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FUNCTIONAL ANALYSIS

Characterization of Globally Lipschitz Nemytskiĭ Operators Between Spaces of Set-Valued Functions of Bounded φ-Variation in the Sense of Riesz

by

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Summary. Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be two normed spaces and K be a convex cone in X. Let $CC(Y)$ be the family of all non-empty convex compact subsets of Y. We consider the Nemytskiĭ operators, i.e. the composition operators defined by $(Nu)(t) = H(t, u(t))$, where H is a given set-valued function. It is shown that if the operator N maps the space $RV_{\varphi_1}([a,b]; K)$ into $RW_{\varphi_2}([a,b]; CC(Y))$ (both are spaces of functions of bounded φ-variation in the sense of Riesz), and if it is globally Lipschitz, then it has to be of the form $H(t, u(t)) = A(t)u(t) + B(t)$, where $A(t)$ is a linear continuous set-valued function and B is a set-valued function of bounded φ_2-variation in the sense of Riesz. This generalizes results of G. Zawadzka [12], A. Smajdor and W. Smajdor [11], N. Merentes and K. Nikodem [5], and N. Merentes and S. Rivas [7].

1. Introduction. In [11] A. Smajdor and W. Smajdor proved that every globally Lipschitz Nemytskiĭ operator $(Nu)(t) = H(t, u(t))$ mapping the space $\mathrm{Lip}([a,b]; CC(Y))$ into itself admits the following representation:

$$(Nu)(t) = A(t)u(t) + B(t), \quad u \in \mathrm{Lip}([a,b]; CC(Y)),\ t \in [a,b],$$

where $A(t)$ is a linear continuous set-valued function and B is a set-valued function belonging to the space $\mathrm{Lip}([a,b]; CC(Y))$. The first such theorem for single-valued functions was proved by J. Matkowski [3] in the space of Lipschitz functions. A similar characterization of the Nemytskiĭ operator has also been obtained by G. Zawadzka [12] in the space of set-valued functions of bounded variation in the classical Jordan sense. For single-valued functions

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it was proved by J. Matkowski and J. Miś [4]. N. Merentes and K. Nikodem [5] and N. Merentes and S. Rivas [7] proved an analogous theorem in the space of set-valued functions of bounded p-variation in the sense of Riesz. Also, they proved a similar result in the case that the Nemytskiĭ operator N maps the space of functions of bounded p-variation in the sense of Riesz into the space of set-valued functions of bounded q-variation in the sense of Riesz, where $1 \leq q \leq p < \infty$, and N is globally Lipschitz.

The aim of this paper is to prove an analogous result in the case that the Nemytskiĭ operator N maps the space $RV_{\varphi_1}([a,b];K)$ of set-valued functions of bounded φ_1-variation in the sense of Riesz into the space $RW_{\varphi_2}([a,b];CC(Y))$ of set-valued functions of bounded φ_2-variation in the sense of Riesz and N is globally Lipschitz.

2. Preliminary results. In this section we introduce some definitions and recall known results concerning the Riesz φ-variation.

Definition 2.1. By a *φ-function* we mean a non-decreasing continuous function $\varphi : [0,\infty] \to [0,\infty]$ such that $\varphi(x) = 0$ if and only if $x = 0$, and $\varphi(x) \to \infty$ as $x \to \infty$.

Definition 2.2. Let $(X, \|\cdot\|)$ be a normed space and φ be a φ-function. Given a function $u : [a,b] \to X$ and a partition $\pi : a = t_0 < \cdots < t_n = b$ of the interval $[a,b]$, we define

$$\sigma_\varphi(u;\pi) := \sum_{i=1}^{n} \varphi\left(\frac{\|u(t_i) - u(t_{i-1})\|}{|t_i - t_{i-1}|}\right)|t_i - t_{i-1}|. \tag{2.1}$$

Denote by Π the set of all partitions π of $[a,b]$. Then the number

$$V_\varphi(u) = V_\varphi(u,[a,b];X) := \sup\{\sigma_\varphi(u;\pi) : \pi \in \Pi\}, \tag{2.2}$$

is called the *Riesz φ-variation* u on $[a,b]$. The function u is said to be of *bounded φ-variation* if $V_\varphi(u) < \infty$.

Denote by $RV_\varphi([a,b])$ the set of all functions $u : [a,b] \to X$ such that $V_\varphi(\lambda u) < \infty$ for some $\lambda > 0$. If φ is convex, then $RV_\varphi([a,b])$ is a Banach space endowed with the norm

$$\|u\|_\varphi := \|u(a)\| + \inf\{\varepsilon > 0 : V_\varphi(u/\varepsilon) \leq 1\}. \tag{2.3}$$

Also, if $u(a) = 0$ we set

$$\|u\|_{\varphi,0} = \inf\{\varepsilon > 0 : V_\varphi(u/\varepsilon) \leq 1\}. \tag{2.4}$$

