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DIFFERENCE AND FUNCTIONAL EQUATIONS

Periodic Solutions of Periodic Retarded Functional Differential Equations

by

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Summary. The paper presents a geometric method of finding periodic solutions of retarded functional differential equations (RFDE) $x'(t) = f(t, x_t)$, where f is T-periodic in t. We construct a pair of subsets of $\mathbb{R} \times \mathbb{R}^n$ called a T-periodic block and compute its Lefschetz number. If it is nonzero, then there exists a T-periodic solution.

1. Introduction. The geometric method of finding periodic solutions presented here is a generalization of the method introduced by R. Srzednicki in [8] and [9]. Its aim is to find periodic solutions of the equation

$$x'(t) = f(t, x), \tag{$*$}$$

where f is a continuous function, T-periodic in the time variable. The idea is to construct a pair of sets, called a T-periodic block, which depends on the equation $(*)$ and has a simple topological structure, so it is easy to compute its Lefschetz number, and if it is nonzero, then $(*)$ has a periodic orbit.

For retarded functional differential equations (RFDE's), the problem is that the proper phase space is the space of continuous functions from some interval to $\mathbb{R}^n$. To overcome the difficulties which arise in the infinite-dimensional case, we will show that the problem of finding T-periodic solutions of such equations can be translated into a finite-dimensional one. That will enable us to use the methods of [8].

The concept of blocks and a generalization of the Ważewski Principle which we apply to RFDE's was presented by K. Rybakowski in [7]. Condition (ii) in our definition of a T-periodic block follows the definition of a polyfacial set in [7].

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2. Preliminaries. We start by recalling some basic notations used in the theory of RFDE's. For $n \geq 1$ and $r > 0$ let $\mathscr{C} = \mathscr{C}([-r,0];\mathbb{R}^n)$ be the space of all continuous functions from the interval $[-r,0]$ (r is called a *lag*) to $\mathbb{R}^n$, with the supremum norm $|\cdot|$ (i.e. $|\varphi| = \sup_{\theta\in[-r,0]} |\varphi(\theta)|$). With this norm $\mathscr{C}$ is a Banach space.

For a function x: $[-r+a,b) \to \mathbb{R}^n$ and $t \in [a,b)$ define $x_t \in \mathscr{C}$ by

$$x_t(\theta) = x(t+\theta), \quad \theta \in [-r,0].$$

We will consider the equation

$$x'(t) = f(t, x_t), \tag{1}$$

where f: $\mathbb{R} \times \mathscr{C} \to \mathbb{R}^n$ is a continuous function.

A continuous function x: $[-r+a,b) \to \mathbb{R}^n$ is a *solution* of (1) if it satisfies this equation for every $t\in[a,b)$ and is saturated (i.e. for every y: $[-r+a,c) \to \mathbb{R}^n$ such that y satisfies (1) and $x_t = y_t$, where $t \in [a, \min(b,c))$, we have $c \leq b$).

With every solution x of (1) we associate the map $[a,b) \ni t \mapsto x_t \in \mathscr{C}$, and call its image the *orbit* of the solution.

Throughout the paper we will assume that f has the following properties:

1) f is T-periodic in t, i.e.

$$f(t,\varphi) = f(t+T,\varphi), \quad t \in \mathbb{R},\ \varphi \in \mathscr{C},$$

and $T > r$,

2) given $\sigma \in \mathbb{R}$ and $\varphi \in \mathscr{C}$ there exists a unique function x: $[-r+a,b) \to \mathbb{R}^n$ which is a solution of (1) and satisfies $x_\sigma = \varphi$. It will be denoted by $x(\sigma,\varphi)$.

The second condition is not very restrictive. It is satisfied, for example, if f is Lipschitzian with respect to the second variable on each compact subset of $\mathbb{R} \times \mathscr{C}$ ([6, Ch. 2, Th. 2.3]).

Let X be a topological space. In what follows, it will be $\mathbb{R}^n$ or $\mathscr{C}$.

Definition 1. Let φ: $\mathbb{R} \times X \times \mathbb{R} \to X$ be a continuous map. Let $\varphi_{(\sigma,t)}$ denote the map $\varphi(\sigma,\cdot,t)$. The map φ is called a (global) *process* on X if:

(i) $\varphi_{(\sigma,0)} = \mathrm{id}_X$ for every $\sigma \in \mathbb{R}$,
(ii) $\varphi_{(\sigma,s+t)} = \varphi_{(\sigma+s,t)} \circ \varphi_{(\sigma,s)}$ for every $\sigma, s, t \in \mathbb{R}$.

