Tame Köthe Sequence Spaces are Quasi-Normable

by

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Summary. We show that every tame Fréchet space admits a continuous norm and that every tame Köthe sequence space is quasi-normable.

1. Introduction. First we recall definitions and basic properties of the above mentioned classes of spaces. Let $X$ be a Fréchet space with the topology defined by an increasing sequence $(\| \cdot \|_n)_{n \in \mathbb{N}}$ of seminorms. We call $X$ tame if the following condition holds: there is an increasing function $S : \mathbb{N} \to \mathbb{N}$ such that for every continuous linear operator $T : X \to X$ there is a natural $k_0$ such that for every $k \geq k_0$ there is a constant $C_k$ such that

$$\|Tx\|_k \leq C_k \|x\|_{S(k)}$$

for every $x \in X$.

This class of spaces was defined by D. Vogt and E. Dubinsky in [3]. They proved that in a tame infinite type power series space every complemented subspace has a basis. For other papers related to the notion of tameness see [7]–[9]. It is known that every finite type power series space is tame (see [10, Lemma 5.1]). The aim of this paper is to analyze which Köthe sequence spaces are tame.

We call $X$ quasi-normable if for every 0-neighbourhood $U$ there exists another 0-neighbourhood $V$ such that for every $\varepsilon > 0$ we can find a bounded set $B$ in $X$ such that

$$V \subset \varepsilon U + B.$$ 

The class of quasi-normable spaces was introduced by A. Grothendieck in [4]. See also [2], [6]. By $L(X)$ we denote the linear space of all continuous linear

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operators acting on $X$. For any operator $A \in L(X)$ we define
$$
\sigma_A(k) = \inf\{n \in \mathbb{N} : \sup_{\|x\| \leq 1} \|Ax\|_k < \infty\}.
$$
Let $I$ be an arbitrary index set and $A = (a^n_i)_{n \in \mathbb{N}}$ a sequence of nonnegative functions defined on $I$ with the property that $a^n_i \leq a^{n+1}_i$ for all $n \in \mathbb{N}, i \in I$. Let us recall that for $1 \leq p < \infty$ a Kôthe sequence space is defined as follows:
$$
\lambda_p(I, A) = \left\{ x = (x_1, x_2, \ldots) : \|x\|_k := \left( \sum_{i \in I} (a^k_i |x_i|)^p \right)^{1/p} < \infty \ \forall k \in \mathbb{N} \right\}
$$
and
$$
\lambda_\infty(I, A) = \left\{ x = (x_1, x_2, \ldots) : \|x\|_k := \sup_{i \in I} a^k_i |x_i| < \infty \ \forall k \in \mathbb{N} \right\}
$$
(see [5, 27]). For other notions from functional analysis used in this paper see [5].

2. Preliminary results

**Lemma 1.** The space $\omega$ of all sequences is not tame.

**Proof.** Recall that
$$
\omega = \left\{ x = (x_1, x_2, \ldots) : \|x\|_k := \max_{j \leq k} |x_j| < \infty \right\}.
$$
Let $S : \mathbb{N} \to \mathbb{N}$ be an arbitrary increasing function and let $A : \omega \to \omega$ be an operator defined as
$$
A((x_j)_{j \in \mathbb{N}}) = (x_{(j+1)})_{j \in \mathbb{N}}.
$$
Let
$$
x^{(n)} = (0, \ldots, 0, n, 0, \ldots).
$$
Then $\|Ax^{(n)}\|_k = n$ and $\|x^{(n)}\|_{(k)} = 0$. Therefore there is no constant $C$ such that $\|Ax\|_k \leq C\|x\|_{(k)}$ for all $x \in \omega$, which proves that $\omega$ is not tame. $\blacksquare$

**Lemma 2.** Tameness is inherited by complemented subspaces.

**Proof.** Let $P : E \to X$ be a projection. If $A$ is a continuous linear operator on $X$ then the operator $A \circ P : E \to X$ is an element of $L(E)$. Thus
$$
\|Ax\|_k = \|A \circ Px\|_k \leq C_k \|x\|_{\sigma_{AP}(k)}
$$
and $\sigma_A(k) \leq \sigma_{AP}(k)$. If $\sigma_{AP}(k) \leq S(k)$ then $\sigma_A(k) \leq S(k)$ and thus if $E$ is tame then $X$ is tame as well. $\blacksquare$
Let $\phi : \mathbb{N} \to \mathbb{N}$ be an arbitrary increasing function and define spaces of linear continuous operators

$L_\phi(X) = \{ A \in L(X) : \forall k \in \mathbb{N} \exists C_k \forall x \in X \| Ax \|_k \leq C_k \| x \|_{\phi(k)} \}$,

$L_{\phi,n}(X) = \{ A \in L(X) : \forall k \geq n \exists C_k \forall x \in X \| Ax \|_k \leq C_k \| x \|_{\phi(k)} \}$.