It is known (see [1] or [2]) that convex φ-function φ with $\lim_{t\to\infty} \varphi(t)/t = r < \infty$ the following inequality holds:

$$\|u\|_{\varphi,0} \leq rV_1(u) \leq r(\varphi^{-1}(1) + (b-a))\|u\|_{\varphi,0} \quad \text{for all } u \in RV_\varphi[a,b], \tag{2.5}$$

where $V_1(u) := \sup_{\pi\in\Pi} \sum_{i=1}^{n} \|u(t_i) - u(t_{i-1})\|$. Consequently, $RV_\varphi[a, b] = BV[a, b]$, and there exist constants K_1 and K_2 such that

$$\|u\|_\varphi \le K_1\|u\|_{BV[a,b]} \le K_2\|u\|_\varphi \quad \text{for all } u \in RV_\varphi[a, b].$$

We need the following definition

DEFINITION 2.3. Let φ be a φ-function. We say that φ satisfies *condition* ∞_1 if

$$\limsup_{t\to\infty} \frac{\varphi(t)}{t} = \infty. \tag{2.6}$$

For φ convex, (2.6) is just $\lim_{t\to\infty} \varphi(t)/t = \infty$.

Let $CC(X)$ be the family of all non-empty convex compact subsets of X, and let D be the *Pompeiu–Hausdorff metric* in $CC(X)$, i.e.

$$D(A, B) := \inf\{t > 0 : A \subseteq B + tS,\ B \subseteq A + tS\},$$

where $S = \{y \in X : \|y\| \le 1\}$, or equivalently,

$$D(A, B) = \max\{e(A, B), e(B, A) : A, B \in CC(X)\},$$

where

$$e(A, B) = \sup\{d(x, B) : x \in A\}, \qquad d(x, B) = \inf\{d(x, y) : y \in B\}. \tag{2.7}$$

DEFINITION 2.4. Let φ be a *φ-function* and $F : [a, b] \to CC(X)$. We say that F has *bounded φ-variation in the Riesz sense* if

$$W^R_\varphi(F; [a, b]) := \sup_{\pi\in\Pi} \sum_{i=1}^{n} \varphi\biggl(\frac{D(F(t_i), F(t_{i-1}))}{|t_i - t_{i-1}|}\biggr)|t_i - t_{i-1}| < \infty. \tag{2.8}$$

Set

$$RW^*_\varphi[a, b] := \{F : [a, b] \to CC(X) : W^R_\varphi(F; [a, b]) < \infty\}, \tag{2.9}$$

$$RW_\varphi[a, b] := \{F : [a, b] \to CC(X) : W^R_\varphi(\lambda F) < \infty \text{ for some } \lambda > 0\}, \tag{2.10}$$

both equipped with the metric

$$D_\varphi(F_1, F_2) := D(F_1(a), F_2(a)) + \inf\{\varepsilon > 0 : W_\varphi(F_1/\varepsilon, F_2/\varepsilon) \le 1\}, \tag{2.11}$$

where

$$W_\varphi(F_1, F_2) := \sup_{\pi\in\Pi} \sum_{i=1}^{n} \varphi\biggl(\frac{D(F_1(t_i) + F_2(t_{i-1}), F_1(t_{i-1}) + F_2(t_i))}{|t_i - t_{i-1}|}\biggr)|t_i - t_{i-1}|.$$

Now, let $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ be two normed spaces and K be a convex cone in X. Given a set-valued function $H : [a, b] \times K \to CC(Y)$ we consider the *Nemytskiĭ operator* $N : K^{[a,b]} \to Y^{[a,b]}$ generated by H, i.e.

$$(Nu)(t) := H(t, u(t)), \qquad u \in K^{[a,b]}, \quad t \in [a, b]. \tag{2.12}$$

We denote by $L(K; CC(Y))$ the space of all set-valued linear functions $A : K \to CC(Y)$, i.e. additive and positively homogeneous.

In the proof of the main results of this paper we will use some facts which we list here as lemmas.

LEMMA 2.1 (see [9, Lemma 3]). *Let* $(X, \|.\|)$ *be a normed space and let* A, B, C *be subsets of* X. *If* A, B *are convex and* C *is non-empty and bounded, then*

$$D(A+C, B+C) = D(A, B). \tag{2.13}$$

LEMMA 2.2 (see [8, Th. 5.6]). *Let* $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ *be normed spaces and* K *be a convex cone in* X. *A set-valued function* $F : [a,b] \to CC(Y)$ *satisfies the Jensen equation*

$$F\left(\frac{x+y}{2}\right) = \frac{1}{2}(F(x) + F(y)), \qquad x, y \in K, \tag{2.14}$$

if and only if there exists an additive set-valued function $A : K \to CC(Y)$ *and a set* $B \in CC(Y)$ *such that* $F(x) = A(x) + B$, $x \in K$.

We will extend the result of N. Merentes and K. Nikodem [5] to set-valued functions of φ-bounded variation.