If for some fixed $T > 0$ we have $\varphi_{(\sigma+T,t)} = \varphi_{(\sigma,t)}$ ($\sigma, t \in \mathbb{R}$), then we call φ a *T-periodic process.*

For $A \subset \mathbb{R} \times \mathbb{R}^n$ let

$$A_t = \{x \in \mathbb{R}^n : (t,x) \in A\}.$$

Definition 2 ([8, Def. 2.2.2]). A pair (A,B) of closed subsets of $\mathbb{R}\times\mathbb{R}^n$ is called a *T-periodic pair* if

(i) A_t and B_t are compact for all $t \in \mathbb{R}$,
(ii) $A_t = A_{t+T}$, $B_t = B_{t+T}$ for every $t \in \mathbb{R}$ (A and B are T-*periodic*),
(iii) A and B are ANR's,
(iv) there exists a T-periodic process ω on $\mathbb{R}^n$ such that A and B consist of trajectories of ω, that is, for every (or, equivalently, for some) $\sigma \in \mathbb{R}$,

$$A = \bigcup_{x \in A_\sigma} \bigcup_{t \in \mathbb{R}} (t, \omega_{(\sigma,t)}(x)), \qquad B = \bigcup_{x \in B_\sigma} \bigcup_{t \in \mathbb{R}} (t, \omega_{(\sigma,t)}(x)).$$

For the definition of an ANR see [1, Ch. IV, Sec. 1].

Remark 3. Condition (iii) in the definition of a T-periodic pair is equivalent to

(iii′) There exists $\sigma \in \mathbb{R}$ such that A_σ and B_σ are ANR's.

Moreover, for each element $C \subset \mathbb{R} \times \mathbb{R}^n$ of a T-periodic pair there exists a T-periodic open subset $U \subset \mathbb{R} \times \mathbb{R}^n$ such that $C \subset U$ and there exists a retraction $\varrho\colon U \to C$ such that ϱ is T-periodic in t and *invariant with respect to time sections*, i.e. $\varrho(t,x) = (t, \pi \circ \varrho(t,x))$ for every $t \in \mathbb{R}$, where $\pi\colon \mathbb{R} \times \mathbb{R}^n \ni (t,x) \mapsto x \in \mathbb{R}^n$.

Proof. Let $\xi\colon \mathbb{R}\times\mathbb{R}^n \ni (t,x) \mapsto ([t],x) \in (\mathbb{R}/T\mathbb{Z})\times\mathbb{R}^n$, where $[t] = t+T\mathbb{Z}$. Denote by $\pi_1\colon (\mathbb{R}/T\mathbb{Z}) \times \mathbb{R}^n \to \mathbb{R}/T\mathbb{Z}$ and $\pi_2\colon (\mathbb{R}/T\mathbb{Z}) \times \mathbb{R}^n \to \mathbb{R}^n$ the usual projections. If ω is as in the definition of a T-periodic pair, then $\omega^*\colon \mathbb{R} \times ((\mathbb{R}/T\mathbb{Z}) \times \mathbb{R}^n) \ni (t, ([\sigma],x)) \mapsto ([t+\sigma], \omega_{(\sigma,t)}(x)) \in (\mathbb{R}/T\mathbb{Z}) \times \mathbb{R}^n$ defines a dynamical system on $(\mathbb{R}/T\mathbb{Z}) \times \mathbb{R}^n$.

The set $C^* = \xi(C)$ is well defined, because C is T-periodic. Furthermore, C^* is ω^*-invariant, and is an ANR as the space of a locally trivial bundle with base a circle and with fiber C_t. Let $r_1^*\colon U_1^* \to C^*$ be a retraction, where U_1^* is a neighborhood of C^* in $(\mathbb{R}/T\mathbb{Z}) \times \mathbb{R}^n$. Taking a neighborhood $U^* \subset U_1^*$ of C^* small enough, we can assume that for every $([t],x) \in U^*$ there exists $\sigma_{([t],x)} \in (-T/2, T/2)$ such that $\pi_1 \circ r_1^*([t],x) = [t+\sigma_{([t],x)}]$. Then the function $r^*\colon U^* \to C^*$, $r^*([t],x) = \omega^*(-\sigma_{([t],x)}, r_1^*([t],x))$, is a retraction, invariant with respect to time sections.

Let $U = \xi^{-1}(U^*)$ and for $(t,x) \in U$ define the retraction $\varrho\colon U \to C$ as

$$\varrho(t,x) = (t, \pi_2 \circ r^* \circ \xi(t,x)).$$

The equivalence of (iii) and (iii′) is now clear. ∎

Now we will construct a pair of subsets of $\mathbb{R} \times \mathscr{C}$ using pairs contained in $\mathbb{R} \times \mathbb{R}^n$. Let (A,B) be a pair of subsets of $\mathbb{R} \times \mathbb{R}^n$. Define the *functional extension* $\mathrm{ex}(A)$ by

(2a) $$\mathrm{ex}(A) = \{(t,\varphi) \in \mathbb{R} \times \mathscr{C} : (t+\theta, \varphi(\theta)) \in A \text{ for every } \theta \in [-r,0]\},$$

and for $B \subset A$,

$$\mathrm{ex}_A(B) = \{(t,\varphi) \in \mathrm{ex}(A) : (t,\varphi(0)) \in B\}. \tag{2b}$$

