + |y| \leq \lambda + 1$, then

$$(|y| - \lambda - 2)1_{\{|y|>\lambda\}} = |y| - \lambda - 2 \leq -1 < 0 = u_\lambda(x, y).$$

Finally, if $|x| + |y| > \lambda + 1$, then $|y| > \lambda$, and hence

$$u_\lambda(x, y) - (|y| - \lambda - 2)1_{\{|y|>\lambda\}} = \frac{1}{2}(y - 2 - \lambda)^2 + \frac{1}{2}(1 - |x|^2) \geq 0.$$

This proves (2.1).

The inequality (2.2) is trivial when $|x|+|y| > \lambda+1$, and for the remaining $(x, y)$, it is equivalent to $|x| + |y| \leq \lambda + 1$. ∎

LEMMA 2.2. *Suppose that $x, y, h, k \in \mathcal{H}$ satisfy the conditions $|k| \leq |h|$, $(x, y) \in \mathcal{D}_1$ and $(x + h, y + k) \in \mathcal{D}_2$. Then*

$$u_\lambda(x + h, y + k) \leq 0. \tag{2.3}$$

*Proof.* The inequality is equivalent to

$$\big||y + k| - 1 - \lambda\big| \leq |x + h|,$$

and hence we will be done if we show the two bounds

$$-|y + k| + 1 + \lambda \leq |x + h|, \qquad |y + k| - 1 - \lambda \leq |x + h|.$$

The first estimate follows from the inclusion $(x + h, y + k) \in \mathcal{D}_2$. To deal with the second bound, use the triangle inequality and the assumptions $(x, y) \in \mathcal{D}_1$ and $|k| \leq |h|$ to obtain

$$|y + k| - 1 - \lambda \leq |y| + |k| - 1 - \lambda \leq -|x| + |k| \leq -|x| + |h| \leq |x + h|,$$

as desired. ∎

Recall that for any semimartingale $X$ there exists a unique continuous local martingale part $X^c$ of $X$ satisfying

$$[X, X]_t = [X^c, X^c]_t + \sum_{0 \leq s \leq t} |\Delta X_s|^2$$

for all $t \geq 0$. Here $\Delta X_s = X_s - X_{s-}$ is the jump of $X$ at $s$, and we use the convention $X_{0-} = 0$. Furthermore, $[X^c, X^c] = [X, X]^c$, the pathwise continuous part of $[X, X]$. We will need the following simple auxiliary fact, which follows from [12, Lemma 1].

LEMMA 2.3. *If the process $Y$ is differentially subordinate to $X$, then for all $t \geq 0$ we have $|\Delta Y_t| \leq |\Delta X_t|$.*

Let us also make a small comment here. Let $X$ be a martingale satisfying $\|X\|_\infty \le 1$, and suppose that $Y$ is differentially subordinate to $X$. Then $X$ is bounded in $L^2$ and hence so is $Y$, for instance by Burkholder's $L^2$ bound. This implies, by classical martingale convergence theorems, that $Y$ converges in $L^2$; the corresponding limit will be denoted by $Y_\infty$.

Equipped with the above statements, we are ready to show the following intermediate result.

THEOREM 2.4. *Suppose that $X$ and $Y$ are $\mathcal{H}$-valued martingales such that $\|X\|_\infty \le 1$ and $Y$ is differentially subordinate to $X$. Then for any $\lambda \ge 0$ and $t \in [0,\infty]$ we have*

$$\mathbb{E}(|Y_t| - \lambda)_+ \le 2\mathbb{P}(|Y_t| > \lambda). \tag{2.4}$$

*Proof.* Introduce the stopping time $\tau = \inf\{s : |X_s| + |Y_s| > \lambda\}$, with the usual convention $\inf\emptyset = +\infty$. By (2.2), we may write

$$\mathbb{E}u_\lambda(X_t, Y_t)1_{\{\tau\le t\}} \le \frac{1}{2}\mathbb{E}\big[(|Y_t| - 1 - \lambda)^2 - |X_t|^2\big]1_{\{\tau\le t\}}.$$

Now, the processes $(|X_s|^2 - [X,X]_s)_{s\ge0}$ and $(|Y_s|^2 - [Y,Y]_s)_{s\ge0}$ are martingales, so

$$\mathbb{E}(|X_t|^2 - [X,X]_t)1_{\{\tau\le t\}} = \mathbb{E}(|X_\tau|^2 - [X,X]_\tau)1_{\{\tau\le t\}},$$

and similarly for $Y$. Consequently, the differential subordination of $Y$ to $X$ yields

$$\begin{aligned}\mathbb{E}(|Y_t|^2 - |X_t|^2)1_{\{\tau\le t\}} &= \mathbb{E}(|Y_\tau|^2 - |X_\tau|^2)1_{\{\tau\le t\}} + \int_{\tau+}^{t} d([Y,Y]_s - [X,X]_s)\\ &\le \mathbb{E}(|Y_\tau|^2 - |X_\tau|^2)1_{\{\tau\le t\}}.\end{aligned}$$