3. Main results

LEMMA 3.1. *If* φ *is a convex* φ*-function that satisfies condition* ∞_1 *and* $F \in RW_\varphi[a,b]$, *then* $F : [a,b] \to CC(X)$ *is continuous.*

Proof. Since $F \in RW_\varphi[a,b]$, there exists $M > 0$ such that

$$\sum_{i=1}^{n} \varphi\left(\frac{D(F(t_i), F(t_{i-1}))}{|t_i - t_{i-1}|}\right) |t_i - t_{i-1}| \le M \tag{3.15}$$

for all partitions of $[a,b]$; in particular given $t, t_0 \in [a,b]$, we have

$$\varphi\left(\frac{D(F(t), F(t_0))}{|t - t_0|}\right) |t - t_0| \le M. \tag{3.16}$$

Since φ is a convex φ-function, from the last inequality we get

$$D(F(t), F(t_0)) \le \frac{\varphi^{-1}\left(\frac{M}{|t-t_0|}\right)}{\frac{1}{|t-t_0|}}. \tag{3.17}$$

By (2.6),

$$\lim_{t\to t_0} D(F(t), F(t_0)) \le \lim_{t\to t_0} \frac{\varphi^{-1}\left(\frac{M}{|t-t_0|}\right)}{\frac{1}{|t-t_0|}} = \lim_{t\to\infty} \frac{Mt}{\varphi(t)} = 0. \tag{3.18}$$

This proves the continuity of F at t_0. Thus F is continuous on $[a,b]$.

Now, we are ready to formulate the main result of this work.

THEOREM 3.1. *Let* $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ *be normed spaces,* K *be a convex cone in* X *and* φ_1, φ_2 *be two convex* φ*-functions in* X, *strictly increasing,* *satisfying condition* ∞_1 *and such that there exist constants* c *and* T_0 *with* $\varphi_2(t) \leq \varphi_1(ct)$ *for all* $t \geq T_0$. *If the Nemytskiĭ operator* N *generated by a set-valued function* $H : [a,b] \times K \to CC(Y)$ *maps* $RV_{\varphi_1}([a,b];K)$ *into* $RW_{\varphi_2}([a,b];CC(Y))$ *and if it is globally Lipschitz, then the set-valued function* H *satisfies the following conditions*:

(a) *For every* $t \in [a,b]$ *there exists* $M(t) \in [0,\infty)$, *such that*

$$D(H(t,x),H(t,y)) \leq M(t)\|x-y\|, \qquad x,y \in K. \tag{3.19}$$

(b) *There are* $A : [a,b] \to L(K;CC(Y))$ *and* $B \in RW_{\varphi_2}([a,b];CC(Y))$ *such that*

$$H(t,x) = A(t)x + B(t), \qquad t \in [a,b],\ x \in K. \tag{3.20}$$

Proof. (a) Since N is globally Lipschitz, there exists a constant $M \in [0,\infty)$ such that

$$D_{\varphi_2}(Nu,Nv) \leq M\|u-v\|_{\varphi_1}, \qquad u,v \in RV_{\varphi_1}([a,b];K). \tag{3.21}$$

Using the definition of N and D_{φ_2} we obtain

$$\begin{aligned} &D(Nu(a),Nv(a)) \\ &\quad + \inf\left\{\varepsilon > 0 : \sup_\pi \sum_{i=1}^n \varphi_2\left(\frac{D(h_{t_i,t_{i-1}}N_{u,v}, h_{t_{i-1},t_i}N_{u,v})}{\varepsilon|t_i - t_{i-1}|}\right)|t_i - t_{i-1}| \leq 1\right\} \\ &\qquad \leq M\|u-v\|_{\varphi_1}, \qquad u,v \in RV_{\varphi_1}([a,b];K), \end{aligned} \tag{3.22}$$

where $h_{s,t}N_{u,v} := (Nu)(s) + (Nv)(t)$. In particular,

$$\inf\left\{\varepsilon > 0 : \varphi_2\left(\frac{D(d_{u,v}(H,t,\bar{t}), d_{u,v}(H,\bar{t},t))}{\varepsilon|\bar{t}-t|}\right)|\bar{t}-t| \leq 1\right\} \leq M\|u-v\|_{\varphi_1},$$

for all $u,v \in RV_\varphi([a,b];K)$ and $t,\bar{t} \in [a,b]$, $t \neq \bar{t}$, where $d_{u,v}(H,s,t) = H(s,u(s)) + H(t,v(t))$. Since φ_1 and φ_2 satisfy

$$\varphi_i\left(\varphi_i^{-1}\left(\frac{1}{|\bar{t}-t|}\right)\right)|\bar{t}-t| = 1, \qquad i = 1,2, \tag{3.23}$$

we obtain

$$\begin{aligned} &\inf\left\{\varepsilon > 0 : \varphi_2\left(\frac{D(d_{u,v}(H,t,\bar{t}), d_{u,v}(H,\bar{t},t))}{\varepsilon|\bar{t}-t|}\right)|\bar{t}-t| \leq 1\right\} \\ &\qquad\qquad = D(d_{u,v}(H,t,\bar{t}), d_{u,v}(H,\bar{t},t)). \end{aligned}$$

Therefore

$$D(d_{u,v}(H,t,\bar{t}), d_{u,v}(H,\bar{t},t)) \leq M\|u-v\|_{\varphi_1}|\bar{t}-t|\varphi_2^{-1}\left(\frac{1}{|\bar{t}-t|}\right). \tag{3.24}$$

Now, fix $t \in (a,b]$ and consider the function $\alpha : [a,b] \to [0,1]$ defined by

$$\alpha(s) := \begin{cases} \dfrac{s-a}{t-a}, & a \le s \le t, \\ 1, & t \le s \le b. \end{cases} \tag{3.25}$$

Then $\alpha \in RV_{\varphi_1}[a,b]$ and $V_{\varphi_1}(\alpha;[a,b]) = \varphi_1\left(\frac{1}{|t-a|}\right)|t-a|$.