Similarly, define

$$\mathrm{ex}(A)_t = \{\varphi \in \mathscr{C} : (t,\varphi) \in \mathrm{ex}(A)\},$$
$$\mathrm{ex}_A(B)_t = \{\varphi \in \mathrm{ex}(A)_t : \varphi(0) \in B_t\}.$$

Lemma 4. *If (A,B) is a T-periodic pair, then* $\mathrm{ex}(A)$, $\mathrm{ex}_A(B)$, $\mathrm{ex}(A)_t$ *and* $\mathrm{ex}_A(B)_t$ *are closed ANR's.*

Proof. Since A and B are ANR's, there are T-periodic open subsets U and V of $\mathbb{R} \times \mathbb{R}^n$ such that $A \subset U$, $B \subset V$ and, by Remark 3, there exist T-periodic retractions $\varrho\colon U \to A$ and $\varrho^*\colon V \to B$ which preserve the time sections A_t and B_t.

Denote by $\varrho_t\colon U_t \to A_t$ and $\varrho^*_t\colon V_t \to B_t$ the restrictions of ϱ and ϱ^* to the appropriate t-time sections. To show that $\mathrm{ex}(A)$ is an ANR define $\widetilde{\varrho}\colon \mathrm{ex}(U) \to \mathrm{ex}(A)$ by $\widetilde{\varrho}(t,\varphi) = (t,\widetilde{\varrho}_t(\varphi))$, where $\widetilde{\varrho}_t(\varphi)(\theta) = \varrho_t(\varphi(\theta))$. Because U is open, $\mathrm{ex}(U)$ is open in $\mathbb{R} \times \mathscr{C}$, and the continuity of $\widetilde{\varrho}$ follows from the compactness of the graph of φ. Thus $\mathrm{ex}(U)$ is an ANR and $\mathrm{ex}(A)$ is its retract, hence also an ANR.

To show that $\mathrm{ex}_A(B)$ is an ANR, we will use the retraction $\widetilde{\varrho}$ as well as ϱ^*. Let (U_1,V_1) be a pair of T-periodic open subsets of $\mathbb{R} \times \mathbb{R}^n$ such that $A \subset U_1 \subset \overline{U}_1 \subset U$ and $B \subset V_1 \subset \overline{V}_1 \subset V$. Define $\widetilde{\varrho}^*\colon \mathrm{ex}_{U_1}(V_1) \to \mathbb{R} \times \mathscr{C}$ by $\widetilde{\varrho}^*(t,\varphi) = (t,\widetilde{\varrho}^*_t(\varphi))$, where $\widetilde{\varrho}^*_t(\varphi)(\theta) = \varphi(\theta) + \varrho^*_t(\varphi(0)) - \varphi(0)$. We can choose U_1 and V_1 sufficiently small to ensure that $\widetilde{\varrho}^*(\mathrm{ex}_{U_1}(V_1))$ is contained in $\mathrm{ex}(U)$. Then $\widetilde{\varrho} \circ \widetilde{\varrho}^*$ is the required retraction onto $\mathrm{ex}_A(B)$, and so this last set is an ANR.

Since A_t is closed, so is $\mathrm{ex}(A)_t$, and the same applies to the other sets from Lemma 4. ■

The main result of the paper is based on the concept of a Lefschetz map f and its Lefschetz number $\Lambda(f)$. The reader is referred to [2] and [5] for unexplained definitions and basic facts concerning these concepts.

We make the following definition (see [8, Def. 2.2.3, p. 23]):

Definition 5. The *Lefschetz number* of a T-periodic pair (A,B) is

$$\mathrm{Lef}_T(A,B) = \Lambda(\omega_{(\sigma,T)}) \in \mathbb{Z}, \tag{3}$$

where $\omega_{(\sigma,T)}\colon (A_\sigma,B_\sigma) \to (A_{\sigma+T},B_{\sigma+T}) = (A_\sigma,B_\sigma)$, and ω is the T-periodic process from the definition of a T-periodic pair.

This number is well defined, because A_σ and B_σ are compact ANR's. It was shown in [8, p. 22] that this definition does not depend on the choice of ω. Moreover, it does not depend on the choice of the initial time σ.

For a T-periodic pair (A, B) and $\sigma \in \mathbb{R}$ define $\widetilde{\omega}$: $(A_\sigma, B_\sigma) \to (\mathrm{ex}(A)_\sigma, \mathrm{ex}_A(B)_\sigma)$ by

$$\widetilde{\omega}(x)(\theta) = \omega_{(\sigma,\theta)}(x) \quad \text{for } \theta \in [-r, 0], \tag{4}$$

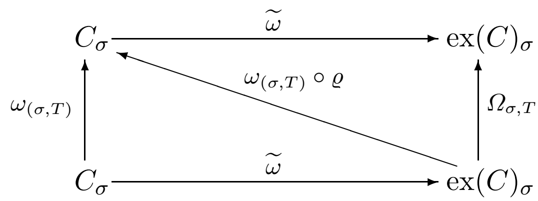
where ω is as above.