Furthermore, the process $(|Y_s|)_{s\ge0}$ is a submartingale, so $\mathbb{E}|Y_t|1_{\{\tau\le t\}} \ge \mathbb{E}|Y_\tau|1_{\{\tau\le t\}}$. Combining the above two estimates with the preceding inequality, we obtain

$$\mathbb{E}u_\lambda(X_t, Y_t)1_{\{\tau\le t\}} \le \mathbb{E}u_\lambda(X_\tau, Y_\tau)1_{\{\tau\le t\}}.$$

However, by Lemma 2.2, we have $u_\lambda(X_\tau, Y_\tau) \le 0$ almost surely. Indeed, if $\tau$ is finite, it suffices to take $x = X_{\tau-}$, $y = Y_{\tau-}$, $h = \Delta X_\tau$ and $k = \Delta Y_\tau$ (and use Lemma 2.3); if $\tau = \infty$, then $|X_\infty| + |Y_\infty| \le \lambda + 1$, so $u_\lambda(X_\infty, Y_\infty) = 0$. Consequently, we have obtained the bound $\mathbb{E}u_\lambda(X_t, Y_t)1_{\{\tau\le t\}} \le 0$, and it remains to combine it with the trivial equality $\mathbb{E}u_\lambda(X_t, Y_t)1_{\{\tau>t\}} = 0$ to get

$$\mathbb{E}u_\lambda(X_t, Y_t) \le 0.$$

By (2.1), this implies $\mathbb{E}(|Y_t| - \lambda - 2)1_{\{|Y_t|>\lambda\}} \le 0$, which is precisely the claim. ∎

We turn our attention to Theorem 1.2.

*Proof of (1.7).* With no loss of generality, we may and do assume that $\|X\|_\infty \le 1$. Pick arbitrary $s \in [0,\infty]$ and $t \in (0,1]$, and recall the alternative definition of $Y_s^{**}$:

$$Y_s^{**}(t) = \sup\left\{\frac{1}{\mathbb{P}(E)}\mathbb{E}|Y_s|1_E : E \in \mathcal{F},\, \mathbb{P}(E) = t\right\}.$$

It follows that

$$Y_s^{**}(t) - Y_s^*(t) = \sup\left\{\frac{1}{\mathbb{P}(E)}\mathbb{E}(|Y_s| - Y_s^*(t))1_E : \mathbb{P}(E) = t\right\}.$$

However, by the definition of $Y_s^*(t)$, we have $\mathbb{P}(|Y_s| > Y_s^*(t)) \le t$. Hence, the above formula implies

$$Y_s^{**}(t) - Y_s^*(t) \le \frac{1}{\mathbb{P}(|Y_s| > Y_s^*(t))}\mathbb{E}(|Y_s| - Y_s^*(t))_+ \le 2,$$

where the latter bound follows from (2.4). This gives the desired bound. ∎

*Sharpness for the Haar system.* We will give an appropriate example. Consider the real-valued function $f = \frac{1}{2}h_0 - \frac{1}{2}h_1 - h_2 = -1_{[0,1/4)} + 1_{[1/4,1)}$; clearly, $\|f\|_\infty = 1$. Furthermore, its transform $g = \frac{1}{2}h_0 + \frac{1}{2}h_1 + h_2 = 2\cdot 1_{[0,1/4)}$ satisfies $g^* = 2\cdot 1_{[0,1/4)}$ and $g^{**}(t) = 2\cdot 1_{(0,1/4]}(t) - (2t)^{-1}1_{(1/4,1]}(t)$. Consequently,

$$\lim_{t\downarrow 1/4}(g^{**}(t) - g^*(t)) = 2,$$

which shows that the constant 2 is indeed the best possible. ∎

**3. Inequalities for harmonic functions.** In this section we will establish weak-$L^\infty$ inequalities for harmonic functions on Euclidean domains. Let $n$ be a positive integer, and let $D$ be an open connected subset of $\mathbb{R}^n$. Fix a point $\xi$ in $D$. For two real-valued harmonic functions $u, v$ on $D$, we say that $v$ is *differentially subordinate* to $u$ if

$$|v(\xi)| \le |u(\xi)| \tag{3.1}$$

and

$$|\nabla v(x)| \le |\nabla u(x)| \quad \text{for any } x \in D. \tag{3.2}$$

This concept was introduced by Burkholder; see [6] for more information and references. Let $D_0$ be a bounded domain satisfying $\xi \in D_0 \subset D_0 \cup \partial D_0 \subset D$. Let $\mu_{D_0}^\xi$ denote the harmonic measure on $\partial D_0$, corresponding to $\xi$. Then the weak-$L^\infty$ norm of $u$ is given by

$$\|u\|_{W(D)} = \sup_{D_0}\sup_{t\in(0,1]}(u_{D_0}^{**}(t) - u_{D_0}^*(t)),$$

where $u_{D_0}$ is the restriction of $u$ to $D_0$, and $u^*_{D_0}$ and $u^{**}_{D_0}$ are the decreasing rearrangement and the associated maximal function of $u_{D_0}$ with respect to the measure $\mu^\xi_{D_0}$.