If we put

$$\| A \|_{\phi(i),i} = \sup_{\| x \|_{\phi(i),i} \leq 1} \| Ax \|_i,$$

then $L_\phi(X)$ and $L_{\phi,n}(X)$ are Fréchet spaces with the sequences of seminorms defined as $\| \|_m = \max_{1 \leq i \leq m} \| \|_{\phi(i),i}$ and $\| \|_m = \max_{n \leq i \leq m} \| \|_{\phi(i),i}$, respectively. Only completeness needs a comment. If $(A_p)_p$ is a Cauchy sequence in $L_\phi(X)$ then for every $x \in X$ the sequence $(A_p x)_p$ is a Cauchy sequence in the complete space $X$. This means that for the operator $Ax = \lim_{p \to \infty} A_p x$ we have

$$\forall k \in \mathbb{N} \exists P \in \mathbb{N} : \| (A - A_P) x \|_k \leq \| x \|_{\phi(k)}.$$

This implies that $\| Ax \|_k \leq (C^P_k + 1) \| x \|_{\phi(k)} = D_k \| x \|_{\phi(k)}$ for all $k$, which shows that $A \in L_\phi(X)$. The proof in the case of $L_{\phi,n}(X)$ is the same.

**Lemma 3.** In every tame Fréchet space $X$ the following condition holds: there exists $\psi : \mathbb{N} \to \mathbb{N}$ such that for any $\phi : \mathbb{N} \to \mathbb{N}$ there exists $k \in \mathbb{N}$ such that for all $m \geq k$ there are $n \in \mathbb{N}$ and a constant $C_m > 0$ such that

(1) $\forall x^* \in X^*, y \in X : \max_{k \leq l \leq m} \| x^* \|_{\psi(l)} \| y \|_l \leq C_m \max_{1 \leq p \leq n} \| x^* \|_{\phi(p)} \| y \|_p$,

where $\| x^* \|_m = \sup_{\| x \|_m \leq 1} | x^*(x) |$.

**Proof.** If the space $X$ is tame with the function $\psi$ then every continuous linear operator is an element of a certain $L_{\psi,k}$ so we may write $L(X) = \bigcup_{k \in \mathbb{N}} L_{\psi,k}(X)$. If we now endow the space $L(X)$ with the topology of pointwise convergence then for every increasing function $\phi : \mathbb{N} \to \mathbb{N}$ we obtain the following diagram where the arrows represent continuous linear mappings:

$$\bigcup_{k} L_{\psi,k} \xrightarrow{id} L \xrightarrow{id} L_{\phi}$$

The continuity of the horizontal arrow comes from the following argument: for every 0-neighbourhood $U(0, x_1, \ldots, x_n, k, \varepsilon)$ of $L$ we define a 0-neighbourhood $V = \{ A \in L_{\phi} : \| A \|_k < \varepsilon/M \}$ in $L_{\phi}$, where $M = \max_{1 \leq i \leq n} \| x_i \|_k$. As is easily seen, $id(V) \subset U$. The continuity of the vertical arrow is proved similarly. Using Grothendieck’s Factorization Theorem [5, 24.33] we find a natural number $k$ such that $L_{\phi}$
embeds continuously in $L_{\psi,k}$. In other words in the tame Fréchet space the following holds:

$$
\exists \psi / \propto \forall \phi / \propto \exists k \forall m \geq k \exists n, C_m \forall T \in L_\phi(X) : \max_{k \leq i \leq m} \|T\|_{\psi(i),i} \leq C_m \max_{1 \leq p \leq n} \|T\|_{\phi(p),p}.
$$

In particular, for one-dimensional operators $T$, $Tx = x^*(x)y$, $x^* \in X$, $y \in X$, we get (1). ■

**Lemma 4.** Let $\lambda_p(I,A)$ be an arbitrary Köthe sequence space. If it is not quasi-normable then, without loss of generality, we may assume that $A$ satisfies the following conditions: $a_i^1 = 1$ for all $i$, and for every natural number $m$ there exists an index subset $J_m = \{i(m,j) : j \in \mathbb{N}\}$ such that

$$
\sup_{j} a^m_{i(m,j)} = c_m < \infty \quad \text{and} \quad \lim_{j} a^{m+1}_{i(m,j)} = \infty.
$$

**Proof.** From [2, Th. 17] it follows that if $\lambda_p(I,A)$ is not quasi-normable then

$$
\exists n \forall m \geq n \exists J \subset I : \inf_{i \in J} a^m_i > 0 \quad \text{and} \quad \inf_{i \in J} a^k_i = 0 \quad \text{for some } k(m) \geq m.
$$

Firstly, we may assume that $n = 1$ and $a_i^1 = 1$ for all $i$ (by dividing by $a_i^1$).