PROPOSITION 6. *If $C \subset \mathbb{R}\times\mathbb{R}^n$ is an element of a T-periodic pair, then for $\Omega_{\sigma,T}$: $\mathrm{ex}(C)_\sigma \to \mathrm{ex}(C)_{\sigma+T} = \mathrm{ex}(C)_\sigma$ defined by*

$$\Omega_{\sigma,T}(\varphi) = \widetilde{\omega}(\omega_{(\sigma,T)}(\varphi(0))),$$

the Lefschetz number $\Lambda(\Omega_{\sigma,T})$ is well defined and $\Lambda(\Omega_{\sigma,T}) = \Lambda(\omega_{(\sigma,T)})$.

Proof. Let ϱ: $\mathrm{ex}(C)_\sigma \ni \varphi \mapsto \varphi(0) \in C_\sigma$. Consider the commutative diagram



[5, Lem. 3.1] implies that the Lefschetz number of $\Omega_{\sigma,T}$ is well defined because it is defined for $\omega_{(\sigma,T)}$. Moreover, $\Lambda(\Omega_{\sigma,T}) = \Lambda(\omega_{(\sigma,T)})$. It is obvious that we can use the same argument for pairs of spaces (see [2]). ■

For a set $W \subset \mathbb{R}\times\mathbb{R}^n$ we introduce the following set, which depends on the equation (1):

$$W^- = \{(t,x) \in \partial W : \text{there exists } \varepsilon > 0 \text{ such that for every } \varphi \in \mathscr{C} \text{ with } (t,\varphi) \in \mathrm{ex}(W) \text{ and } \varphi(0) = x \text{ and for all } \theta \in (0,\varepsilon] \text{ we have } (t+\theta, x(t,\varphi)(t+\theta)) \notin W\}.$$

This is a generalization of the exit set from the theory of dynamical systems.

DEFINITION 7. A pair (W, W^-) of subsets of $\mathbb{R}\times\mathbb{R}^n$ is a *T-periodic block* for the equation (1) if

(i) (W, W^-) is a T-periodic pair,
(ii) $\partial W \setminus W^- = \{(t,x) \in \partial W$: there exists $\varepsilon > 0$ such that for every $\varphi \in \mathscr{C}$ with $(t,\varphi) \in \mathrm{ex}(W)$ and $\varphi(0) = x$ and for all $\theta \in (0,\varepsilon]$ we have $(t+\theta, x(t,\varphi)(t+\theta)) \in \operatorname{int} W\}$.

We recall that according to the definition of a T-periodic pair, both sets in a T-periodic block have to be closed. This is an essential property, which will enable us to use the Ważewski Principle.

3. The main theorem

THEOREM 8. *Let f: $\mathbb{R}\times\mathscr{C} \to \mathbb{R}^n$ be a T-periodic continuous map which satisfies the uniqueness condition. Suppose that there exists a T-periodic* *block* (W,W^-) *for the equation* (1). *Then* $\mathrm{Lef}_T(W,W^-)$ *is well defined and if* $\mathrm{Lef}_T(W,W^-)\neq 0$, *then there exists a* T*-periodic solution of* (1).

The rest of this section is devoted to the proof of Theorem 8. The idea is to reduce the problem from the infinite-dimensional space $\mathscr{C}$ to $\mathbb{R}^n$. The first step will be the change of space and of the *solution operator*

$$\Phi(0,T)\colon \mathrm{ex}(W)_0 \ni \varphi \mapsto x_T(0,\varphi)\in\mathscr{C}.$$

If the initial time is 0 and W^- is not empty, then there are T-time solutions $x_T(0,\varphi)$, where $\varphi\in \mathrm{ex}(W)_0$, which are not contained in $\mathrm{ex}(W)_t$ for some $t\in(0,T]$. To avoid the problem with the range of the solution operator, we will modify the range and domain of $\Phi(0,T)$. But this operation should not create any additional fixed points of the modified operator, denoted by Θ. The next aim is to construct the homotopy between Θ and $\Omega_{0,T}$ defined in Proposition 6. The last step is to apply Proposition 6.

To simplify notation we will assume, without loss of generality, that the initial time is 0. Define

$$\tau\colon \mathrm{ex}(W)\ni(\sigma,\varphi)\mapsto \sup\{t\geq 0 : (s,x(\sigma,\varphi)(s))\in W \\ \text{for all } s\in[\sigma,\sigma+t]\}\in[0,\infty].$$

The next lemma is a version of the Ważewski Principle [10] presented in a modern way by C. Conley [3]. For RFDE's a generalization of the Ważewski Principle was given by K. Rybakowski [7].