The harmonic analogue of Theorem 1.2 is the following.

THEOREM 3.1. *Suppose $v$ is differentially subordinate to $u$. Then*

$$\|v\|_{W(D)} \leq 2\|u\|_{L^\infty(D)}, \tag{3.3}$$

*and the constant* 2 *is the best possible.*

*Proof.* The proof of (3.3) is standard. Fix a domain $D_0 \subset D$ as above. Let $B = (B_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}^n$ starting from $\xi$, and introduce the stopping time $\tau = \tau_{D_0} = \inf\{t : B_t \in \partial D_0\}$. Let $X$ and $Y$ be martingales defined by

$$X_t = u(B_{\tau\wedge t}) \quad \text{and} \quad Y_t = v(B_{\tau\wedge t}), \qquad t \geq 0. \tag{3.4}$$

The property (3.1) gives $|Y_0| \leq |X_0|$, and (3.2) implies that $Y$ is differentially subordinate to $X$. This follows at once from the identities

$$[X,X]_t = |X_0|^2 + \int_0^{\tau\wedge t} |\nabla u(B_s)|^2\, ds,$$

$$[Y,Y]_t = |Y_0|^2 + \int_0^{\tau\wedge t} |\nabla v(B_s)|^2\, ds.$$

Consequently, by (1.7), we have the estimate $\|Y\|_{W(\Omega)} \leq 2\|X\|_\infty$. However, $\|X\|_\infty \leq \|u\|_{L^\infty(D)}$ and, for each $s$, we have $\mathbb{P}(|Y_\infty| \geq s) = \mu^\xi_{D_0}(\{x \in \partial D_0 : |v(x)| \geq s\})$. The latter identity implies that the nonincreasing rearrangements of $Y_\infty$ and $v_{D_0}$ coincide, and hence

$$\sup_{t\in(0,1]} (v^{**}_{D_0}(t) - v^*_{D_0}(t)) \leq \|Y\|_{W(\Omega)}.$$

This proves (3.3), since $D_0$ was arbitrary.

It remains to show that the constant 2 cannot be improved; we will provide an appropriate example in dimension 1. Let $D = (-1, 3)$, $\xi = 0$ and let $u, v : D \to \mathbb{R}$ be given by

$$u(x) = -x + 1, \qquad v(x) = x + 1.$$

We have $u(0) = v(0) = 1$ and $|\nabla u(x)| = |\nabla v(x)|$ for all $x \in D$, so $v$ is differentially subordinate to $u$. Furthermore, note that $\|u\|_\infty = 2$; to handle the weak norm of $v$, pick a subdomain $D_0 = (-a, 3a)$, where $a \in (0, 1)$ is a fixed parameter. It is clear that the harmonic measure $\mu^\xi_{D_0}$ on $\{-a, 3a\}$ is given by $\mu^\xi_{D_0}(\{-a\}) = 3/4$ and $\mu^\xi_{D_0}(\{3a\}) = 1/4$. Now, $v(-a) = -a + 1$ and

$v(3a) = 3a + 1$, which implies that

$$v^*_{D_0}(t) = \begin{cases} 3a + 1 & \text{if } t \in (0, 1/4], \\ -a + 1 & \text{if } t \in (1/4, 1] \end{cases}$$

and

$$v^{**}_{D_0}(t) = \begin{cases} 3a + 1 & \text{if } t \in (0, 1/4], \\ \dfrac{1}{t}\left(\dfrac{3a+1}{4} + (1-a)\left(t - \dfrac{1}{4}\right)\right) & \text{if } t \in (1/4, 1]. \end{cases}$$

Therefore, we see that

$$\lim_{t \downarrow 1/4} (v^{**}_{D_0}(t) - v^*_{D_0}(t)) = 3a + 1 - (-a + 1) = 4a,$$

and hence $\|v\|_{W(D)}/\|u\|_{L^\infty(D)} \geq 2a$. Since $a \in (0, 1)$ was arbitrary, the constant 2 in (3.3) is indeed the best possible. ■

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Adam Osękowski
Department of Mathematics, Informatics and Mechanics
University of Warsaw
Banacha 2
02-097 Warszawa, Poland
E-mail: ados@mimuw.edu.pl

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