Fix $x, y \in K$ and define $u, v : [a,b] \to K$ by

$$u(s) := x, \quad v(s) := \alpha(s)(y-x) + x, \quad s \in [a,b]. \tag{3.26}$$

Then $u, v \in RV_{\varphi_1}([a,b];K)$ and

$$\|u-v\|_\varphi = \|u(a) - v(a)\| + \inf\left\{\varepsilon > 0 : \sup_\pi \sum_{i=1}^n \varphi_1\left(\frac{\|(u-v)(t_i) - (u-v)(t_{i-1})\|}{\varepsilon|t_i - t_{i-1}|}\right)|t_i - t_{i-1}| \le 1\right\}. \tag{3.27}$$

From the definition of u and v we have

$$\|u-v\|_{\varphi_1} = \inf\left\{\varepsilon > 0 : \varphi_1\left(\frac{\|x-y\|}{\varepsilon|t-a|}\right)|t-a| \le 1\right\}. \tag{3.28}$$

From (3.23) we get

$$\inf\left\{\varepsilon > 0 : \varphi\left(\frac{\|x-y\|}{\varepsilon|t-a|}\right)|t-a| \le 1\right\} = \frac{\|x-y\|}{|t-a|\varphi_1^{-1}\left(\frac{1}{|t-a|}\right)}. \tag{3.29}$$

Hence,

$$D(d_{u,v}(H,t,\bar{t}), d_{u,v}(H,\bar{t},t)) \le \frac{M|\bar{t}-t|\varphi_2^{-1}\left(\frac{1}{|\bar{t}-t|}\right)\|x-y\|}{|t-a|\varphi_1^{-1}\left(\frac{1}{|t-a|}\right)}. \tag{3.30}$$

Hence, substituting in (3.24) the particular functions u and v defined above, and taking $\bar{t} = a$ in (3.30), we get

$$D(H(t,x) + H(a,x), H(a,x) + H(t,y)) \le M\,\frac{\varphi_2^{-1}\left(\frac{1}{|t-a|}\right)}{\varphi_1^{-1}\left(\frac{1}{|t-a|}\right)}\|x-y\| \tag{3.31}$$

for all $t \in (a,b]$ and $x, y \in K$. By Lemma 2.1 and the above inequality,

$$D(H(t,x), H(t,y)) \le M\,\frac{\varphi_2^{-1}\left(\frac{1}{|t-a|}\right)}{\varphi_1^{-1}\left(\frac{1}{|t-a|}\right)}\|x-y\| \tag{3.32}$$

for all $t \in (a,b]$ and $x, y \in K$.

Now, we have to consider the case $t = a$. Define $\beta : [a,b] \to [0,1]$ by

$$\beta(s) := \frac{s-a}{b-a}, \quad s \in [a,b]. \tag{3.33}$$

Then $\beta \in RV_{\varphi_1}[a,b]$ and $V_{\varphi_1}(\beta;[a,b]) = \varphi_1\left(\frac{1}{|b-a|}\right)|b-a|$.

Fix $x, y \in K$ and define $u, v : [a,b] \to K$ by

$$u(s) := x, \quad v(s) := \beta(s)(x-y) + y, \quad s \in [a,b]. \tag{3.34}$$

Then $u, v \in RV_{\varphi_1}([a,b];K)$ and

$$\begin{aligned}
(3.35)\quad \|u-v\|_{\varphi_1} &= \|x-y\| + \inf\left\{\varepsilon>0 : \varphi_1\left(\frac{\|x-y\|}{\varepsilon|b-a|}\right)|b-a|\le 1\right\}\\
&= \|x-y\| + \frac{\|x-y\|}{||b-a|\varphi_1^{-1}\left(\frac{1}{|b-a|}\right)}\\
&= \|x-y\|\left(1+\frac{1}{|b-a|\varphi_1^{-1}\left(\frac{1}{|b-a|}\right)}\right).
\end{aligned}$$

Substituting $\bar{t}=a$ and $t=b$ in (3.24), we get

$$(3.36)\qquad D(H(b,x)+H(a,y), H(a,x)+H(b,x)) \le MK(a,b,x,y,\varphi_1^{-1},\varphi_2^{-1})$$

for all $x, y \in K$, where

$$\begin{aligned}
&K(a,b,x,y,\varphi_1^{-1},\varphi_2^{-1})\\
&\qquad = |b-a|\varphi_2^{-1}\left(\frac{1}{|b-a|}\right)\|x-y\|\left(1+\frac{1}{|b-a|\varphi_1^{-1}\left(\frac{1}{|b-a|}\right)}\right).
\end{aligned}$$

By Lemma 2.1 and the last inequality we get

$$\begin{aligned}
(3.37)\qquad &D(H(a,x),H(a,y))\\
&\qquad \le M|b-a|\varphi_2^{-1}\left(\frac{1}{|b-a|}\right)\|x-y\|\left(1+\frac{1}{|b-a|\varphi_1^{-1}\left(\frac{1}{|b-a|}\right)}\right)
\end{aligned}$$

for all $x, y \in K$.