Lemma 9. *The map* τ *is continuous.*

Proof. We consider two cases:

If $\tau(\sigma,\varphi)<\infty$ then $x_{\sigma+\tau(\sigma,\varphi)}(\sigma,\varphi)\in \mathrm{ex}_W(W^-)_{\sigma+\tau(\sigma,\varphi)}$. By the definition of W^- we can choose $\varepsilon>0$ such that $x(\sigma,\varphi)(\sigma+\tau(\sigma,\varphi)+\varepsilon)\notin W_{\sigma+\tau(\sigma,\varphi)+\varepsilon}$.

Let U_1 be a neighborhood of $(\sigma+\tau(\sigma,\varphi)+\varepsilon, x(\sigma,\varphi)(\sigma+\tau(\sigma,\varphi)+\varepsilon))$ in $\mathbb{R}\times\mathbb{R}^n$ such that $U_1\cap W=\emptyset$. Let $\widehat{U}_1$ be a neighborhood of $(\sigma+\tau(\sigma,\varphi)+\varepsilon, x_{\sigma+\tau(\sigma,\varphi)+\varepsilon}(\sigma,\varphi))$ in $\mathbb{R}\times\mathscr{C}$ with $(s,\psi(0))\in U_1$ for every $(s,\psi)\in\widehat{U}_1$. The continuous dependence on initial conditions implies that there exists a neighborhood $\widehat{V}_1$ of (σ,φ) in $\mathbb{R}\times\mathscr{C}$ such that $(s+\tau(\sigma,\varphi)+\varepsilon, x_{s+\tau(\sigma,\varphi)+\varepsilon}(s,\psi))\in\widehat{U}_1$ for every $(s,\psi)\in\widehat{V}_1$. Because the value at 0 of every function from $\widehat{U}_1$ does not belong to W, for every $(s,\psi)\in\widehat{V}_1$ we have $\tau(s,\psi)<\tau(\sigma,\varphi)+\varepsilon$.

Let U_2 be a neighborhood of $(\sigma+\tau(\sigma,\varphi)-\varepsilon, x(\sigma,\varphi)(\sigma+\tau(\sigma,\varphi)-\varepsilon))$ in $\mathbb{R}\times\mathbb{R}^n$ disjoint from W^- (we can assume that $\tau(\sigma,\varphi)>0$ and $\varepsilon<\tau(\sigma,\varphi)$). We define $\widehat{U}_2$ as above. Let U_3 be an open neighborhood of $\{(t,x(\sigma,\varphi)(t)) : t\in[\sigma,\sigma+\tau(\sigma,\varphi)-\varepsilon]\}$ in $\mathbb{R}\times\mathbb{R}^n$ such that $U_3\cap W^-=\emptyset$.

Using the continuous dependence again, we find a neighborhood $\widehat{V}_2$ of (σ,φ) such that $(s+\tau(\sigma,\varphi)-\varepsilon, x_{s+\tau(\sigma,\varphi)-\varepsilon}(s,\psi))\in\widehat{U}_2$ for every $(s,\psi)\in\widehat{V}_2$.

Taking $\widehat{V}_2$ small enough, we can assume that $(t, x(s,\psi)(t))$ is in U_3 for all $(s,\psi) \in \widehat{V}_2$ and $t \in [\sigma, \sigma + \tau(\sigma,\varphi) - \varepsilon]$. This gives the opposite inequality $\tau(s,\psi) > \tau(\sigma,\varphi) - \varepsilon$, which proves the continuity in the first case.

We only sketch the proof in the case $\tau(\sigma,\varphi) = \infty$, because it is similar. We choose $N > \sigma$ and construct an open neighborhood U of $(N, x(\sigma,\varphi)(N))$ in $\mathbb{R}\times\mathbb{R}^n$ disjoint from W. Then we choose a neighborhood $\widehat{U}$ of $(N, x_N(\sigma,\varphi))$ in $\mathbb{R} \times \mathscr{C}$ and a neighborhood U' of the trajectory $(t, x(\sigma,\varphi)(t))$ on the interval $[\sigma, N]$, and finally we conclude that for some neighborhood $\widehat{V}$ of (σ,φ) and for every $(s,\psi) \in \widehat{V}$ we have $\tau(s,\psi) > N$. ∎

To simplify notation, for $\varphi \in \mathrm{ex}(W)_0$ we will write $\tau(\varphi)$ instead of $\tau(0,\varphi)$.

Let $\mathscr{D}$ be the topological quotient space obtained from the union $\mathrm{ex}(W)_0 \cup (\mathrm{ex}_W(W^-)_0 \times \mathbb{S}^1)$ ($\mathbb{S}^1$ is the unit circle in the complex plane), where every $\varphi \in \mathrm{ex}_W(W^-)_0 \subset \mathrm{ex}(W)_0$ is identified with $(\varphi, 1) \in \mathrm{ex}_W(W^-)_0 \times \mathbb{S}^1$.