Secondly, every set $J_m$ is infinite so we may write $J_m = \{i(m,j) : j \in \mathbb{N}\}$.

Finally, omitting rows of the matrix $A$ suitably, numbers $k(m)$ can be chosen as $k(m) = m + 1$ for $m \in \mathbb{N}$. ■

2. Main results

**Proposition 5.** Every tame Fréchet space has a continuous norm.

**Proof.** If the space does not admit a continuous norm then from [1, Lemmas 1 and 2] it contains $\omega$ as a complemented subspace; but then from our assumption and Lemma 2, $\omega$ is tame, which contradicts Lemma 1. ■

**Theorem 6.** Tame Köthe sequence spaces are quasi-normable.

**Proof.** By Proposition 5 we may assume that $a_i^k > 0$ for all $i \in I$, $k \in \mathbb{N}$. Suppose that $\lambda_p(I,A)$ is a tame Köthe space which is not quasi-normable. Using Lemma 3 we may write

$$
\|x^*\|_{\psi(k)} \|y\|_k \leq C_k \max_{1 \leq p \leq n} \|x^*\|_{\phi(p)} \|y\|_p.
$$

Without losing of generality we may assume that $n \geq k$. For all $j, v \in \mathbb{N}$ define

$$
x_i^v x = x_{i(\phi(k-1),v)} \quad \text{and} \quad y_j = e_{i(k-1,j)},
$$

where $x_i$ denotes the $i$th coordinate of the vector $x$, $e_i$ is the $i$th vector of
the standard basis, and \( i(k, j) \) denotes the index of number \( j \) from the index set \( J_k \). Since \( \|y_j\|_p = a^p_{i(k, j)} \) and \( \|x^*_v\|^* = (a^l_1(\phi(k-1), v))^{-1} \), we obtain for all \( j, v \in \mathbb{N} \) the inequality

\[
\frac{a^k_{i(k, j)}}{a^p_{i(\phi(k-1), v)}} \leq C_k \max_{1 \leq p \leq n} \frac{a^p_{i(\phi(k-1), v)}}{a^p_{i(\phi(k-1), v)}}.
\]

The function \( \phi \) has been arbitrary so far but from now on we choose \( \phi(k-1) = \psi(k) \). Without loss of generality we may assume that \( \psi \) is strictly increasing, which, combined with Lemma 4, gives us

\[
a^k_{i(\phi(k-1), v)} = a^\phi_{i(\phi(k-1), v)} \leq c^\phi(k-1)
\]

for all \( v \) and

\[
a^k_{i(k-1, j)} \xrightarrow[j \to \infty]{} \infty.
\]

Equivalently we may write

\[
\frac{1}{c^\phi(k-1)} a^k_{i(k-1, j)} \leq \frac{a^k_{i(k-1, j)}}{a^\psi_{i(\phi(k-1), v)}}.
\]

The estimates of the right hand side of (4) will be divided into two cases. If \( p \leq k - 1 \) then

\[
a^p_{i(k-1, j)} \leq a^{k-1}_{i(k-1, j)} \leq c_{k-1} \quad \text{and} \quad a^\phi_{i(\phi(k-1), v)} \geq a^1_{i(\phi(k-1), v)} = 1,
\]

for all \( j, v \). If \( p \geq k \) then also \( \phi(p) \geq \phi(k) \geq \phi(k-1) + 1 \), which leads to

\[
a^\phi_{i(\phi(k-1), v)} \geq a^\phi_{i(\phi(k-1), v)} \xrightarrow[v \to \infty]{} \infty
\]

and

\[
a^p_{i(k-1, j)} \geq a^k_{i(k-1, j)} \xrightarrow[j \to \infty]{} \infty.
\]

This implies that for every natural number \( j \) there is an index \( v_j \in \mathbb{N} \) depending on \( k \) but not on \( p \) such that \( a^\phi_{i(\phi(k-1), v_j)} \geq a^p_{i(k-1, j)} \). If we now extract from \( \{x^*_v\}^\infty_{v=1} \) the subsequence \( \{x^*_v\}^\infty_{v=1} \), then we obtain the inequality

\[
\max_{1 \leq p \leq n} \frac{a^p_{i(k-1, j)}}{a^\phi_{i(\phi(k-1), v_j)}} \leq \max \{c_{k-1}, 1\} = d_k.
\]

Combining the inequalities (4), (6) and (7) we finally get

\[
a^k_{i(k-1, j)} \leq C_k c_{i(\phi(k-1), v_j)} a^p_{i(k-1, j)} < \infty \quad \text{for all} \; j;
\]

but, by (5), \( \lim_j a^k_{i(k-1, j)} = \infty \), a contradiction. This completes the proof. ■
References


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