Define $M : [a,b] \to \mathbb{R}$ by

$$(3.38)\quad M(t) := \begin{cases} M\,\dfrac{\varphi_2^{-1}\left(\frac{1}{|t-a|}\right)}{\varphi_1^{-1}\left(\frac{1}{|t-a|}\right)}, & t\in(a,b],\\[2ex] M|b-a|\varphi_2^{-1}\left(\dfrac{1}{|b-a|}\right)\left(1+\dfrac{1}{|b-a|\varphi_1^{-1}\left(\frac{1}{|b-a|}\right)}\right), & t=a.\end{cases}$$

Hence

$$(3.39)\qquad D(H(t,x),H(t,y)) \le M(t)\|x-y\|, \qquad x,y\in X,\ t\in[a,b].$$

Consequently, for every $t \in [a,b]$ the function $H(t,\cdot) : K \to CC(Y)$ is continuous.

(b) Fix $t, t_0 \in [a,b]$ such that $t_0 < t$. Since N is globally Lipschitz, there exists a constant $M > 0$ such that

$$(3.40)\qquad D(d_{u,v}(H,t,t_0), d_{u,v}(H,t_0,t)) \le M|t_0-t|\varphi_2^{-1}\left(\frac{1}{|t_0-t|}\right)\|u-v\|_{\varphi_1},$$

where $d_{u,v}(H,s,t) = H(s,u(s)) + H(t,v(t))$. Define $\gamma : [a,b] \to [0,1]$ by

$$\gamma(s) = \begin{cases} \dfrac{s-a}{t_0-a}, & a \le s \le t_0, \\ \dfrac{t-s}{t-t_0}, & t_0 \le s \le t, \\ 0, & t \le s \le b. \end{cases}$$

Then $\gamma \in RV_{\varphi_1}[a,b]$. Fix $x, y \in K$ and define $u, v : [a,b] \to K$ by

$$\begin{aligned} u(s) &:= \frac{\gamma(s)}{2}x + \left(1 - \frac{\gamma(s)}{2}\right)y, \\ v(s) &:= \left(\frac{1+\gamma(s)}{2}\right)x + \left(\frac{1-\gamma(s)}{2}\right)y, \qquad s \in [a,b]. \end{aligned} \tag{3.41}$$

Then $u, v \in RV_{\varphi_1}([a,b];K)$ and

$$\|u - v\| = \|u(a) - v(a)\| = \frac{\|x-y\|}{2}. \tag{3.42}$$

Hence, substituting in (3.40) the particular functions u, v defined in (3.41), we obtain

$$D\left(H(t_0,x) + H(t,y), H\left(t_0, \frac{x+y}{2}\right) + H\left(t, \frac{x+y}{2}\right)\right) \le M|t-t_0|\varphi_2^{-1}\left(\frac{1}{|t-t_0|}\right)\frac{\|x-y\|}{2}. \tag{3.43}$$

Since N maps $RV_{\varphi_1}([a,b];K)$ into $RW_{\varphi_2}([a,b];CC(Y))$, it follows that $H(\cdot,z)$ is continuous for all $z \in K$. Hence, letting $t_0 \uparrow t$ in (3.43), we obtain

$$D\left(H(t,x) + H(t,y), H\left(t, \frac{x+y}{2}\right) + H\left(t, \frac{x+y}{2}\right)\right) = 0 \tag{3.44}$$

for all $t \in [a,b]$ and $x, y \in K$. Thus

$$H\left(t, \frac{x+y}{2}\right) + H\left(t, \frac{x+y}{2}\right) = H(t,x) + H(t,y). \tag{3.45}$$

Since the values of H are convex, we have

$$H\left(t, \frac{x+y}{2}\right) = \frac{1}{2}\{H(t,x) + H(t,y)\} \tag{3.46}$$

for all $t \in [a,b]$ and $x, y \in K$. Thus for all $t \in [a,b]$, the set-valued function $H(t,\cdot) : K \to CC(Y)$ satisfies the Jensen equation (2.14). Now by Lemma 2.2, there exist an additive set-valued function $A(t) : K \to CC(Y)$ and a set $B(t) \in CC(Y)$ such that

$$H(t,x) = A(t)(x) + B(t), \qquad x \in K,\ t \in [a,b]. \tag{3.47}$$

From (3.47) and (3.39) we deduce that for all $t \in [a,b]$ there exists $M(t) \in [0,\infty)$ such that

$$D(A(t)x, A(t)y) \le M(t)\|x-y\|, \quad x, y \in K. \tag{3.48}$$

Consequently, for every $t \in [a,b]$ the set-valued function $A(t) : K \to CC(Y)$ is continuous, and $A(t) \in L(K; CC(Y))$. Since $A(t)$ is additive and $0 \in K$, we have $A(t) = 0$ for all $t \in [a,b]$ and $H(\cdot, 0) = B(\cdot)$.