We now define $\Theta\colon \mathscr{D} \to \mathscr{D}$. For $\varphi \in \mathrm{ex}(W)_0$ we set

$$\Theta(\varphi) = \begin{cases} x_T(0,\varphi) & \text{if } \tau(\varphi) > T, \\ (\widetilde{\omega}(\omega_{(\tau(\varphi), T-\tau(\varphi))}(x(0,\varphi)(\tau(\varphi)))), e^{\frac{T-\tau(\varphi)}{T}\pi i}) & \text{if } \tau(\varphi) \le T-r, \\ (\Theta_1(\varphi), e^{\frac{T-\tau(\varphi)}{T}\pi i}) & \text{if } \tau(\varphi) \in (T-r, T], \end{cases}$$

where

$$\Theta_1(\varphi)(t) = \begin{cases} x_T(0,\varphi)(t) & \text{if } T+t \le \tau(\varphi), \\ \widetilde{\omega}(\omega_{(\tau(\varphi), T-\tau(\varphi))}(x(0,\varphi)(\tau(\varphi))))(t) & \text{if } T+t > \tau(\varphi), \end{cases}$$

and $\widetilde{\omega}$ is given by (4). For $(\varphi, u) \in \mathrm{ex}_W(W^-)_0 \times \mathbb{S}^1$ we define

$$\Theta(\varphi, u) = (\widetilde{\omega}(\omega_{(0,T)}(\varphi(0))), ue^{\pi i}).$$

Lemma 10. *The map Θ is continuous and compact* (*i.e. there exists a compact set $\mathscr{K} \subset \mathscr{D}$ such that $\Theta(\mathscr{D}) \subset \mathscr{K}$*).

Proof. The continuity of Θ follows from the continuity of τ. From [6, Ch. 3, Cor. 6.1] we know that the solution map $\Phi(0,T)\colon \mathrm{ex}(W)_0 \ni \varphi \mapsto x_T(0,\varphi) \in \mathscr{C}$ is compact. The map $\widetilde{\omega}\colon W_0 \to \mathrm{ex}(W)_0$ is also compact, because $\widetilde{\omega}(W_0)$ is an equicontinuous and bounded family of functions. ∎

Proof of the main theorem. If $\varphi \in \mathscr{D}$ is a fixed point of Θ, then it is a fixed point of the solution map $\Phi(0,T)\colon \mathrm{ex}(W)_0 \ni \varphi \mapsto x_T(0,\varphi) \in \mathscr{C}$. This implies that $\mathrm{Fix}\,\Theta \subset \mathrm{ex}(W)_0$.

The map $\Theta\colon \mathscr{D} \to \mathscr{D}$ satisfies the conditions of [5, Th. 9.5]. The set $\mathscr{D}$ is an ANR, because $\mathrm{ex}(W)_0$, $\mathrm{ex}_W(W^-)_0$ and $\mathrm{ex}_W(W^-)_0 \times \mathbb{S}^1$ are, and $\mathscr{D}$ is the union of $\mathrm{ex}(W)_0$ and $(\mathrm{ex}_W(W^-)_0 \times \mathbb{S}^1)$ with intersection $\mathrm{ex}_W(W^-)_0$ (see [1, Th. IV.6.1]).

If $\Lambda(\Theta) \neq 0$, then Θ has a fixed point φ^*, which is also a fixed point of $\Phi(0,T)$. Because the solution operator $\Phi(0,T)$ for (1) induces a T-periodic process on $\mathscr{C}$, we obtain

$$x_t(0,\varphi^*) = x_t(T,\varphi^*) = x_t(T, x_T(0,\varphi^*)) = x_{t+T}(0,\varphi^*).$$

This ends the proof that $x(0,\varphi^*)$ is a T-periodic solution of (1).

If we know that the Lefschetz number $\Lambda(\Theta)$ is defined, then we can easily compute it using information about the geometry of (W, W^-) only. We will construct two homotopies to show that Θ is homotopic to Ω: $\mathscr{D} \to \mathscr{D}$, where

$$\Omega(\varphi) = \widetilde{\omega}(\omega_{(0,T)}(\varphi(0))), \qquad \Omega(\varphi, u) = (\widetilde{\omega}(\omega_{(0,T)}(\varphi(0))), u).$$

Let $\varphi \in \mathscr{D}$. Then, roughly speaking, the first homotopy will move $\Theta(\varphi)$ to the function defined by $\widetilde{\omega}$, without changing the ending point $\Theta(\varphi)(0)$. Define χ: $\mathrm{ex}(W)_0 \times [0,1] \to \mathrm{ex}(W)_0$ by

$$\chi(\varphi, s)(t) = \begin{cases} \varphi(t) & \text{if } t \geq -r(1-s), \\ \widetilde{\omega}(\omega_{(-r(1-s), r(1-s))}(\varphi(-r(1-s))))(t) & \text{if } t < -r(1-s). \end{cases}$$