The Nemytskiĭ operator N maps $RV_{\varphi_1}([a,b]; K)$ into $RW_{\varphi_2}([a,b]; CC(Y))$. Therefore $H(\cdot,0) = B(\cdot) \in RW_{\varphi_2}([a,b]; K)$. Consequently, the set-valued function H has to be of the form $H(t,x) = A(t)x + B(t)$ for all $t \in [a,b]$ and $x \in K$, where $A(t) \in L(K; CC(Y))$ and $B \in RW_{\varphi_2}([a,b]; CC(Y))$.

THEOREM 3.2. *Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ be normed spaces, K be a convex cone in X and φ_1, φ_2 be two convex φ-functions in X, strictly increasing, satifying condition ∞_1 and $\lim_{t\to\infty} \varphi_2^{-1}(\varphi_1(t))/t = \infty$. If the Nemytskiĭ operator N generated by a set-valued function $H : [a,b] \times K \to CC(Y)$ maps $RV_{\varphi_2}([a,b]; K)$ into $RW_{\varphi_1}([a,b]; CC(Y))$ and is globally Lipschitz, then*

$$H(t,x) = H(t,0) \quad \text{for } t \in [a,b],\ x \in K;$$

i.e. the Nemytskiĭ operator is constant.

Proof. Since N is globally Lipschitz from $RV_{\varphi_2}([a,b]; K)$ to $RW_{\varphi_1}([a,b]; CC(Y))$, there exists a constant M such that

$$D_{\varphi_1}(Nu, Nv) \le M\|u-v\|_{\varphi_2}, \quad u, v \in RV_{\varphi_2}([a,b]; K). \tag{3.49}$$

Fix $t, t_0 \in [a,b]$ such that $t_0 < t$. Using the definition of N and D_{φ_1} we obtain

$$D(d_{u,v}(H,t,t_0), d_{u,v}(H,t_0,t)) \le M\|u-v\|_{\varphi_2}|t-t_0|\varphi_1^{-1}\left(\frac{1}{|t-t_0|}\right) \tag{3.50}$$

for $u, v \in RV_{\varphi_2}([a,b]; K)$.

Define $\alpha : [a,b] \to [0,1]$ by

$$\alpha(s) := \begin{cases} 1, & a \le s \le t_0, \\ -\dfrac{s-t}{t-t_0}, & t_0 \le s \le t, \\ 0, & t \le s \le b. \end{cases}$$

Then $\alpha \in RV_{\varphi_2}[a,b]$ and $V_{\varphi_2}(\alpha; [a,b]) = |t-t_0|\varphi_2^{-1}\left(\frac{1}{|t-t_0|}\right)$.

Fix $x \in K$ and define $u, v : [a,b] \to K$ by

$$u(s) := x, \quad v(s) := \alpha(s)x, \quad s \in [a,b]. \tag{3.51}$$

Then $u, v \in RV_{\varphi_2}([a,b]; K)$ and

$$\begin{aligned}(3.52)\qquad \|u-v\|_{\varphi_2} &= \|u(a)-v(a)\| \\ &+ \inf\left\{\varepsilon>0 : \sup_{\pi}\sum_{i=1}^{n}\varphi_2\left(\frac{\|(u-v)(t_i)-(u-v)(t_{i-1}\|}{\varepsilon|t_i-t_{i-1}|}\right)|t_i-t_{i-1}|\le 1\right\} \\ &= \inf\left\{\varepsilon>0 : \varphi_2\left(\frac{\|x\|}{\varepsilon|t-t_0|}\right)|t-t_0|\le 1\right\} = \frac{\|x\|}{|t-t_0|\varphi_2^{-1}\left(\frac{1}{|t-t_0|}\right)}.\end{aligned}$$

Hence, substituting in (3.50) the particular functions u, v defined in (3.51), we obtain

$$\begin{aligned}(3.53)\qquad & D(H(t,x)+H(t_0,x), H(t_0,x)+H(t,0)) \\ &\qquad \le M|t-t_0|\,\frac{\varphi_1^{-1}\left(\frac{1}{|t-t_0|}\right)}{|t-t_0|\varphi_2^{-1}\left(\frac{1}{|t-t_0|}\right)}\,\|x\|.\end{aligned}$$

Then

$$(3.54)\qquad D(H(t,x)+H(t_0,x), H(t_0,x)+H(t,0)) \le M\,\frac{\varphi_1^{-1}\left(\frac{1}{|t-t_0|}\right)}{\varphi_2^{-1}\left(\frac{1}{|t-t_0|}\right)}\,\|x\|.$$

By Lemma 2.1 and the above inequality, we get

$$(3.55)\qquad D(H(t,x), H(t,0)) \le M\|x\|\,\frac{\varphi_1^{-1}\left(\frac{1}{|t-t_0|}\right)}{\varphi_2^{-1}\left(\frac{1}{|t-t_0|}\right)}.$$

Since $\lim_{t\to\infty}\varphi_2^{-1}(\varphi_1(t))/t=\infty$, letting $t_0\uparrow t$ in the last inequality, we have

$$(3.56)\qquad D(H(t,x), H(t,0)) = 0.$$

Thus for all $t\in[a,b]$ and for all $x\in K$, we get

$$(3.57)\qquad H(t,x) = H(t,0).$$

THEOREM 3.3. *Let $(X,\|\cdot\|)$, $(Y,\|\cdot\|)$ be normed spaces, K be a convex cone in X and φ be a convex φ-function in X satisfying condition ∞_1. If the Nemytskiĭ operator N generated by a set-valued function $H:[a,b]\times K\to CC(Y)$ maps $RV_\varphi([a,b];K)$ into $BW([a,b];CC(Y))$ and is globally Lipschitz, then the left regularization $H^*:[a,b]\times K\to CC(Y)$ of H defined by*

$$H^*(t,x) = \begin{cases} H^-(t,x), & t\in(a,b],\ x\in K, \\ \lim\limits_{s\downarrow a} H(s,x), & t=a,\ x\in K, \end{cases}$$

satisfies the following conditions:

(a) *For every $t\in[a,b]$ there exists $M(t)\in[0,\infty)$ such that*

$$(3.58)\qquad D_1(H^*(t,x), H^*(t,y)) \le M(t)\|x-y\|,\qquad x,y\in K.$$

(b) *There are functions* $A : [a,b] \to L(K; CC(Y))$ *and* $B \in BW([a,b]; CC(Y))$ *such that*

$$H^*(t,x) = A(t)x + B(t), \qquad t \in [a,b],\ x \in K. \tag{3.59}$$

Proof. Fix $t \in [a,b]$ and consider $\alpha : [a,b] \to [0,1]$ defined by

$$\alpha(s) := \begin{cases} 1, & a \le s \le t, \\ \dfrac{s-b}{t-b}, & t \le s \le b. \end{cases} \tag{3.60}$$

Then $\alpha \in RV_\varphi[a,b]$ and $V_\varphi(\alpha; [a,b]) = \varphi\big(\frac{1}{|t-b|}\big)|t-b|$.

Fix $x, y \in K$ and define $u, v : [a,b] \to K$ by

$$u(s) := x, \quad v(s) := \alpha(s)(y-x) + x, \qquad s \in [a,b]. \tag{3.61}$$

Then $u, v \in RV_\varphi([a,b]; K)$ and

$$\|u-v\|_\varphi = \|u(a) - v(a)\| + \inf\left\{\varepsilon > 0 : \sup_\pi \sum_{i=1}^n \varphi\left(\frac{\|(u-v)(t_i) - (u-v)(t_{i-1})\|}{\varepsilon|t_i - t_{i-1}|}\right)|t_i - t_{i-1}| \le 1\right\}. \tag{3.62}$$

From the definition of u and v we obtain

$$\|u-v\|_\varphi = \|x-y\| + \left(1 + \frac{1}{|b-t|\varphi^{-1}\big(\frac{1}{|b-t|}\big)}\right). \tag{3.63}$$

Since N is globally Lipschitz, there exists a constant $M > 0$ such that

$$D(H(b,u(b)) + H(t,v(t)), H(t,u(t)) + H(b,v(b))) \le M\|u-v\|_\varphi$$

for $u, v \in RV_\varphi([a,b]; K)$. Substituting u, v defined by (3.61), we obtain

$$D(H(b,x)+H(t,y), H(b,x)+H(t,x)) \le M(t)\|x-y\|_\varphi\left(1+\frac{1}{|b-t|\varphi^{-1}\big(\frac{1}{|b-t|}\big)}\right)$$

for all $t \in [a,b)$ and $x, y \in K$. By Lemma 2.1 we get

$$D(H(t,x), H(t,y)) \le M(t)\|x-y\|_\varphi\left(1 + \frac{1}{|b-t|\varphi^{-1}\big(\frac{1}{|b-t|}\big)}\right) \tag{3.64}$$

for all $t \in [a,b)$ and $x, y \in K$.