We define the first homotopy, h_1: $\mathscr{D} \times [0,1] \to \mathscr{D}$, as follows. On $\mathrm{ex}(W)_0 \times [0,1]$ and if $\tau(\varphi) > T$, we set

$$h_1(\varphi, s) = \chi\big(x_T(0,\varphi), s\big).$$

If $\varphi \in \mathrm{ex}(W)_0$ and $\tau(\varphi) \in (T-r, T]$, then

$$h_1(\varphi, s) = \big(\chi(\Theta_1(\varphi), s), e^{\frac{T-\tau(\varphi)}{T}\pi i}\big).$$

On the rest of $\mathscr{D}$ the map h_1 does not depend on the second variable and is equal to Θ:

$$\begin{aligned} h_1(\varphi, s) &= \Theta(\varphi) && \text{if } \varphi \in \mathrm{ex}(W)_0 \text{ and } \tau(\varphi) \in [0, T-r], \\ h_1((\varphi, u), s) &= \Theta(\varphi, u) && \text{if } (\varphi, u) \in \mathrm{ex}_W(W^-)_0 \times \mathbb{S}^1. \end{aligned}$$

The second homotopy, denoted by h_2: $\mathscr{D} \times [0,1] \to \mathscr{D}$, is defined as follows. If $\varphi \in \mathrm{ex}(W)_0$ and $\tau(\varphi) \geq T(1-s)$, then

$$h_2(\varphi, s) = \widetilde{\omega}(\omega_{(T(1-s), Ts)}(x(0,\varphi)(T(1-s)))).$$

If $\varphi \in \mathrm{ex}(W)_0$ and $\tau(\varphi) < T(1-s)$, then

$$h_2(\varphi, s) = (\widetilde{\omega}(\omega_{(\tau(\varphi), T-\tau(\varphi))}(x(0,\varphi)(\tau(\varphi)))), e^{(\frac{T-\tau(\varphi)}{T} - s)\pi i}).$$

On $\mathrm{ex}_W(W^-)_0 \times \mathbb{S}^1$ we set

$$h_2((\varphi, u), s) = (\widetilde{\omega}(\omega_{(0,T)}(\varphi(0))), u e^{(1-s)\pi i}).$$

Clearly $h_1(\cdot, 1) = h_2(\cdot, 0)$, and the homotopy H: $\mathscr{D} \times [0,1] \to \mathscr{D}$ defined by

$$H(\varphi, s) = \begin{cases} h_1(\varphi, 2s) & \text{if } s \leq 1/2, \\ h_2(\varphi, 2s-1) & \text{if } s > 1/2, \end{cases}$$

joins Θ to Ω.

Now let

Ω_1: $\mathrm{ex}(W)_0 \ni \varphi \mapsto \widetilde{\omega}(\omega_{(0,T)}(\varphi(0))) \in \mathrm{ex}(W)_0$,

Ω_2: $\mathrm{ex}_W(W^-)_0 \times \mathbb{S}^1 \ni (\varphi, u) \mapsto (\widetilde{\omega}(\omega_{(0,T)}(\varphi(0))), u) \in \mathrm{ex}_W(W^-)_0 \times \mathbb{S}^1$,

Ω_3: $\mathrm{ex}_W(W^-)_0 \ni \varphi \mapsto \widetilde{\omega}(\omega_{(0,T)}(\varphi(0))) \in \mathrm{ex}_W(W^-)_0$.

Set $\Delta = W_0 \cup (W_0^- \times \mathbb{S}^1)/\sim$, where $\sim$ identifies $x \in W_0^-$ with $(x, 1) \in W_0^- \times \mathbb{S}^1$. Furthermore, let γ: $\Delta \to \Delta$ be defined by $\gamma(x) = \omega_{(0,T)}(x)$ and $\gamma(x, s) = (\omega_{(0,T)}(x), s)$. The restrictions of γ denoted by γ_1: $W_0 \to W_0$, γ_2: $W_0^- \times \mathbb{S}^1 \to W_0^- \times \mathbb{S}^1$, γ_3: $W_0^- \to W_0^-$ are well defined. Proposition 6 implies that $\Lambda(\Omega) = \Lambda(\gamma)$ and $\Lambda(\Omega_i) = \Lambda(\gamma_i)$ for $i = 1, 2, 3$.

The next part of the proof is similar to the proof of the main theorem of [8, p. 27]. By [2, Prop. 4.1], $\Lambda(\gamma) = \Lambda(\gamma_2) + \Lambda(\gamma_4)$, where γ_4: $(\Delta, W_0^- \times \mathbb{S}^1) \to (\Delta, W_0^- \times \mathbb{S}^1)$ is determined by γ. The Lefschetz number of γ_2 is 0, because γ_2 is homotopic to a map without fixed points, and it follows that $\Lambda(\gamma) = \Lambda(\gamma_4)$. Since the triad $(\Delta; W_0^- \times \mathbb{S}^1, W_0)$ is excisive (see, for example, [4, III.8.23 Ex. 1(b)] for details), the relative singular homology groups (with rational coefficients) $H(\Delta, W_0^- \times \mathbb{S}^1)$ and $H(W_0, W_0^-)$ are equal. Hence $\Lambda(\gamma_4) = \mathrm{Lef}_T(W_0, W_0^-)$ and $\Lambda(\gamma) = \mathrm{Lef}_T(W_0, W_0^-)$.

The Lefschetz number is invariant under homotopy, therefore $\Lambda(\Theta) = \mathrm{Lef}_T(W_0, W_0^-)$. If $\mathrm{Lef}_T(W_0, W_0^-) \neq 0$, then $\Lambda(\Theta) \neq 0$ and, by [5, Th. 9.5], Θ has a fixed point φ^*. This implies that φ^* is a fixed point of $\Phi(0, T)$, because it follows from the construction that Θ has no fixed point in $\mathrm{ex}_W(W^-)_0 \times \mathbb{S}^1$. We conclude that $\varphi^* \in \{\varphi \in \mathscr{C} : (\theta, \varphi(\theta)) \in \mathrm{int}\, W$ for every $\theta \in [-r, 0]\}$ and $x(\sigma, \varphi^*)$ is a periodic solution of (1) contained in $\mathrm{int}\, W$. ∎

4. Examples. Our first example is a simple delay equation in $\mathbb{R}^n$:

$$\dot{x} = |x|x + x(t - r) + f(t), \tag{5}$$

where f is a T-periodic continuous function ($T > r$).

If B_d is a ball with center at the origin and its radius d is sufficiently large, then for every point of ∂B_d and every $\varphi \in \mathrm{ex}(B_d)$ such that $\varphi(0) \in \partial B_d$, solutions of the equation (5) are directed outside the ball. Using the previous notation, $\partial B_d = B_d^-$. We can take $(\mathbb{R} \times B_d, \mathbb{R} \times \partial B_d)$ as a T-periodic block. Since $\mathrm{Lef}_T(\mathbb{R} \times B_d, \mathbb{R} \times \partial B_d) = 1$, there exists a T-periodic orbit of (5).

The next example is an equation on the complex plane $\mathbb{C}$:

$$\dot{z} = \bar{z}^q + z(t - r) + p(t), \tag{6}$$

where $q \geq 2$ and p: $\mathbb{R} \to \mathbb{C}$ is a $2k\pi$-periodic continuous function ($k \in \mathbb{Z}$).

If we omit the term $z(t - r)$, then it was shown in [8, Ex. 6.4.1] that this (ordinary) differential equation has a $2k\pi$-periodic solution. For $z \in \mathbb{C}$ with $|z|$ large, we can estimate the term with delay, and it does not change the structure of a (sufficiently large) $2k\pi$-periodic block. Therefore (6) has a $2k\pi$-periodic solution.

The same reasoning applies to the equation

$$\dot{z} = \bar{z}^r e^{it} + (z(t-r))^q + p(t), \tag{7}$$

where $2 \le q < r$ and p is a $2k\pi$-periodic, continuous complex-valued function ($k \in \mathbb{Z}$).

If (W, W^-) is an appropriate $2k\pi$-periodic block for the equation $\dot{z} = \bar{z}^r e^{it} + z^q + p(t)$, then the exit set W^- does not change if we replace z^q by $(z(t-r))^q$. Thus (7) has a $2k\pi$-periodic solution.

To be more specific, consider the equation

$$\dot{z} = \bar{z}^2 e^{3it} + z(t-r) + e^{it}. \tag{8}$$

The $2\pi/3$-periodic block (W, W^-) for this equation has the form of a skewed prism with a hexagonal base, and the exit set consists of three stripes winding around this prism. This was shown in [8] in the case of the ordinary differential equation which arises from (8) by neglecting the term with delay. For (W, W^-) sufficiently large we obtain $\mathrm{Lef}_{2\pi}(W, W^-) = -2$, and so there exists a 2π-periodic solution of (8).

We can modify these equations to obtain equations like

$$\dot{z} = \bar{z}^q + \int_{-r}^{0} e^{-r/(r+\theta)} z(t-\theta)\, d\theta + p(t), \tag{9}$$

where p is a T-periodic, continuous complex-valued map and $q \ge 2$.

As in the second example, we can choose a T-periodic block for the equation $\dot{z} = \bar{z}^q + z + p(t)$, and if it is sufficiently large, then (9) has a T-periodic solution.

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