For $t = b$, by a similar reasoning, we show that there exists a constant $M(b) > 0$ such that

$$D(H(t,x), H(t,y)) \le M(b)\|x-y\|_\varphi \quad \text{for } x, y \in K.$$

Define the function $M : [a,b] \to \mathbb{R}$ by

$$M(t) := \begin{cases} M\left(1 + \dfrac{1}{|b-t|\varphi^{-1}\big(\frac{1}{|b-t|}\big)}\right), & a \le t < b, \\ M(b), & t = b. \end{cases} \tag{3.65}$$

Hence

$$D(H(t,x),H(t,y)) \le M(t)\|x-y\|_{\varphi}, \quad t\in[a,b],\ x,y\in K. \tag{3.66}$$

Passing to the limit in (3.64) and by the inequality (3.66) and the definition of H^* we conclude that for all $t\in[a,b]$ there exists $M(t)\in[0,\infty)$ such that

$$D(H^*(t,x),H^*(t,y)) \le M(t)\|x-y\| \quad \text{for } x,y\in K.$$

Now we shall prove that

$$H^*(t,x)=A(t)x+B(t), \quad t\in[a,b], x\in K,$$

where $A(t)$ is a linear continuous set-valued function, and $B\in BW([a,b];CC(Y))$.

Fix $t,t_0\in[a,b]$ with $t_0<t$ and $n\in\mathbb{N}$. Define the partition π_n of the interval $[t_0,t]$ by

$$\pi_n : a<t_0<t_1<\cdots<t_{2n-1}<t_{2n}=t, \quad \text{where} \quad t_i-t_{i-1}=\frac{t-t_0}{2n},$$

$i=1,\dots,2n$.

Since N is globally Lipschitz from $RV_{\varphi}([a,b];K)$ to $BW([a,b];CC(Y))$, there exists a constant $M>0$ such that

$$\sum_{i=1}^{n} D\left(d_{u,v}(H,t_{2i},t_{2i-1}),d_{u,v}(H,t_{2i-1},t_{2i})\right)\right) \le M\|u-v\|_{\varphi}, \tag{3.67}$$

where $u,v\in RV_{\varphi}([a,b];K)$ and $d_{u,v}(H,s,t)=H(s,u(s))+H(t,v(t))$.

Define $\alpha:[a,b]\to[0,1]$ in the following way:

$$\alpha(s):=\begin{cases} 0, & a\le s\le t_{i-1},\\ \dfrac{s-t_{i-1}}{t_i-t_{i-1}}, & t_{i-1}\le s\le t_i,\ i=1,3,\dots,2n-1,\\ -\dfrac{s-t_i}{t_i-t_{i-1}}, & t_{i-1}\le s\le t_i,\ i=2,4,\dots,2n,\\ 0, & t_i\le s\le b. \end{cases} \tag{3.68}$$

Then $\alpha\in RV_{\varphi}([a,b];K)$ and $V_{\varphi}(\alpha;[a,b])=|t-t_0|\varphi\big(\frac{2n}{|t-t_0|}\big)$.

Fix $x,y\in K$ and define $u,v:[a,b]\to K$ by

$$\begin{aligned} u(s)&:=\frac{\alpha(s)}{2}x+\left(1-\frac{\alpha(s)}{2}\right)y, \quad s\in[a,b].\\ v(s)&:=\frac{1+\alpha(s)}{2}x+\frac{1-\alpha(s)}{2}y, \quad s\in[a,b]. \end{aligned} \tag{3.69}$$

Then $u,v\in RV_{\varphi}([a,b];K)$ and $\|u-v\|_{\varphi}=\|x-y\|/2$. Hence, substituting in (3.67) the particular functions u,v defined in (3.69), we obtain

$$\sum_{i=1}^{n} D\left(H(t_{2i},y)+H(t_{2i-1},x), H\left(t_{2i},\frac{x+y}{2}\right)+H\left(t_{2i-1},\frac{x+y}{2}\right)\right) \le M\frac{\|x-y\|}{2} \tag{3.70}$$

for all $x, y \in K$. Since N maps $RV_\varphi([a,b];K)$ into $BW([a,b];CC(Y))$, it follows that $H(\cdot,z) \in BW([a,b];CC(Y))$ for all $z \in K$. Hence, letting $t_0 \uparrow t$ in (3.70), we get

$$D\left(H^*(t,y)+H^*(t,x), H^*\left(t,\frac{x+y}{2}\right)+H^*\left(t,\frac{x+y}{2}\right)\right) \le M\frac{\|x-y\|}{2n} \tag{3.71}$$

for all $x, y \in K$ and $n \in \mathbb{N}$. Passing to the limit as $n \to \infty$ we get

$$H^*\left(t,\frac{x+y}{2}\right)+H^*\left(t,\frac{x+y}{2}\right) = H^*(t,y)+H^*(t,x) \tag{3.72}$$

for all $x, y \in K$ and $t \in [a,b]$.

Since $H^*(t,x)$ is a convex set, we have

$$H^*\left(t,\frac{x+y}{2}\right) = \frac{1}{2}(H^*(t,y)+H^*(t,x)) \tag{3.73}$$

for all $x, y \in K$ and $t \in [a,b]$.

Thus for every $t \in [a,b]$ the set-valued function $H^*(t,\cdot) : K \to CC(Y)$ satisfies the Jensen equation (2.14). By Lemma 2.2, for all $t \in [a,b]$ there exist an additive set-valued function $A(t) : K \to CC(Y)$ and a set $B(t) \in CC(Y)$ such that

$$H^*(t,x) = A(t)x + B(t) \quad \text{for } t \in [a,b] \text{ and } x \in K.$$

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